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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF TAMPERE

REPORT A-1996-2

UNIVERSITY OF TAMPERE
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SERIES OF PUBLICATIONS A
A-1996-2, MARCH 1996 (REVISED MARCH 1997)

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ISBN 951-44-3949-X
ISSN 0783-6910

Abstract: The thickness of a graph is the minimum number of planar subgraphs into which the graph can be decomposed. This note discusses some recent attempts to determine upper bounds for the thickness of a graph as a function of the number of edges or as a function of its maximum degree.

1. Introduction

One way to characterize the embeddability of a graph G is to determine its *thickness*, $\theta(G)$, i.e., the minimum number of planar subgraphs into which G can be decomposed. The thickness of complete graphs and complete bipartite graphs is known [1-4], but on the other hand, very little is known about the thickness of an arbitrary graph. (We consider simple graphs only.) Recently, Dean et al. [6], Halton [7] and Cimikowski [5] have studied the thickness of a graph as a function of the number of edges or as a function of its maximum degree. The present note continues this study. A somewhat different line of research is followed by Jünger et al. [9].

2. Halton's theorem

We say that a graph has degree d, if d is the maximum degree of its nodes. Halton [7] has shown that any graph G of degree d has $\theta(G) \leq \lceil d/2 \rceil$. Halton's proof is based on Petersen's theorem. (If a graph is regular and of even degree, then it is 2-factorable [8, p. 90].) Hence, in order to prove the theorem, Halton first constructs a regular graph containing the given graph as a subgraph. The result then follows by Petersen's theorem. Next we give a new simple proof for Halton's theorem using only the most elementary concepts of graph theory.

We first need a little lemma. Let $G = (V,E)$ be a (connected) graph and let T be a spanning tree of T. A node in V is said to be an *end-node with respect to T* if it has degree one in T.

Lemma. Let $G = (V,E)$ be a (connected) graph of degree d, and let V' be the set of nodes having degree d. Then each biconnected component of G has a spanning tree T such that at most two nodes of V' are end-nodes with respect to T.

Proof. (Induction on $|V|$.) If G has two nodes, then the only spanning tree has two end-nodes. Suppose now (the induction hypothesis), that the lemma holds for all graphs having m ($m < n$) nodes. Let $G = (V,E)$ be of degree d and have n nodes. Suppose first that $|V'| < |V|$, i.e., there is at least one node x in V having degree less than d. We can apply the induction hypothesis to the graph $G-x$. The node x can be joined to the spanning tree in question with any edge adjacent to x. Otherwise ($V' = V$), G is regular. In each biconnected component there is a simple path containing all the nodes of the component. The first and the last node of the path are end-nodes, the others are not. ⊠

Theorem (Halton). If $G = (V,E)$ has degree d, then $\theta(G) \leq \lceil d/2 \rceil$.

Proof. (Induction on d.) If d is 1 or 2, then G is planar and its thickness is 1. Suppose (the induction hypothesis) that the theorem holds for all degrees k ($k < d$). Consider now a graph $G = (V,E)$ with degree d. According to the lemma, each biconnected component $G_i = (V_i,E_i)$ has a spanning tree $T_i = (V_i,E_i \cap V')$ with at most two end-nodes having degree d in G. Let H be the graph obtained as the union of all T_i's and all the bridges of G. Let x and y be the end-nodes in T_i having degree d in G. (The cases in which we have one or zero such end-nodes are treated analogously.) If x (respectively y) is an end-node of a bridge, then its degree in H is at least two. Otherwise we can add to H one edge e adjacent to x (resp. adjacent to y) from $E-E_i$ without losing the planarity. Such an edge always exists, since x (resp.

y) is of degree d and not adjacent to a bridge. The planar graph so obtained is denoted by $H^* = (V,E^*)$.

Now the graph $G' = (V,E-E^*)$ is of degree at most d-2. This holds because all nodes of V' have decreased their degree by at least 2 and all other nodes by at least 1 compared with the original graph G. We can apply the induction hypothesis to G'. Hence, $\theta(G') \leq \lceil (d-2)/2 \rceil$, and further, $\theta(G) \leq \theta(G') + \theta(T') \leq \lceil (d-2)/2 \rceil + 1 = \lceil d/2 \rceil$. ⊠

3. Dean et al.'s proof technique

Dean et al. have proved that if $G = (V,E)$ is a graph with e edges (and n nodes), then $\theta(G) \leq \lfloor \sqrt{e/3} + 3/2 \rfloor$. The proof is an induction on n + e. The induction step proceeds as follows. If the degree of each node is more than $\sqrt{e/3}$, then it is sufficient to approximate $\theta(G)$ by the thickness of the complete graph having n nodes. Otherwise (there is a node v with degree at most $\sqrt{e/3}$), we can apply the induction hypothesis to G-v. Hence, G-v can be decomposed into $k = \lfloor \sqrt{e/3} + 3/2 \rfloor$ planar subgraphs $H_1, \ldots, H_k$. The node v and one of its adjacent edges can now be inserted to each of the H_i, i = 1,..., k, without breaking the planarity of the subgraphs. Hence, G has thickness at most $\lfloor \sqrt{e/3} + 3/2 \rfloor$.

Cimikowski [5] has later proposed another proof using the above technique ending up in a new bound $\lfloor \sqrt{2e/3} + 3/2 \rfloor$. We shall show that such an improvement is not possible, i.e., Cimikowski's proof is not valid.

We aim at a bound $\lfloor \sqrt{xe/3} + 3/2 \rfloor$, where x is a positive real number to be minimized in order to obtain as sharp bound as possible. Note that Dean et al. have x = 3 and Cimikowski has x = 2. The approximation using the thickness of complete graphs is possible when $\lfloor (n+9)/6 \rfloor \leq \lfloor \sqrt{xe/3} + 3/2 \rfloor$. This holds when G does not have a node of degree $\sqrt{e/x}$ or less (the sum of the degrees (2e) is now more than $n\sqrt{e/x}$). In order to complete the proof, the boundary degree $\sqrt{e/x}$ cannot be bigger than the bound to be obtained. Hence, we must have $\sqrt{e/x} \le \lfloor \sqrt{xe/3} + 3/2 \rfloor$. If we now set x = 2 (as Cimikowski suggested) we notice that the inequality does not hold when e > 35. Only a marginal improvement over Dean et al.'s x = 3 is possible. This shows that Cimikowski's choice x = 2 cannot be proved by using this technique.

Dean et al. have conjectured that $\theta(G) \le \sqrt{e/16} + O(1)$ for any graph G.

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