Tilastollinen päättely I   Harj. 10   vko 13   2012

10.1. Ks. Alaluku 13.2.2

Otos jakaumasta $Poi(\lambda)$: $f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, \ldots$

$L(\lambda) = \dfrac{\lambda^{\sum x_i} e^{-\lambda n}}{\prod_{i=1}^{n} x_i!}$, $\sum x_i = n\bar{x} = 35 \cdot 4.26 = 149.1$

$H_0 : \lambda = 2.1$ vs. $H_1 : \lambda = 4.0$

uskottavuussuhde

$$\lambda(x_1, \ldots, x_n) = \frac{L(2.1)}{L(4.0)} = \frac{2.1^{149.1} e^{-2.1 \cdot 35}}{4.0^{149.1} e^{-4 \cdot 35}} = \left(\frac{2.1}{4.0}\right)^{149.1} e^{1.9 \cdot 35} \approx 0$$

H_0 hylätään, koska H_0:n uskottavuus suhteessa H_1:n uskottavuuteen on 0.

10.2. Ks. Alaluku 13.3.1

$H_0 : \lambda = 2.1$ vs. $H_1 : \lambda \neq 2.1$

Wilksin uskottavuustestisuure W

$$W = -2 \log \frac{L(\lambda_0)}{L(\hat{\lambda})} = 2[\ell(\hat{\lambda}) - \ell(\lambda_0)] \qquad W \xrightarrow{d} \chi^2(1)$$

$$\ell(\lambda) = -n\lambda + \underbrace{\sum x_i}_{n\bar{x}} \log \lambda = n(-\lambda + \bar{x} \log \lambda)$$

$$\ell'(\lambda) = -n + \frac{\sum x_i}{\lambda} = 0, \text{ kun } \lambda = \frac{\sum x_i}{n} = \bar{x} \qquad \hat{\lambda} = 4.26$$

$$W = 2[n(-\hat{\lambda} + \bar{x} \cdot \log \hat{\lambda}) - n(\lambda_0 + \bar{x} \cdot \log \lambda_0)] = 2n[(-\hat{\lambda} + \bar{x} \cdot \log \hat{\lambda}) - (-\lambda_0 + \bar{x} \log \lambda_0)]$$

$(n = 35 \quad \bar{x} = 4.26 \quad \hat{\lambda} = 4.26 \quad \lambda_0 = 2.1)$

$= 2 \cdot 35[(-4.26 + 4.26 \cdot \log 4.26) - (-2.1 + 4.26 \log 2.1)]$

≈ 59.72635

p-arvo:

$P(W \geq 59.73) = 1 - \text{pchisq}(59.73, 1) = 0 \quad < 0.05$

$\therefore$ H_0 hylätään.

10.3. (ks. Esimerkki 13.3) $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$

Hylkäysalue $C = \{\bar{x} \mid \bar{x} \geq c\}$ $\quad \bar{x} \sim N(\theta, \frac{1}{9})$

Testin voimakkuusfunktio $\gamma(\theta) = P(\bar{x} \geq c) = P\left(\dfrac{\bar{x} - \theta}{\sqrt{1/9}} \geq \dfrac{c - \theta}{\sqrt{1/9}}\right)$

$= 1 - \Phi\left(\dfrac{c - \theta}{1/3}\right) = 1 - \Phi(3(c - \theta)) = \Phi[3(\theta - c)]$

θ:n kasvava funktio, c kiinnitetty

kun $H_0 : \theta \leq 0$ on tosi $\quad \theta \in \Theta_0 = \{\theta \mid \theta \leq 0\}$

testin merkitsevyystaso $\alpha = 0.05 = \max_{\theta \in \Theta_0} \Phi(3(\theta - c)) = \Phi(-3c)$

Nyt $\Phi(-3c) = 0.05$

$\text{qnorm}(0.05) = -1.645 = -3c \longrightarrow c = \dfrac{1.645}{3} = 0.548$

Hylkäysalue: $C = \{\bar{x} \mid \bar{x} \geq 0.548\}$

Tässä $\bar{x} = 0.9 > 0.548$ eli $\bar{x} \in C \Rightarrow H_0$ hylätään ainakin 5%:n riskitasolla.

