Tilastollinen päättely I   Harj. 6   vk 8   2012

(6.1.) $P(X=x;\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ Estimoidaan Poissonin jak. parametri λ

$$L(\lambda) = \prod_{i=1}^{55} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{55} x_i} e^{-n\lambda}}{\prod_{i=1} x_i!}$$

$$l(\lambda) = \sum_{i=1}^{55} x_i \log\lambda - n\lambda - \prod_{i=1}^{55} \log x_i!$$

$$l'(\lambda) = \frac{\sum x_i}{\lambda} - n = 0, \text{ kun } \frac{\sum x_i}{\lambda} = n \text{ eli kun } \lambda = \frac{\sum x_i}{n} = \bar{x}$$

$$\hat{\lambda} = \bar{x}$$

Nyt $P(X=2;\lambda) = \dfrac{\lambda^2 e^{-\lambda}}{2!}$

$$\hat{\lambda} = \bar{x} = \frac{0\cdot 7 + 1\cdot 14 + 2\cdot 12 + \dots + 5\cdot 3}{55} = 2.109091$$

ks. uskottavuuden invarianssi (Alaluku 10.7 ja Esim. 10.14)

$$\widehat{P(X=2;\lambda)} = P(X=2;\bar{x}) = \frac{\bar{x}^2 e^{-\bar{x}}}{2} \approx \frac{2.1^2 e^{-2.1}}{2} \approx 0.2699$$

(6.2.) Otos $X_i \sim Ber(\pi)$ $i=1,\dots,90$ $\sum_{i=1}^{90} x_i = 41$ $1/2 \le \pi \le 1$

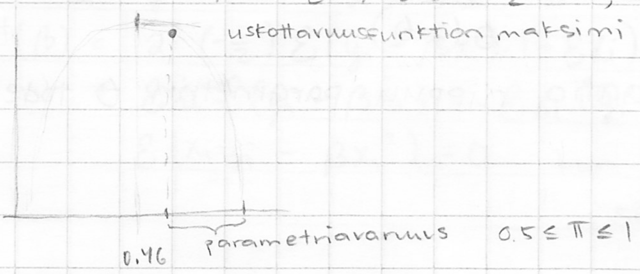
$\sum X_i \sim Bin(90,\pi)$ olk. $y = \sum_{i=1}^{90} x_i$

$$L(\pi) = \pi^y (1-\theta)^{90-y}$$

$$l(\pi) = y \log\pi + (90-y)\log(1-\pi)$$

$$l'(\pi) = \frac{y}{\pi} - \frac{90-y}{1-\pi} \qquad l'(\pi) = 0, \text{ kun } \pi = \frac{y}{90} \text{ eli } \frac{\sum x_i}{n}$$

π:n SUE: $\max\{\bar{x}, 1/2\} = \dfrac{1}{2}$, koska $\bar{x} = \dfrac{41}{90} \approx 0.46$



(6.3.) $X_i \sim Poi(\beta i)$ $i=1,\dots,20$ $X_i \perp\!\!\!\perp X_j$ $\beta > 0$ $f(x_i) = \dfrac{e^{-\beta i} \beta i^{x_i}}{x_i!}$ } ei riipu β:sta

β:n SUE (vrt. Esim. 10.11)

uskottavuusf. $L(\beta) = \prod_{i=1}^{20} e^{-\beta i} \beta i^{x_i} = e^{-\beta \sum_{i=1}^{20} i} \prod_{i=1}^{20} \beta_i^{x_i}$

Logaritmoitu us. $l(\beta) = \log L(\beta) = -\beta \sum_{i=1}^{20} i + \sum(x_i \log(\beta i))$

Pistefunktio $S(\beta) = l'(\beta) = -\sum_{i=1}^{20} i + \sum\left(\dfrac{x_i}{\beta i} \cdot i\right) = \dfrac{\sum x_i}{\beta} - \sum i$

$S(\beta) = 0$, kun $\dfrac{\sum x_i}{\beta} = \sum i$ eli $\beta = \dfrac{\sum x_i}{\sum i}$

$$\hat{\beta} = \frac{\sum x_i}{\sum i} = \frac{195}{210} \approx 0.93$$

6.4. 15 hav. otos $Ber(\theta)$ $0 \le \theta \le 1$

Otoksen yhteisjakauman tnf.

