Tilastollinen päättely I   Harj. 8   Vk 11   2012

(8.1) $a_1 X_1 + \dots + a_n X_n = \sum_{i=1}^{n} a_i X_i$ on μ:n harhaton estimaattori, jos $E(\sum_{i=1}^{n} a_i X_i) = \mu$

$$E(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i E(X_i) = \sum_{i=1}^{n} a_i \mu = (\sum_{i=1}^{n} a_i)\mu = \mu \quad \text{kaikilla } \mu \in \mathbb{R}$$

Siis on oltava $\boxed{\sum_{i=1}^{n} a_i = 1}$ lisäksi asetetaan ehto $a_i > 0$, $i = 1, 2, \dots, n$

(8.2.) $\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \xrightarrow{d} N(0,1)$, kun $n \to \infty$

ts. $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2)$, kun $n \to \infty$ (ks. Lause 11.1 ja Esimerkki 11.8)

Nyt
$g(\bar{X}) = \exp(\bar{X})$

$g(\mu) = e^{\mu}$
$g'(\mu) = e^{\mu}$
$[g'(\mu)]^2 = e^{2\mu}$

$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$
$E(g(\bar{X})) = g(E(\bar{X})) = g(\mu) = e^{\mu}$
$\mathrm{Var}(g(\bar{X})) = \mathrm{Var}(\bar{X})[g'(\mu)]^2$
$= \frac{\sigma^2}{n} e^{2\mu}$

Siis lauseen 11.1 mukaan
$\exp(\bar{X})$ noudattaa likimain normaalijakaumaa
$N(e^{\mu}, \frac{\sigma^2}{n} e^{2\mu})$

(8.3.) $g(\bar{X}) = \sqrt{\bar{X}}$
$g(\lambda) = \lambda^{\frac{1}{2}}$
$g'(\lambda) = \frac{1}{2}\lambda^{-\frac{1}{2}}$

$E(\bar{X}) = \lambda$
$\sigma^2 = \mathrm{Var}(\bar{X}) = \frac{\lambda}{n}$
$\sigma^2[g'(\lambda)]^2 = \frac{\lambda}{n}[\frac{1}{2}\lambda^{-\frac{1}{2}}] = \frac{\lambda}{n}[\frac{1}{4}\lambda^{-1}]$
$= \frac{1}{4n}$

$$\therefore \frac{\sqrt{\bar{X}} - \sqrt{\lambda}}{\sqrt{\frac{1}{4n}}} = 2\sqrt{n}(\sqrt{\bar{X}} - \sqrt{\lambda}) \xrightarrow{d} N(0,1)$$

ts. $\sqrt{\bar{X}} \overset{\text{likim.}}{\sim} N(\sqrt{\lambda}, \frac{1}{4n})$ (Lause 11.1)

(8.4) $Ber(\theta)$ hav. otos $x_1=0,\ x_2=1,\ x_3=1,\ x_4=1$. $T=\sum_{i=1}^{4} X_i$

Merk.

$A = (X_1=0,\ X_2=1,\ X_3=1,\ X_4=1)$

$B = (T=3) = \{(X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4) \mid \sum_{i=1}^{4} x_i = 3\}$

ts. tapahtuma B toteutuu otoksilla

$(0,1,1,1),\quad (1,0,1,1),\quad (1,1,0,1),\quad (1,1,1,0)$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \overset{*}{=} \frac{P(A)}{P(B)} = \frac{\theta^3(1-\theta)}{\binom{4}{3}\theta^3(1-\theta)} = \frac{1}{\binom{4}{3}} = \frac{1}{4}$$

Huom. $T=\sum_{i=1}^{4} X_i \sim Bin(4,\theta)$, joten

$P(B) = P(T=3) = \binom{4}{3}\theta^2(1-\theta)$

$\Rightarrow P(A|B)$ ei riipu θ:sta

ja

$A\subset B$, joten $A\cap B = A$ $*$

Jos $B=(T=t)$ ja $t\neq 3$, niin $A\cap B=\emptyset$

ja $P(A|B)=0 \Rightarrow$ todennäköisyys ei riipu θ:sta.