(10.4) $X_1, \dots, X_n$ otos eksponenttijakaumasta, jonka keskiarvo on θ.

$T = \frac{2n\bar{x}}{\theta} \sim \text{Khi2}(2n)$

a) $100(1-\alpha)\%$:n lv. θ:lle

$P(a \le T \le b) = 1-\alpha \qquad 0<\alpha<1$

Määritetään Khi2(2n)-jakaumasta fraktiilit a ja b s.e.

$P(T \le b) = 1-\frac{\alpha}{2}$ ja $P(T \le a) = \frac{\alpha}{2}$ eli $b = \chi^2_{\frac{\alpha}{2};2n}$ $a = \chi^2_{1-\frac{\alpha}{2},2n}$

$$P\left(\chi^2_{1-\frac{\alpha}{2};2n} \le \frac{2n\bar{x}}{\theta} \le \chi^2_{\frac{\alpha}{2};2n}\right) = P\left(\frac{\chi^2_{1-\frac{\alpha}{2};2n}}{2n\bar{x}} \le \frac{1}{\theta} \le \frac{\chi^2_{\frac{\alpha}{2};2n}}{2n\bar{x}}\right)$$

$$= P\left(\frac{2n\bar{x}}{\chi^2_{\frac{\alpha}{2};2n}} \le \theta \le \frac{2n\bar{x}}{\chi^2_{1-\frac{\alpha}{2};2n}}\right)$$ joten θ:n $100(1-\alpha)\%$:n lv. on

$$\left[\frac{2n\bar{x}}{\chi^2_{\frac{\alpha}{2};2n}}, \frac{2n\bar{x}}{\chi^2_{1-\frac{\alpha}{2};2n}}\right]$$

b) $H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$

Hylätään H_0, kun $\begin{cases} \frac{2n\bar{x}}{\chi^2_{1-\frac{\alpha}{2};2n}} \le \theta_0 & \text{tai} \\ \frac{2n\bar{x}}{\chi^2_{\frac{\alpha}{2};2n}} \ge \theta_0 \end{cases}$

Muutoin H_0 hyväksytään.
Testin merkitsevyystaso on α.

(10.5) onnistumisten lkm
$X \sim \text{Bin}(10, \pi)$ hav. $x = 9$

$H_0: \pi = 0.5$ vs. $H_1: \pi \ne 0.5$

Waldin testisuure (kaava 13.4.1)

$\chi^2_W = (\hat{\pi} - \pi_0)^2 \mathcal{J}(\hat{\pi}) \qquad \hat{\pi} = \frac{x}{n} = \frac{9}{10} \qquad \pi_0 = 0.5$

$f(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x} \qquad n = 10$

$L(\pi) = \pi^x (1-\pi)^{10-x} \cdot c$, missä c on vakio, joka ei riipu π:stä

$\ell(\pi) = x \log \pi + (10-x)\log(1-\pi) + \log c$

$\ell'(\pi) = \frac{x}{\pi} + \frac{10-x}{1-\pi} \cdot (-1) = S(\pi)$

$I(\pi) = -S'(\pi) = -\ell''(\pi) = \frac{x}{\pi^2} + \frac{10-x}{(1-\pi)^2} \qquad 0<\pi<1$

$\mathcal{J}(\pi) = E(I(\pi)) = E\left(\frac{x}{\pi^2}\right) + E\left(\frac{10-x}{(1-\pi)^2}\right) = \frac{10 \cdot \cancel{\pi}}{\pi^{\cancel{2}}} + \frac{10-10\pi}{(1-\pi)^2} = \frac{10}{\pi} + \frac{10\cancel{(1-\pi)}}{(1-\pi)^{\cancel{2}}}$

$= \frac{10}{\pi} + \frac{10}{(1-\pi)} = \frac{10}{\pi(1-\pi)}$

$\chi^2_W = (0.9-0.5)^2 \mathcal{J}(0.9) = 0.16 \cdot \frac{10}{0.9(1-0.9)} = 0.16 \cdot \frac{10}{0.09} = \underline{17.8}$