$$f(x_1,\dots,x_{15};\theta) = \prod_{i=1}^{15} f(x_i;\theta) = \prod_{i=1}^{15} \theta^{x_i}(1-\theta)^{1-x_i}, \quad x_i = 0 \text{ tai } 1$$

$$= \theta^{\sum_{i=1}^{15} x_i}(1-\theta)^{15-\sum_{i=1}^{15} x_i}$$

θ:in uskottavuusfunktio

$$L(\theta) = f(x_1,\dots,x_{15};\theta) = \theta^{\sum_{i=1}^{15} x_i}(1-\theta)^{15-\sum_{i=1}^{15} x_i}$$

$$l(\theta) = \log L(\theta) = \sum_{i=1}^{15} x_i \log\theta + \left(15 - \sum_{i=1}^{15} x_i\right)\log(1-\theta)$$

$$S(\theta) = l'(\theta) = \frac{\sum x_i}{\theta} - \frac{15-\sum x_i}{1-\theta} = 0, \quad \text{kun}$$

$$\frac{(1-\theta)\sum x_i}{(1-\theta)\theta} - \frac{(15-\sum x_i)\theta}{(1-\theta)\theta} = 0 \iff \frac{\sum x_i - \cancel{\theta\sum x_i} - \theta 15 + \cancel{\theta \sum x_i}}{(1-\theta)\theta} = 0$$

eli kun $\sum x_i - 15\theta = 0 \iff \hat{\theta} = \frac{\sum x_i}{15} = \frac{7}{15}$ (SUE)

Havaittu (Fisherin) informaatio $I(\hat\theta) = -l''(\hat\theta)$

$$I(\theta;x) = -l''(\theta;x) = -\left(-\frac{\sum x_i}{\theta^2} - \frac{15-\sum x_i}{(1-\theta)^2}\right) = \frac{\sum x_i}{\theta^2} + \frac{15-\sum x_i}{(1-\theta)^2}$$

$$I(\hat\theta;x) = \frac{7}{(7/15)^2} + \frac{15-7}{(1-7/15)^2} = \frac{15^2}{7} + \frac{8}{(8/15)^2} = \frac{15^2}{7} + \frac{15^2}{8} = \frac{8\cdot 15^2}{8\cdot 7} + \frac{7\cdot 15^2}{7\cdot 8} = \frac{15^3}{56}$$

$$\approx 60{,}26786$$

Nyt tarkastellaan havaintoja satunnaismuuttujina:

Fisherin informaatio $\mathcal{I}(\theta;x) = E(I(\theta;x))$

$$\mathcal{I}(\theta;x) = E(I(\theta;x)) = E\left(\frac{\sum x_i}{\theta^2} + \frac{15-\sum x_i}{(1-\theta)^2}\right) = \frac{E(\sum x_i)}{\theta^2} + \frac{15 - E(\sum x_i)}{(1-\theta)^2}$$

$\boxed{\text{Nyt } E(\sum x_i) = n\cdot\theta \quad n = 15}$

$$= \frac{15\theta}{\theta^2} + \frac{15-15\theta}{(1-\theta)^2} = \frac{15}{\theta} + \frac{15\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}} = \frac{15(1-\theta)}{\theta(1-\theta)} + \frac{15\theta}{(1-\theta)\theta}$$

$$= \frac{15(1-\theta)+15\theta}{\theta(1-\theta)} = \frac{15}{\theta(1-\theta)}$$

Fisherin informaatio riippuu parametrin θ todellisesta arvosta.

(6.5.) ks. Esim. 10.4 s. 278 $x_1, \dots, x_n$ otos $N(1, \sigma^2)$

uf. $L(\sigma^2; x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-1)^2}{2\sigma^2}\right] = 2\pi^{-\frac{n}{2}} \sigma^{2^{-\frac{n}{2}}} \cdot \exp\left[-\frac{\Sigma(x_i-1)^2}{2\sigma^2}\right]$

log.uf. $\ell(\sigma^2; x_1, \dots, x_n) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-1)^2$

pistef. $S(\sigma^2; x_1, \dots, x_n) = \ell'(\sigma^2; x_1, \dots, x_n) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\Sigma(x_i-1)^2$

SUE: $S(\sigma^2; x_1, \dots, x_n) = 0$, kun $-\frac{n}{2\sigma^2} + \frac{\Sigma(x_i-1)^2}{2(\sigma^2)^2} = 0 \Rightarrow -\frac{n\sigma^2}{2(\sigma^2)^2} + \frac{\Sigma(x_i-1)^2}{2(\sigma^2)^2} = 0$

eli kun $-n\sigma^2 + \Sigma(x_i-1)^2 = 0$

$$\sigma^2 = \frac{\Sigma(x_i-1)^2}{n} = \frac{1}{n}\sum_{i=1}^{n}(x_i-1)^2$$

SUE $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-1)^2$

inf. funktio $I(\sigma^2; x_1, \dots, x_n) = -\ell''(\sigma^2) = -\left(\frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\Sigma(x_i-1)^2\right)$

hav. inf. $I(\hat{\sigma}^2) = -\ell''(\hat{\sigma}^2) = -\frac{n}{2(\hat{\sigma}^2)^2} + \frac{1}{(\hat{\sigma}^2)^3}\underbrace{\Sigma(x_i-1)^2}_{n\hat{\sigma}^2}$

$$= -\frac{n}{2(\hat{\sigma}^2)^2} + \frac{n\hat{\sigma}^2}{(\hat{\sigma}^2)^3} = -\frac{n}{2(\hat{\sigma}^2)^2} + \frac{n}{(\hat{\sigma}^2)^2}$$

$$= \frac{-n+2n}{2(\hat{\sigma}^2)^2} = \frac{n}{2(\hat{\sigma}^2)^2} = \frac{n}{2\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-1)^2\right)^2}$$