(8.5) $X_1,\dots,X_n$ otos $N(\mu,\sigma^2)$ $\qquad f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

a) $\sigma^2=1$ $\qquad f(x_1,\dots,x_n;\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}} = \frac{1}{\sqrt{2\pi}^{\,n}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2}$

$$= (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left[(x_i-\bar{x})^2 + (\bar{x}-\mu)^2\right]\right)$$

$$= (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum(x_i-\bar{x})^2 - \frac{n}{2}(\bar{x}-\mu)^2\right)$$

$$= \underbrace{\exp\left(-\frac{n}{2}(\bar{x}-\mu)^2\right)}_{g(\bar{x},\mu)} \cdot \underbrace{(2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\sum(x_i-\bar{x})^2\right)}_{h(x_1,\dots,x_n)}$$

ei riipu μ:stä

$\therefore \bar{X}$ on μ:n tyhjentävä tunnusluku

b) $\mu=0$ $\qquad f(x_1,\dots,x_n;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}} \exp\left(-\frac{\sum x_i^2}{2\sigma^2}\right)$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum x_i^2}{2\sigma^2}\right]$$

$$= \underbrace{(\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum x_i^2}{2\sigma^2}\right]}_{g(\sum x_i^2,\ \sigma^2)} \underbrace{(2\pi)^{-\frac{n}{2}}}_{h(x_1,\dots,x_n)=(2\pi)^{-\frac{n}{2}}}$$

ei riipu σ^2:sta

c) $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ on σ^2:n harhaton estimaattori, jos $\mu=0$, koska

$E\left(\frac{1}{n}\sum x_i^2\right) = \frac{1}{n}\sum E(x_i^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2) = \sigma^2+\mu^2 = \sigma^2$, kun $\mu=0$

$\hat{\sigma}^2$ on tyhjentävä, koska se on tyhjentävän tunnusluvun $\sum x_i^2$ funktio.

Jos $\mu\neq 0$ $\quad S^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$ on harhaton estimaattori.

8.6 a) $g(\bar{x},\mu) = g(5,\mu) = \exp\left[-\frac{n}{2}(5-\mu)^2\right]$

$L(\mu) = \text{vakio} * g(5,\mu)$

$\ell(\mu) = -\frac{n}{2}(5-\mu)^2$

$\ell'(\mu) = 2\cdot\left(-\frac{n}{2}\right)(5-\mu)\cdot(-1) = n(5-\mu) = 0$, kun $\mu = 5$

$\Rightarrow \hat{\mu} = 5$

b) 1. otos: $\bar{x} = \frac{3+6+4+3+9}{5} = \frac{25}{5} = 5$

$g(\bar{x},\mu) = g(5,\mu) = \exp\left[-\frac{5}{2}(5-\mu)^2\right]$

$h(3,6,4,3,9) = (2\pi)^{-\frac{5}{2}} \exp\left[-\frac{1}{2}\underbrace{\Sigma(x_i-\bar{x})^2}_{26}\right] \approx 0{,}000$

2. otos: $\bar{x} = \frac{5+9+1+6+4}{5} = 5$

$g(\bar{x},\mu) = g(5,\mu) = \exp\left(-\frac{5}{2}(5-\mu)^2\right)$

$h(5,9,1,6,4) = (2\pi)^{-\frac{5}{2}} \exp\left(-\frac{1}{2}\underbrace{\Sigma(x_i-\bar{x})^2}_{34}\right) \approx 0{,}000$

SUE $\hat{\mu} = \underline{\underline{5}} = \bar{x}$ kummastakin otoksesta.

8.7 Esim. 10.7 Lauseke (10.3.10) $\sigma^2 = 1$

$\ell(\alpha,\beta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\Sigma(y_i-\alpha-\beta x_i)^2$

$= -\frac{1}{2}\Sigma(\alpha+\beta x_i)^2 + \alpha\underbrace{\Sigma(y_i)}_{t_1} + \beta\underbrace{\Sigma x_i y_i}_{t_2} - \frac{n}{2}\log(2\pi) - \frac{1}{2}\Sigma y_i^2$

siis

$L(\alpha,\beta) = \underbrace{\exp\left[-\frac{1}{2}\Sigma(\alpha+\beta x_i)^2 + \alpha t_1 + \beta t_2\right]}_{g(t_1,t_2,\alpha,\beta)} \cdot \underbrace{2\pi^{-\frac{n}{2}} e^{-\frac{1}{2}\Sigma y_i^2}}_{h(y_1,\ldots,y_n)}$

8.8. $A(\eta) = \log(1+e^{\eta})$

$A'(\eta) = \frac{e^{\eta}}{1+e^{\eta}} = \theta = E(X)$, sillä $\eta = \log\frac{\theta}{1-\theta} \Rightarrow \theta = \frac{e^{\eta}}{1+e^{\eta}}$

$A''(\eta) = \frac{e^{\eta}(1+e^{\eta}) - e^{\eta}e^{\eta}}{(1+e^{\eta})^2} = \frac{e^{\eta}\cancel{(1+e^{\eta})}}{(1+e^{\eta})^{\cancel{2}}} - \frac{(e^{\eta})^2}{(1+e^{\eta})^2}$

$= \theta - \theta^2 = \theta(1-\theta) = Var(X)$