SIMPLE GABOR FEATURE SPACE FOR INVARIANT OBJECT RECOGNITION

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Abstract

Invariant object recognition is one of the most challenging problems in computer vision. The authors propose a simple Gabor feature space, which has been successfully applied to applications, e.g., in invariant face detection to extract facial features in demanding environments. In the proposed feature space, illumination, rotation, scale, and translation invariant recognition of objects can be realized within a reasonable amount of computation. In this study, fundamental properties of Gabor features, construction of the simple feature space, and invariant search operations in the feature space are discussed in more detail.

Key words: Gabor filter, invariant object recognition, feature extraction

1 Introduction

Invariant object recognition has been one of the most important topics in computer vision research for decades. Often the research has focused on invariant features. They make invariant recognition possible in a straightforward manner, but also present some pitfalls as an invariant feature is always a generalization and may lack some useful information. In addition, a single feature rarely allows an accurate recognition but relationships between several features need to be examined. It seems that requirements of a truly invariant system cannot be exclusively met with a global invariant feature, such as the moment invariants; at least not within a reasonable computing complexity.

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Since the study of a general image processing operator by Granlund (1978), Gabor filters have been used in feature extraction from images. With properly chosen filter parameters, Gabor filters have similar characteristics to multi-resolution techniques, such as Gabor expansion and wavelets (e.g. Bastiaans, 1980; Daubechies, 1990; Lee, 1996). Thus, due to duality between Gabor expansion and Gabor filters it is reasonable to assume that similar properties can be established also for Gabor filters. The authors have recently proposed a novel framework utilizing Gabor filter based local features in rotation, scale, and translation invariant recognition of objects (Kamarainen et al., 2002a; Hamouz et al., 2003). The framework is based on a feature space constructed from Gabor filter responses and invariant search operations established in the feature space. In this study preceding research is concluded by showing the most important properties of Gabor filters inducing invariance and by describing the feature space in more detail. The feature space assumes that objects or their parts can be distinguished by localized features, and in this study particularly, at only one location. Despite this restriction, the approach performs outstandingly well in applications, and there is evidence that several local features can be combined effectively. It should be noted that features themselves are not invariant, but simple operations can be established to achieve illumination, rotation, scale, and translation invariance; some or all of them simultaneously.

2 Feature Extraction with Gabor Filters

There are two general approaches to use Gabor functions in feature extraction: i) Gabor expansion and ii) Gabor filtering. In the expansion, Gabor functions represent distinct, not necessarily orthogonal, pieces of image information. A distinct advantage of Gabor functions is their optimality in time and frequency, or space and spatial-frequency in two dimensions, providing the smallest possible pieces of information about time-frequency events (Gabor, 1946). Any well-behaving function can be represented as a linear combination of Gabor functions (Bastiaans, 1980; Gabor, 1946). However, Gabor expansion requires a computation of biorthogonal analysis functions, which may turn to a very time consuming task. Thus, it has been more common to use Gabor functions as analysis filters in image processing. Lades et al. (1993) use Gabor filter responses at single spatial location, but do not take advantage of existing invariance properties. Use of Gabor filters is also motivated by the physiology of mammalian visual system (Daugman, 1985). Nevertheless, image analysis with biorthogonal counterparts of Gabor functions can be thought as a dual to the analysis with Gabor filters, and thus, also Gabor filters extract space–spatial-frequency events from images.

The Gabor's original 1-d function of time and frequency has been generalized to a 2-d function of space and spatial-frequency and several forms have been proposed (e.g. Granlund, 1978; Daugman, 1985). The authors have defined the following form of a normalized 2-d Gabor filter function in the continuous spatial domain (Kyrki, 2002)

$$\begin{aligned} \psi(x, y; f, \theta) &= \frac{f^2}{\pi\gamma\eta} e^{-\left(\frac{f^2}{\gamma^2}x'^2 + \frac{f^2}{\eta^2}y'^2\right)} e^{j2\pi f x'} \\ x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \tag{1}$$

where f is the frequency of a sinusoidal plane wave, θ is the anti-clockwise rotation of the Gaussian envelope and the sinusoid, γ is the spatial width of the filter along the plane wave, and η the spatial width perpendicular to the wave.

