Matemaattisen tilastotieteen perusteet Harj. 6 vk 49 2011

(6.1) (X,Y) satunnaisvektori, jonka arvojoukko on $S=\{(0,1),(0,2),(1,0),(1,1),(2,0)\}$

$f(x,y)=\frac{x+2y}{12}$

$f(y|x)=\frac{f(x,y)}{f_X(x)}$ ja $f_X(x)=\sum_{y=0}^{2} f(x,y)$, $f_X(1)=\frac{1}{12}+\frac{3}{12}=\frac{4}{12}=\frac{1}{3}$

a) $f(y|X=1)=\frac{f(1,y)}{f_X(1)}=\frac{(1+2y)/12}{1/3}=\frac{1+2y}{4}$, $y=0,1$

joten

$E(Y|X=1)=\sum_{y=0}^{1} y\, f(y|1)=0\cdot\frac{1}{4}+1\cdot\frac{3}{4}=\underline{\underline{\frac{3}{4}}}$

b) $E(XY)=x\cdot y\cdot f(x,y)=\frac{3}{12}=\frac{1}{4}$ $E(X)=E(X|Y=y)$, $y\in S_Y$

Ks. s.195 $E(X)=E(X|Y)=0+1\cdot\left(\frac{1+2\cdot0}{12}+\frac{1+2\cdot1}{12}\right)+2\cdot\left(\frac{2+2\cdot0}{12}\right)$

$=\frac{4}{12}+\frac{4}{12}=\frac{8}{12}=\frac{2}{3}$

$E(Y)=E(Y|X)=0+1\cdot\left(\frac{0+2\cdot1}{12}+\frac{1+2\cdot1}{12}\right)+2\cdot\left(\frac{0+2\cdot2}{12}\right)=\frac{5}{12}+\frac{8}{12}=\frac{13}{12}$

$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{4}-\frac{2}{3}\cdot\frac{13}{12}=-\frac{17}{36}\approx-0,4722$

(6.2.) a) $X \perp\!\!\!\perp Y$, jos $P(X=x,Y=y)=P(X=x)P(Y=y)$ kaikilla x,y.

$f(x,y)=p_{xy}$

$f(x,y)$	$y=0$	$y=1$	$f_X(x)$
$x=0$	$p_{00}=\frac{1}{8}$	$p_{01}=\frac{1}{4}$	$1-p_1$ $\frac{3}{8}$
$x=1$	$p_{10}=\frac{3}{8}$	$p_{11}=\frac{2}{8}$	$p_1=\frac{5}{8}$
$f_Y(y)$	$1-p_2=\frac{1}{2}$	$p_2=\frac{1}{2}$	1

esim.

$P(X=1,Y=1)=p_{11}=\frac{2}{8}=\frac{1}{4}$

$P(X=1)P(Y=1)=\left(\frac{3}{8}+\frac{2}{8}\right)\left(\frac{1}{4}+\frac{2}{8}\right)$

$=\frac{5}{8}\cdot\frac{1}{2}=\frac{5}{16}$

eli $P(X=1,Y=1)\neq P(X=1)P(Y=1)$ $\therefore$ X ja Y eivät riippumattomat

b) $Cov(X,Y)=E(XY)-E(X)E(Y)$

$=p_{11}-p_1p_2=\frac{2}{8}-\frac{5}{8}\cdot\frac{1}{2}=\frac{4}{16}-\frac{5}{16}=\underline{\underline{-\frac{1}{16}}}=-0,0625$

X = ässien lkm Y = jätkien lkm, 5 korttia 52:sta. 52 − 8 = 44

(6.3) a) $f(x,y)=\frac{\binom{4}{x}\binom{4}{y}\binom{44}{5-(x+y)}}{\binom{52}{5}}$, $0\le x+y\le 5$ vrt. (7.4.3) s. 208

b) $P(X=x|Y=2)=\frac{P(X=x,Y=2)}{P(Y=2)}=\frac{f(x,2)}{f_Y(2)}=f_1(x|2)$

52 − 4 = 48 $=\frac{\frac{\binom{4}{x}\binom{4}{2}\binom{44}{3-x}}{\binom{52}{5}}}{\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}}=\frac{\binom{4}{x}\binom{44}{3-x}}{\binom{48}{3}}$, $x=0,1,2,3$

6.4. $\int 1$ X = osuman etäisyys origosta

$$F_X(x) = P(X \le x) = x^2, \quad 0 \le x \le 1$$
$$F_X(x) = 0, \quad x < 0$$
$$F_X(x) = 1, \quad x > 1$$

$$f_X(x) = F_X'(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{muualla} \end{cases}$$

6.5. a) $F_{X,Y}(0{,}6, 0.8) = \int_0^{0.6}\int_0^{0.8} dx\,dy = 0{,}6 \cdot 0.8 = 0.48$ s. 187.

