Tilastollinen päättely I Harj. 2 vko 4 2012

2.1. a) $P(Y < \frac{3}{4}) = P(X_1 < \frac{3}{4}, X_2 < \frac{3}{4}) \overset{\perp\!\!\perp}{=} P(X_1 < \frac{3}{4}) P(X_2 < \frac{3}{4})$

$$= \left(\int_0^{3/4} 3x^2\right)^2 = \left(\Big/_0^{3/4} x^3\right)^2 = \left[\left(\tfrac{3}{4}\right)^3\right]^2 = \frac{729}{4096} \approx 0.1779$$

b) $E(Y) = E(X_1 + X_2) = E(X_1) + E(X_2)$

$$= 2 \cdot \int_0^1 x \cdot 3x^2 \, dx = 2 \cdot 3 \cdot \int_0^1 x^3 \, dx = 2 \cdot 3 \cdot \Big/_0^1 \frac{x^4}{4} = 2 \cdot 3 \cdot \frac{1}{4} = \frac{6}{4} = \underline{\underline{\frac{3}{2}}}$$

$$Var(Y) = Var(X_1 + X_2) = Var(X_1) + Var(X_2) = 2 \cdot \int_0^1 (x - \tfrac{3}{4})^2 \cdot 3x^2 \, dx$$

$$= 2\left[E(X^2) - (\tfrac{3}{4})^2\right] = 2\left[\int_0^1 x^2 \cdot 3x^2 \, dx - (\tfrac{3}{4})^2\right]$$

$$= 2\left[3 \Big/_0^1 \frac{x^5}{5} - (\tfrac{3}{4})^2\right] = 2 \cdot \left[\frac{3}{5} - (\tfrac{3}{4})^2\right] = 2 \cdot \frac{3}{80} = \frac{3}{40} \approx \underline{\underline{0.075}}$$

2.2. a) $P(X_1 = 1, X_2 = 3, X_3 = 1) = f(1) f(3) f(1) = f^2(1) \cdot f(3)$

$$= (\tfrac{3}{4})^2 \cdot \tfrac{3}{4} (\tfrac{1}{4})^2 = (\tfrac{3}{4})^3 (\tfrac{1}{4})^2 \approx 0.026367$$

b) $X_1 + X_2 + X_3 = 5$ toteutuu kun

{täsmälleen 2 sat.mjista X_1, X_2, X_3 saa arvon 1 ja yksi saa arvon 3}

tai {täsmälleen 2 saa arvon 2 ja yksi saa arvon 1}

$$\therefore P(X_1 + X_2 + X_3 = 5) = \binom{3}{2} P(X_1 = 1, X_2 = 1, X_3 = 3) + \binom{3}{2} P(X_1 = 2, X_2 = 2, X_3 = 1)$$

$$= 3 f^2(1) f(3) + 3 f^2(2) f(1) = 3 \cdot \frac{3^3}{4^5} + 3 \cdot \frac{3^3}{4^5} = 2 \frac{3^4}{4^5} \approx 0.158$$

c) $P(Y \le 2) = P(X_1 \le 2, X_2 \le 2, X_3 \le 2) = \prod_{i=1}^{3} P(X_i \le 2) = F^3(2) \approx 0.82397$

$$= (\underbrace{\tfrac{3}{4}}_{f(1)} + \underbrace{\tfrac{3}{4} \cdot \tfrac{1}{4}}_{f(2)})^3 = (\tfrac{15}{16})^3$$

$\boxed{F(2) = \sum_{i=1}^{2} f(x)}$

2.3. a) $f(x_i; \theta) = \frac{1}{\theta} e^{-\frac{x_i}{\theta}}, \quad i = 1, \ldots, n \quad \begin{matrix} x_i > 0 \\ \theta > 0 \end{matrix}$

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} \frac{x_i}{\theta}}$$ (otoksen yhteisjak. tf.)

b) $M_{X_i}(t) = \frac{1}{1 - \theta t}, \quad t < \frac{1}{\theta}$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(\frac{1}{1 - \theta t}\right)^n$$

$M_Y(t)$ on gammajakauman Gamma(n, θ) mf.

siis $Y \sim$ Gamma(n, θ)

(2.4.) Koska $Y \sim Gamma(n, \theta)$, niin $E(Y) = n\theta$

a) $E(aY) = an\theta = \theta \Rightarrow \underline{\underline{a = \frac{1}{n}}}$

b) Kun $n=5$, $Y \sim Gamma(5, \theta)$

$M_Y(t) = \left(\frac{1}{1-\theta t}\right)^5$

$M_{\frac{Y}{\theta}}(t) = M_Y\left(\frac{t}{\theta}\right) = \left(\frac{1}{1-t}\right)^5, \quad t < 1$

$\therefore \frac{Y}{\theta} \sim Gamma(5, 1)$ scale = 1

$P(9.59 < \frac{Y}{\theta} < 34.2) = pgamma(34.2, 5, 1) - pgamma(9.59, 5, 1)$

$= 0.038$

(2.5.) $M_X(t) = E(e^{tX}) = \frac{1}{4}\sum_{i=1}^{4} e^{it} = \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t})$

$M_Y(t) = E(e^{tY}) = \frac{1}{6}\sum_{j=1}^{6} e^{jt} = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$