The form in Eq. (1) is centered to the origin and its response at any location (x, y) for an image function $\xi(x, y)$ can be calculated with the convolution

$$\begin{aligned} r_\xi(x, y; f, \theta) &= \psi(x, y; f, \theta) * \xi(x, y) \\ &= \iint_{-\infty}^{\infty} \psi(x - x_\tau, y - y_\tau; f, \theta)\xi(x_\tau, y_\tau)dx_\tau dy_\tau \end{aligned} \tag{2}$$

The response of Gabor filter in Eq. (2) is the low-level Gabor feature used in this study. Next, it is shown how Gabor features can be used to extract exact information of pose and scale changes when a geometric manipulation of an input image $\xi(x, y)$ is a function of the Gabor filter parameters: (x, y) for translation, θ for rotation, and f for scale. Finally, in a feature space constructed from these Gabor filter responses, invariant search operations can be established based on the rotation, scale, and translation properties of Gabor features, which are considered next.

3 Invariance properties of Gabor features

Invariance theorems in this section are not complicated as they result from the properties of convolution, but will be briefly revisited since the literature on Gabor filtering seems to lack them and they form the theoretical basis for the feature space. Similar theorems generally hold for any function, which maintains the same shape in all translations, scales, and orientations; probably the most popular examples being wavelets over scales and translations (e.g. Daubechies, 1990). The Gabor feature in Eq. (2) may provide properties similar to multi-resolution analysis, and in addition, the feature behaves smoothly (Kamarainen et al., 2002d), which is difficult to achieve with wavelets. In this sense Gabor filters approximate joint shiftability conditions better than orthogonal wavelets (Simoncelli et al., 1992). The following theorems are given in the continuous domain, where for example the translation invariance is trivial, but the shiftability concept becomes important in the discrete case where the discrete sampling of parameters may destroy invariance of non-shiftable functions and transforms. The smooth behavior of Gabor filter responses also provides tolerance for small object deformations, noise, and image distortions (Kamarainen et al., 2002c,d).

3.1 Rotation

For an image $\xi_1(x, y)$ Gabor feature, the response of normalized Gabor filter in Eq. (2), at location (x_0, y_0) is

$$r_{\xi_1}(x_0, y_0; f, \theta) = \iint_{-\infty}^{\infty} \psi(x_0 - x_\tau, y_0 - y_\tau; f, \theta)\xi_1(x_\tau, y_\tau)dx_\tau dy_\tau \tag{3}$$

Next, a rotated version $\xi_2(x, y)$ of an image $\xi_1(x, y)$ is considered, where the image $\xi_1(x, y)$ is rotated anti-clockwise around the spatial location (x_0, y_0) by an angle ϕ as

$$\begin{aligned} \xi_2(x, y) &= \xi_1(\hat{x}, \hat{y}) \\ \hat{x} &= (x - x_0)\cos\phi + (y - y_0)\sin\phi + x_0 \\ \hat{y} &= -(x - x_0)\sin\phi + (y - y_0)\cos\phi + y_0 \end{aligned} \tag{4}$$

By substituting the rotated image in Eq. (4) into the convolution formula in Eq. (3) and by integrating over integration axes (x'_τ, y'_τ) rotated around the same point (x_0, y_0) by the angle $-\phi$ it is straightforward to show that

$$\begin{aligned} r_{\xi_2}(x_0, y_0; f, \theta) &= \iint_{-\infty}^{\infty} \psi(x_0 - x'_\tau, y_0 - y'_\tau; f, \theta - \phi)\xi_1(x'_\tau, y'_\tau)dx'_\tau dy'_\tau \\ &= r_{\xi_1}(x_0, y_0; f, \theta - \phi) \end{aligned} \tag{5}$$

which is due to the rotation property of Gabor features. The rotation property appearing as the equivalence in Eq. (5) proves that the response of Gabor filter for a rotated image is equal to the response of correspondingly rotated filter for the original image without rotation. This result holds for any continuous domain function that retains the same shape regardless of orientation parameter.