$\chi^2_W \overset{\text{likim.}}{\sim} \text{Khi2}(1)$ p-arvo: $P(\chi^2_W \ge 17.8) < 0.01$ $\underline{H_0 \text{ hylätään}}$

(1-pchisq(17.8, 1) ≈ 0.0000245)

(10.6.) Raon pistetestisuure (kaava 13.4.3)

$$U = \frac{S(\pi_0)^2}{J(\pi_0)} \overset{\text{likim}}{\sim} \chi^2(1) \qquad x = 9, \quad \pi_0 = 0.5$$

$$S(\pi_0)^2 = \left(\frac{x}{\pi_0} - \frac{10-x}{1-\pi_0}\right)^2 = \left(\frac{9}{0.5} - \frac{10-9}{1-0.5}\right)^2 = (18-2)^2 = 16^2$$

$$J(\pi_0) = \frac{10}{\pi_0(1-\pi_0)} = \frac{10}{0.5 \cdot 0.5} = 40$$

$$U = \frac{16^2}{40} = \underline{6.4} \qquad \text{p-arvo: } P(U \geq 6.4) = \underline{\underline{0.011}} \qquad \overset{\text{R:llä}}{(1\text{-pchisq}(6.4, 1))}$$

Wilksin uskottavuustestisuure

$$W = -2 \log \frac{L(\pi_0; x)}{L(\hat{\pi}; x)} = -2 \log \frac{L(0.5)}{L(0.9)} = -2 \log \left(\frac{0.5^9 (1-0.5)^{10-9}}{0.9^9 (1-0.9)^{10-9}}\right)$$

$$= -2 \log (0.0252) = 7.36$$

$$W \sim \chi^2(1) \qquad \text{p-arvo: } P(W \geq 7.36) = 1 - \text{pchisq}(7.36, 1) = \underline{\underline{0.007}}$$

(10.7.) $H_0: \pi = 0.50$ vs. $H_1: \pi > 0.50$

tarkka p-arvo binomijakauman avulla

$$P(X \geq 9; \pi = 0.5) = \sum_{x=9}^{10} \binom{10}{x} 0.5^{10} = 0.011$$

$H_0: \pi = 0.5$ vs. $H_1: \pi \neq 0.5$

$$P(X \geq 9 \text{ tai } X \leq 1; \pi = 0.5) = 2 \cdot P(X \geq 9; \pi = 0.5) = 0.021$$

(10.8.) Alaluku 13.5.4

$H_0: \theta_A = \theta_L = \theta$ (aspiriinilla ei vaikutusta) $\qquad H_1: \theta_A \neq \theta_L$

Ryhmä	Sydänk. (y)	Yht (n)
ASP	139	11 037
Lume	239	11 034
Yht	378	22 071

H_0:n vallitessa sydänkoht. tn:n SUE

$$\hat{\theta} = \frac{y_A + y_L}{n_A + n_L} = \frac{y}{n} = \frac{378}{22071} = 0.017$$

infarktien lkm $Y \sim Bin(n, \theta)$ ja

$$l(\theta) = y \log \theta + (n-y) \log (1-\theta)$$

H_1:n vallitessa

$Y_A \perp\!\!\!\perp Y_L$ (infarktien lkm ryhmissä) ja $Y_A \sim Bin(\theta_A, n_A)$ ja $Y_L \sim Bin(\theta_L, n_L)$

$$l(\theta_A, \theta_L) = y_A \log \theta_A + y_L \log \theta_L + (n_A - y_A) \log (1-\theta_A) + (n_L - y_L) \log (1-\theta_L)$$

$$\hat{\theta}_A = \frac{139}{11037} \quad \text{ja} \quad \hat{\theta}_L = \frac{239}{11034}$$

Wilksin uskottavuustestisuure: $W = 2[l(\hat{\theta}_A, \hat{\theta}_L) - l(\hat{\theta})] = 2[13.64] = 27.26$

$W \sim \text{Khi2}(1)$ $\quad$ p-arvo: $P(W \geq 27.26) \approx 0$ $\quad$ Ho hylätään, aspiriini vaikuttaa.