b) $P(0.25 \le X \le 0.75, 0.1 \le Y \le 0.75)$
$= \Big/_{0.25}^{0.75} x \Big/_{0.1}^{0.75} y = (0.75 - 0.25)(0.75 - 0.1) = 0.5 \cdot 0.65 = 0.325$

6.6. $X, Y \sim N(0,1)$

a) merk. $V = X + 2Y \Rightarrow E(V) = \overset{=0}{E(X)} + 2\overset{=0}{E(Y)} = 0$ ja
$Var(V) = \underset{=1}{Var(X)} + 2^2 \underset{=1}{Var(Y)} = 5$
$\therefore V \sim N(0,5)$
$P(X + 2Y \le 3) = P(V \le 3) = P\left(\frac{V-0}{\sqrt{5}} \le \frac{3-0}{\sqrt{5}}\right) = \Phi\left(\frac{3}{\sqrt{5}}\right) = 0{,}91$

b) $Cor(X,Y) = \frac{1}{2} = \rho \qquad \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} \qquad \sigma_{12} = \rho\sigma_1\sigma_2$

$E(V) = 0$
$Var(V) = Var(X) + 4 \cdot Var(Y) + 2 \cdot Cov(X, 2Y)$
$= 5 + 4 \cdot Cov(X,Y) = 5 + 4 \cdot \underbrace{\rho \cdot \sigma_X \sigma_Y}_{\frac{1}{2}} = 5 + 2 = \underline{7}$

$P(V \le 3) = P\left(\frac{V-0}{\sqrt{7}} \le \frac{3-0}{\sqrt{7}}\right) = \Phi\left(\frac{3}{\sqrt{7}}\right) = 0{,}87$

N(0,1) kertymäfunktion arvot taulukosta tai R:llä $\xrightarrow{\text{esim.}}$ pnorm$\left(\frac{3}{\sqrt{7}}\right)$

(6.7.) Esimerkin 7.6 (s. 192) mukaan $\mu_X = \frac{1}{3}$ $\mu_Y = \frac{2}{3}$

$E(Y^2) = \frac{1}{2}$

$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)\,dx = 2 \Big/_0^1 \left(\frac{x^3}{3} - \frac{x^4}{4}\right) = 2\left(\frac{1}{3} - \frac{1}{4}\right) = 2\left(\frac{4}{12} - \frac{3}{12}\right) = \frac{1}{6}$

$Var(X) = E(X^2) - \mu_X^2 = \frac{1}{6} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18} = \frac{1}{18} = \hat{\sigma}_X^2$

$Var(Y) = E(Y^2) - \mu_Y^2 = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$

$\boxed{E(XY) = \frac{1}{4} \text{ annettu tentävaanannossa}}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{4} - \frac{2}{9} = \frac{9}{36} - \frac{8}{36} = \frac{1}{36} = \hat{\sigma}_{12}$

$E(Y|X) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ $\quad \boxed{\rho = \frac{\hat{\sigma}_{12}}{\hat{\sigma}_X \hat{\sigma}_Y}}$

$= \mu_Y + \frac{\hat{\sigma}_{12}}{\hat{\sigma}_X \cancel{\hat{\sigma}_Y}} \cdot \frac{\cancel{\hat{\sigma}_Y}}{\hat{\sigma}_X}(x - \mu_X)$

$= \frac{2}{3} + \frac{1/36}{1/18}\left(x - \frac{1}{3}\right) = \frac{2}{3} + \frac{1}{2}\left(x - \frac{1}{3}\right) = \frac{2}{3} + \frac{1}{2}x - \frac{1}{6}$

$= \underline{\underline{\frac{1}{2} + \frac{1}{2}x}}$

(6.8.) kts. s. 215 lause 7.9

a) $E(Y|X=x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 2.4 + \frac{0.6 \cdot 0.6}{0.4}(x - 3.2)$

$= 2.4 + 0.9x - 2.88 = 0.9x - 0.48$

b) $P(Y < 1.8) = P\left(\frac{Y - 2.4}{0.6} < \frac{1.8 - 2.4}{0.6}\right) = P(Z < -1) = 1 - \Phi(1) = 0.1586$

c) $\mu_{Y|X=2.5} = E(Y|X=2.5) = 0.9 \cdot 2.5 - 0.48 = 1.77$

$\sigma^2_{Y|X=2.5} = Var(Y|X=2.5) = \sigma_Y^2(1 - \rho^2) = 0.6^2(1 - 0.6^2) = 0.6^2 - 0.6^4$

$\sigma_{Y|X=2.5} = \sqrt{0.6^2 - 0.6^4} = 0.48$

$P(Y < 1.8 \,|\, X = 2.5) = P\left(\frac{Y - 1.77}{0.48} < \frac{1.8 - 1.77}{0.48}\right) = \Phi(0.0625)$

≈ 0.5249