3.2 Scaling

Next, an image which is homogeneously scaled version of the image $\xi_1(x, y)$ is considered. The new image is scaled by a factor a as $\xi_3 = \xi_1(ax, ay)$. A response for the scaled image is

$$\begin{aligned} r_{\xi_3}(x, y; f, \theta) &= \iint_{-\infty}^{\infty} \psi(x - x_\tau, y - y_\tau; f, \theta) \xi_3(x_\tau, y_\tau) dx_\tau dy_\tau \\ &= \iint_{-\infty}^{\infty} \xi_3(x - x_\tau, y - y_\tau) \psi(x_\tau, y_\tau; f, \theta) dx_\tau dy_\tau \\ &= \iint_{-\infty}^{\infty} \xi_1(ax - ax_\tau, ay - ay_\tau) \psi(x_\tau, y_\tau; f, \theta) dx_\tau dy_\tau \end{aligned} \tag{6}$$

By making substitutions

$$\hat{x}_\tau = ax_\tau \Rightarrow dx_\tau = \frac{d\hat{x}_\tau}{a}, \quad \hat{y}_\tau = ay_\tau \Rightarrow dy_\tau = \frac{d\hat{y}_\tau}{a} \tag{7}$$

and by respectively re-ordering the variables in Eq. (1), Eq. (6) can be rewritten as

$$\begin{aligned} r_{\xi_3}(x, y; f, \theta) &= \iint_{-\infty}^{\infty} \xi_1(ax - \hat{x}_\tau, ay - \hat{y}_\tau) \psi(\hat{x}_\tau, \hat{y}_\tau; \frac{f}{a}, \theta) d\hat{x}_\tau d\hat{y}_\tau \\ &= r_{\xi_1}(ax, ay; \frac{f}{a}, \theta) \end{aligned} \tag{8}$$

The scale property of Gabor features, shown by the equality in Eq. (8), proves the response of Gabor filter for a scaled image to be equal to the response of a correspondingly scaled Gabor filter at the same relative location of the original image. The scale property is fundamental for wavelets (Daubechies, 1990) and holds also for the normalized Gabor filter in Eq. (1) (Porat and Zeevi, 1988). It is important to note that this property depends on the normalization of the filter and is not valid for all forms of Gabor filters presented in the literature.

3.3 Translation

In the case of the centered filter in Eq. (1) translation invariant search is trivial as features using Eq. (2) can be calculated at any location (x, y). For a translated image

$$\xi_4 = \xi_1(x - x_1, x - y_1) \tag{9}$$

the response at (x, y) is

$$\begin{aligned} r_{\xi_4}(x, y; f, \theta) &= \iint_{-\infty}^{\infty} \xi_4(x - x_\tau, y - y_\tau) \psi(x_\tau, y_\tau; f, \theta) dx_\tau dy_\tau \\ &= \iint_{-\infty}^{\infty} \xi_1(x - x_1 - x_\tau, y - y_1 - y_\tau) \psi(x_\tau, y_\tau; f, \theta) dx_\tau dy_\tau \\ &= r_{\xi_1}(x - x_1, y - y_1; f, \theta) \end{aligned} \tag{10}$$

which implies the translation property.

3.4 Illumination

Uniform illumination change can be modeled as a multiplication by a constant. Let $\xi_5(x, y) = c\,\xi_1(x, y)$. By the linearity of the convolution, it can be written

$$r_{\xi_5}(x, y; f, \theta) = c\, r_{\xi_1}(x, y; f, \theta) \tag{11}$$

4 Simple Gabor feature space

According to the properties defined in the previous section it can be stated for an image ξ_6 which is equal to ξ_1 but rotated by ϕ, scaled by a and intensity multiplied by c, that

$$r_{\xi_6}(x_0, y_0; f, \theta) = c\, r_{\xi_1}(ax_0, ay_0; \frac{f}{a}, \theta - \phi) \tag{12}$$

In general, an equality similar to Eq. (12) can be obtained for all functions, which retain the same shape regardless of translation, scale or orientation. Yet the result cannot be easily obtained by e.g. discrete wavelets but it is possible to inherit important wavelet properties to Gabor functions (Lee, 1996).