$W = X + Y \qquad X \perp\!\!\!\perp Y$

a) W:n momenttif. $M_W(t) = E(e^{tW}) = E(e^{tX+tY}) = E(e^{tX} \cdot e^{tY})$

$\overset{\perp\!\!\!\perp}{=} E(e^{tX}) \cdot E(e^{tY}) = M_X(t) M_Y(t) = \frac{1}{4}(e^t + e^{2t} + e^{3t} + e^{4t}) \cdot \frac{1}{6}(e^t + \dots + e^{6t})$

$= \frac{1}{24}\left[(e^{2t} + e^{3t} + e^{4t} + \dots + e^{7t}) + (e^{3t} + e^{4t} + \dots + e^{8t}) + (e^{4t} + \dots + e^{9t}) + (e^{5t} + \dots + e^{10t})\right]$

$= \frac{1}{24}(e^{2t} + 2e^{3t} + 3e^{4t} + 4e^{5t} + 4e^{6t} + 4e^{7t} + 3e^{8t} + 2e^{9t} + e^{10t})$

b) W:n tnf.

$$f(w) = \begin{cases} \frac{1}{24}, & w = 2, 10 \\ \frac{2}{24}, & w = 3, 9 \\ \frac{3}{24}, & w = 4, 8 \\ \frac{4}{24}, & w = 5, 6, 7 \end{cases}$$

(2.6.) a) $X \sim Bin(3, 0.6)$ ja $V \sim Bin(9, 0.6)$ $\quad Y = X + V$

$M_X(t) = (0.4 + 0.6e^t)^3$

$M_V(t) = (0.4 + 0.6e^t)^9$

$M_{X+V}(t) = M_X(t) \cdot M_V(t) = (0.4 + 0.6e^t)^{12}$

$\therefore X + V \sim Bin(12, 0.6)$

b) Konvoluutiokaavalla $\quad X \sim Bin(2, 0.2) \quad V \sim Bin(2, 0.6)$

$f_Y(y) = P(X+V = y) = \sum_{k=0}^{y} f_X(k) f_V(y-k)$

$y = 0, 1, 2, 3, 4 \qquad = \sum_{k=0}^{y} \binom{2}{k} 0.2^k\, 0.8^{2-k} \cdot \binom{2}{y-k} 0.6^{y-k}\, 0.4^{2-(y-k)}$

$f_Y(0) = 0.8^2 \cdot 0.4^2 = 0.1024$

$f_Y(1) = 0.8^2 \cdot 2 \cdot 0.6 \cdot 0.4 + 2 \cdot 0.2 \cdot 0.8 \cdot 0.4^2 = 0.3584$

$\vdots$

$f_Y(4) = \dots = 0.0144$

<u>R:llä</u> vuorollaan $y \leftarrow 0$ tai $y \leftarrow 1, \dots, y \leftarrow 4$

```
fy <- sum(dbinom(0:y, 2, 0.2)*dbinom(y:0, 2, 0.6))
```

(2.7) $E(X_1X_2) \overset{\perp}{=} E(X_1)E(X_2) = \underline{\mu_1\mu_2}$

$$Var(X_1X_2) = E[(X_1X_2)^2] - (\mu_1\mu_2)^2 \overset{X_1^2 \perp X_2^2}{=} E(X_1^2)E(X_2^2) - \mu_1^2\mu_2^2$$

$$Var(X_1) = \sigma_1^2 = E(X_1^2) - \underbrace{[E(X_1)]^2}_{\mu_1^2} \text{ joten } \boxed{E(X_1^2) = \sigma_1^2 + \mu_1^2} \text{ ja } \boxed{E(X_2^2) = \sigma_2^2 + \mu_2^2}$$

silloin

$$Var(X_1X_2) = (\sigma_1^2+\mu_1^2)(\sigma_2^2+\mu_2^2) - \mu_1^2\mu_2^2$$
$$= \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \cancel{\mu_1^2\mu_2^2} - \cancel{\mu_1^2\mu_2^2}$$
$$= \underline{\sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2}$$

(2.8) $Y \sim Poi(\lambda)$, $\lambda = 4$

Olk. N perheiden lkm. (suuri)

Lasten lkm:n odotusarvo on $N \cdot 4$

k:n lapsen perheitä on keskimäärin $N \cdot k \cdot p_k$, missä

$$p_k = P(Y=k) = \frac{4^k}{k!}e^{-4}$$

Todennäköisyys, että satunnaisesti valittu lapsi on perheestä, jossa k lasta, on

$$\frac{kNp_k}{N \cdot 4} = \frac{kp_k}{4}, \quad k = 1, 2, \dots$$

Lapsen sisarusten lkm on $k-1$

Sisarusten lkm:n odotusarvo on

$$q = \sum_{k=1}^{\infty}(k-1)\cdot\frac{kp_k}{\lambda} = \frac{1}{\lambda}\Big(\underbrace{\sum k^2 p_k}_{E(Y^2)} - \underbrace{\sum kp_k}_{E(Y)}\Big)$$
$$= \frac{1}{\lambda}(\lambda + \lambda^2 - \lambda) = \lambda = \underline{\underline{4}}$$

$$Var(Y) = E(Y^2) - (E(Y))^2$$
$$E(Y^2) = Var(Y) + (E(Y))^2$$
$$= \lambda + \lambda^2$$