(6.6.) vrt. Esim. 10.7 s. 281

$\ell(\beta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\Sigma(y_i - \beta x_i)^2$ s. 281 (10.3.10) $\sigma^2 = 1$

$S(\beta) = \ell'(\beta) = 2\cdot\left(-\frac{1}{2}\right)\Sigma(y_i - \beta x_i)\cdot(-\Sigma x_i)$

$= \Sigma x_i \Sigma(y_i - \beta x_i) = \Sigma x_i(y_i - \beta x_i) = 0$

$\Sigma(x_i y_i - \beta x_i^2) = 0$ kun $\beta\Sigma x_i^2 = \Sigma(x_i y_i)$

SUE $\hat{\beta} = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$

$\hat{\beta} = \frac{13}{6}$

y_i	x_i	$y_i x_i$	x_i^2
1	-1	-1	1
3	-1	-3	1
3	-1	-3	1
5	1	5	1
7	1	7	1
8	1	8	1
	Σ	13	6
		$\Sigma x_i y_i$	Σx_i^2

Informaatiofunktio

$I(\beta) = -\ell''(\beta) = -(-\Sigma x_i^2) = \Sigma x_i^2 = 6$

Havaittu informaatio: informaatiofunktio ei riipu β:sta, joten

$I(\hat{\beta}) = I(\beta) = 6$ kaikilla β

(6.7.) $X_1, \dots, X_n$ otos jak. Poi(3) (ks. Esim. 10.16)

$f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \dots$

$L(\lambda) = \lambda^x e^{-\lambda}$, $\ell(\lambda) = x \log \lambda - \lambda$, $S(\lambda; x_i) = \dfrac{x_i}{\lambda} - 1$

$S(3; x_i) = \dfrac{x_i}{3} - 1$

Pistesuure $S(3; X_1, \dots, X_n) = \sum_{i=1}^{n} S(3; x_i) = \frac{1}{3} \sum x_i - n$

Pistesuureen odotusarvo $E(S(3; X_1, \dots, X_n)) = \frac{1}{3} E(\overbrace{\textstyle\sum X_i}^{n \cdot 3}) - n = \frac{1}{3} n \cdot 3 - n = 0$

— " — varianssi $\mathrm{Var}(S(3; X_1, \dots, X_n)) = \frac{1}{3^2} \mathrm{Var}(\sum X_i) = \frac{1}{3^2} \sum \underbrace{\mathrm{Var}(X_i)}_{3} = \frac{1}{3^2} n \cdot 3$

$= \underline{\underline{\dfrac{n}{3}}}$

(6.8.) $f(x) = \theta / (\theta + x)^2$

$L(\theta; x_i) = \theta / (\theta + x_i)^2$

$\ell(\theta; x_i) = \log \theta - 2 \log(\theta + x_i)$

Logaritmoitu uskottavuusf. $\ell(\theta; x_1, \dots, x_n) = \sum_{i=1}^{n} \ell(\theta; x_i) = n \log \theta - 2 \sum \log(\theta + x_i) = \ell(\theta)$

Pistefunktio $S(\theta) = \ell'(\theta) = \dfrac{n}{\theta} - 2 \sum \dfrac{1}{\theta + x_i}$

Informaatiof. $I(\theta) = -\ell''(\theta) = -\left(-\dfrac{n}{\theta^2} + 2 \sum_{i=1}^{n} \dfrac{1}{(\theta + x_i)^2}\right) = \dfrac{n}{\theta^2} - 2 \sum_{i=1}^{n} \dfrac{1}{(\theta + x_i)^2}$

Määritetään SUE numeerisesti optimointifunktiosta

$n = 5$ $x1 = 1.5$ $x_2 = 4.8$ $x3 = 7.0$ $x4 = 3.3$ $x5 = 5.5$

Maksimoidaan logaritmoitua uskottavuusfunktiota

$\ell(\theta) = 5 \log \theta - 2 \sum_{i=1}^{5} \log(\theta + x_i)$

R:llä

```
n <- 5
x1 <- 1.5
:
x5 <- 5.5
loglike <- function(theta) {5 * log(theta) - 2 * (log(theta + x1) +
        log(theta + x2) + ... + log(theta + x5))}
optimize(f = loglike, interval = c(0, 100), maximum = TRUE)
curve(loglike, 0, 15)
```

$\Rightarrow \hat{\theta} = 3,950875$ loglike($\hat{\theta}$) = -14.10147