So far only features at a single location (x, y) have been considered, but even then Gabor filters capture information from a remarkable large area, and thus, by combining responses of several filters in different orientations and frequencies, surprisingly complex objects can be represented (e.g. Krüger and Sommer, 2002). For more complicated object structures and objects where discriminative information at one location is not sufficient, features must be combined from several spatial locations (Lades et al., 1993; Krüger and Sommer, 2002). In that case the invariant properties are more difficult to use, yet still possible. Since the presented feature space is restricted to a single spatial location, it is referred to as simple, but still being surprisingly efficient (Kamarainen et al., 2002a; Hamouz et al., 2003).

4.1 Sampling filter parameters

It is clear that a filter bank, consisting of several filters, needs to be used, as relationships between responses of filters provide the basis for distinguishing objects. Next, it is considered how the filter parameters should be chosen for a bank of filters.

The selection of discrete rotation angles θ_k has already been demonstrated by Kyrki et al. (2001) and Park and Yang (2001), where it was shown how orientations must be spaced uniformly, that is,

$$\theta_k = \frac{k2\pi}{n}, \;\; k = \{0, \ldots, n-1\} \tag{13}$$

where θ_k is the kth orientation and n is the number of orientations to be used. However, often the computation can be reduced to half since responses on angles $[\pi, 2\pi[$ are 90° phase shifted from responses on $[0, \pi[$ in a case of a real valued input.

In the selection of discrete frequencies f_k, exponential sampling must be used (e.g. Kamarainen et al., 2002a; Daugman, 1988), that is,

$$f_k = a^{-k} f_{max}, \;\; k = \{0, \ldots, m-1\} \tag{14}$$

where f_k is the kth frequency, $f_0 = f_{max}$ is the highest frequency desired, and a is the frequency scaling factor ($a > 1$). Useful values for a include $a = 2$ for octave spacing and $a = \sqrt{2}$ for half-octave spacing.

4.2 Feature matrix

Now, using the features in Eq. (2) and the parameter selection schemes in Eq. (13) and Eq. (14) to cover frequencies of interest $f_0, \ldots, f_{m-1}$ and the orientations for desired angular discrimination, one can construct a feature matrix $\mathbf{G}$ at an image location (x_0, y_0)

$$\mathbf{G} = \begin{pmatrix} r(x_0, y_0; f_0, \theta_0) & r(x_0, y_0; f_0, \theta_1) & \cdots & r(x_0, y_0; f_0, \theta_{n-1}) \\ r(x_0, y_0; f_1, \theta_0) & r(x_0, y_0; f_1, \theta_1) & \cdots & r(x_0, y_0; f_1, \theta_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ r(x_0, y_0; f_{m-1}, \theta_0) & r(x_0, y_0; f_{m-1}, \theta_1) & \cdots & r(x_0, y_0; f_{m-1}, \theta_{n-1}) \end{pmatrix} \tag{15}$$

For convenience, typical matrix indexing is used in $\mathbf{G}$, i.e., $\mathbf{G}(1,1) = g_{1,1} = r(x_0, y_0; f_0, \theta_0)$ and $\mathbf{G}(m,n) = g_{m,n} = r(x_0, y_0; f_{m-1}, \theta_{n-1})$. For illumination invariance, the response matrix can be normalized as

$$\mathbf{G}' = \frac{\mathbf{G}}{\sqrt{\sum_{i,j} |g_{i,j}|^2}} \tag{16}$$

The feature matrix in Eq. (15) or Eq. (16) can be used as an input feature for any classifier, e.g., as was used in face evidence detection by Kamarainen et al. (2002a). If features are extracted using Gabor feature matrix from objects in a standard pose, it is possible to introduce matrix manipulations that allow invariant search. First, a column-wise circular shift of feature matrix can be defined as

$$\mathbf{G}^{(\theta+k)} = \Big(\mathbf{G}(1:m, k:n)\ \mathbf{G}(1:m, 1:k-1) \Big) \tag{17}$$

where $\mathbf{G}(i:j, u:v)$ represents a sub-matrix of $\mathbf{G}$ containing rows $i \ldots j$ and columns $u \ldots v$. It should be noted that if the responses are calculated only for half orientation space, e.g., $[0, \pi[$, the phase wrapping must be taken into consideration in the column-wise shift. Similarly a row-wise shift for scale manipulation can be defined

$$\mathbf{G}_{(f+k)} = \Big(\mathbf{G}(k+1:m, 1:n)\ \mathbf{G}(m+1:m+k, 1:n) \Big) \tag{18}$$

The column-wise circular shift in Eq. (17) corresponds to search over all rotation angles, and the row-wise shift in Eq. (18) corresponds to search over all down-scales. Note that the row-wise shift is not circular but the highest frequencies (f_0, ..., f_{k-1}) vanish and new lower frequencies (f_{m+1}, ..., f_{m+k}) are mapped to the Gabor feature matrix as replacements. A simple method for performing an invariant classification of objects is demonstrated in Algorithm 1. It should be noted that the complexity of the algorithm depends on both size of an input image and number of row and column elements in the feature matrix. Furthermore, the invariance sensitivity can be adjusted to a desired level by selecting a proper number of row and column elements and the invariance degree can be adjusted by allowing a proper amount of shift operations, both these affecting to the complexity of the algorithm.

Algorithm 1 *Invariant search of object class at location* (x, y)

```
Compute feature matrix G at (x, y)
Compute normalized feature matrix G'
for all column shifts k do
   for all row shifts l do
      (Class, Confidence) = classify( G'^(θ+k)_(f+l) )
      if Confidence > bestConfidence then
         bestConfidence ← Confidence
         bestClass ← Class
      end if
   end for
end for
```

To provide a smooth behavior of features, the sharpness parameters γ and η must be adjusted to have a sufficient overlap of Gabor filters in space and frequency (Daugman, 1988; Lee, 1996). As a conclusion it can be said that a very versatile recognition is possible, as rotation, scaling, and illumination invariance can be achieved.

5 Experimental results

The simple Gabor feature space introduced in this study has already been used in several applications, e.g., in rotation invariant and noise robust recognition of electric components by Kamarainen et al. (2002b), and illumination, scale, translation, and orientation invariant detection of facial evidences by Kamarainen et al. (2002a) and Hamouz et al. (2003).

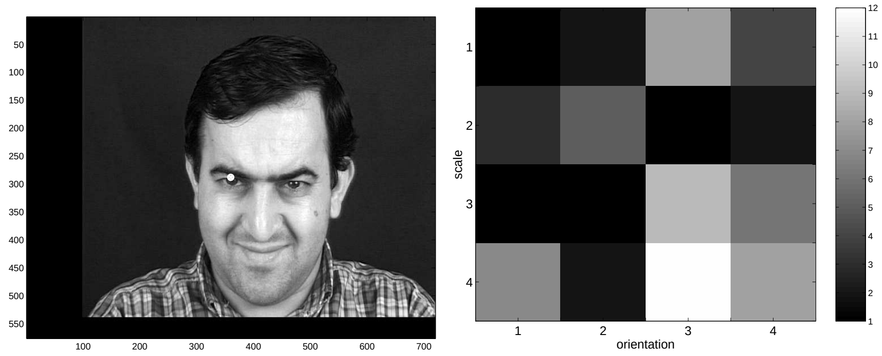
In this study, the main contributions are the formulation of the response matrix, which represents the simple Gabor feature space, and the row-wise and column-wise shifts of matrix corresponding orientation and scale manipulation of an image. A response matrix and the shift operations are illustrated in Fig. 1, which demonstrates how an invariant detection of facial evidences can be performed. Rotation of an image induces a circular column-wise shift of the response matrix, Fig. 1(b), and scaling induces a row-wise shift, Fig. 1(c).

The representation power of Gabor features is demonstrated in Fig. 2 which shows a reconstruction of an eye image using 16 Gabor filter responses at a single point. The reconstruction was performed using the coefficients shown in Fig. 1(a). It can be seen that the filters are able to capture more detail in the immediate neighborhood of the filter centroid but still preserve coarser structures in a wide area.

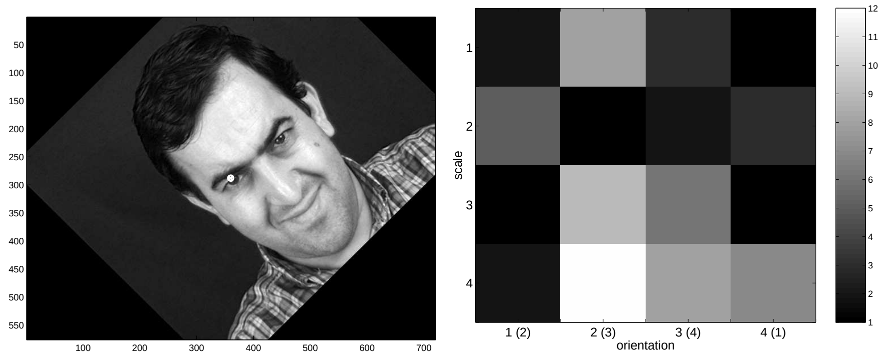
6 Discussion

Motivated by the promising results in applications, the authors have introduced the simple Gabor feature space and its theoretical framework in this study. The proposed feature space is especially useful in the scale, rotation, and translation invariant recognition of objects and furthermore it provides robustness to noise and illumination changes. The feature space is considered simple since it is spatially localized and thus is aimed to be used with sufficiently distinguishable objects. Sometimes objects are too complex to be distinguished in the simple Gabor feature space, but in that case it may be possible to distinguish salient sub-parts, which can be used as input to a more general classification systems. This approach has already been successfully utilized in the face detection presented by Hamouz et al. (2003).

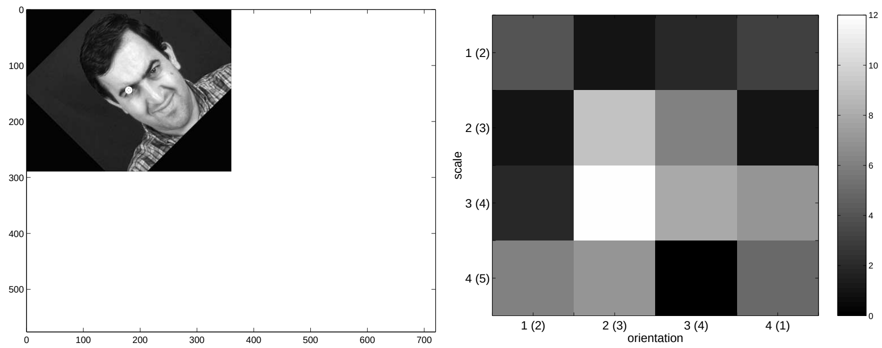
In this study, the motivation was to inspect the feature space itself and to present the theory behind the methods, including the invariant properties in Eqs. (5), (6) and (10), the construction of the feature matrix at one location in Eq. (15), and the rotation and scale invariant search operations by formulas



(a)



(b)

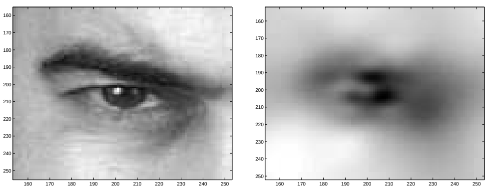


(c)

Fig. 1. Original image and its rotated and scaled versions, and corresponding feature matrice magnitudes computed at the centroid of the right eye ($f_k = 1/14, 1/28, 1/56, 1/112$, $\gamma = 1$, $\eta = 1$, $\theta_k = 0°, 45°, 90°, 135°$). For rotation note that the image origin is in the upper left corner. (a) original image. (b) rotated anti-clockwise by 45°. (c) rotated and scaled by 0.5×.

in Eqs. (17) and (18).

The proposed Gabor features lie somewhere between shiftable operators (Simoncelli et al., 1992) and orthogonal wavelets. In the future it would be intriguing to study what kind of advantages can be achieved with this kind of features, which approximate the shiftability of steerable functions and the



(a) (b)

Fig. 2. Reconstructed image using 16 Gabor filter coefficients. (a) original. (b) reconstruction.

orthogonality of separable wavelets.

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