Matemaattisen tilastotieteen perusteet 2011 Harj. 5 vk 48

(5.1) $Z \sim N(0,1)$ $\quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$Y = e^Z$

$$F_Y(y) = P(Y \le y) = P(e^Z \le y), \quad y > 0$$
$$= P(Z \le \log y) \qquad \ln$$
$$= \int_0^{\log(y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$\therefore \frac{d}{dy} F_Y(y) = F'(y) = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(\log y)^2}{2}} = f_Y(y)$$

Kts. Lause 6.3 s. 165

$$f_Y(y) = f_X(g(y)) \, |g'(y)| \qquad y \in S_Y$$

$Y = e^Z \qquad Z = \log(y)$ eli $g(y) = \log(y)$

$g'(y) = \frac{1}{y}$

(5.2) $X \sim Tas(0,1)$ $\quad f(x) = \begin{cases} 1, & \text{kun } x \in [0,1] \\ 0, & \text{muualla} \end{cases}$ $\quad F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$

$Y = 2X - 1$

$$F_Y(y) = P(Y \le y) = P(2X - 1 \le y) = P(2X \le y+1)$$
$$= P\left(X \le \frac{y+1}{2}\right) = F_X\left(\frac{y+1}{2}\right) = \begin{cases} 0, & \frac{y+1}{2} < 0 \Leftrightarrow y < -1 \\ \frac{y+1}{2}, & 0 \le \frac{y+1}{2} \le 1 \Leftrightarrow -1 \le y \le 1 \\ 1, & \frac{y+1}{2} > 1 \Leftrightarrow y > 1 \end{cases}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} 0, & y < -1 \\ \frac{1}{2}, & -1 \le y \le 1 \\ 0, & y > 1 \end{cases}$$

(5.3.) X noudattaa logistista jakaumaa Esim. 6.1 $\quad F_X(x) = \frac{1}{1+e^{-x}}$

$Y = \frac{1}{1+e^{-x}}$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{1+e^{-x}} \le y\right), \quad 0 < \frac{1}{1+e^{-x}} < 1$$
$$= P\left(X \le -\log\left(\frac{1-y}{y}\right)\right) = F_X\left(-\log\left(\frac{1-y}{y}\right)\right)$$
$$= \frac{1}{1+e^{-\left(-\log\left(\frac{1-y}{y}\right)\right)}} = \frac{1}{1+\frac{1-y}{y}} = y$$

$f_Y(y) = F_Y'(y) = 1, \qquad 0 < y < 1 \qquad \therefore Y \sim Tas(0,1)$

(5.4.) $f(x) = \begin{cases} \frac{1}{4}, & -2 < x < 2 \\ 0, & \text{muualla} \end{cases}$ $\quad F(x) = \begin{cases} 0, & x \le -2 \\ \frac{x+2}{4}, & -2 < x < 2 \\ 1, & x \ge 2 \end{cases}$

a) $Y = X^3$

$F_Y(y) = P(Y \le y) = P(X^3 \le y) \qquad -8 < y < 8$

$= P(X \le \sqrt[3]{y}) = F_X(y^{\frac{1}{3}})$

$= \frac{y^{\frac{1}{3}} + 2}{4}$

$f_Y(y) = F_Y'(y) = \frac{1}{4} \cdot \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{12} \cdot \frac{1}{\sqrt[3]{y^2}} = \frac{1}{12\sqrt[3]{y^2}}, \quad -8 < y < 8$

b) $Z = X^4$

$F_Z(z) = P(Z \le z) = P(X^4 \le z)$

$= P(-z^{\frac{1}{4}} \le X \le z^{\frac{1}{4}}) = F_X(z^{\frac{1}{4}}) - F_X(-z^{\frac{1}{4}})$

$= \frac{1}{4}\left[z^{\frac{1}{4}} + 2 + z^{\frac{1}{4}} - 2\right] = \frac{1}{4} \cdot 2 \cdot z^{\frac{1}{4}} = \frac{z^{\frac{1}{4}}}{2}$

$f_Z(z) = F_Z'(z) = \frac{1}{2} \cdot \frac{1}{4} \cdot z^{-\frac{3}{4}} = \frac{1}{8} z^{-\frac{3}{4}}, \quad 0 < z < 16$

(5.5.) $X \sim N(100, 14^2)$

$P(89{,}5 < X < 97{,}5) = P\left(\frac{89{,}5 - 100}{14} < \frac{X - 100}{14} < \frac{97{,}5 - 100}{14}\right)$

$= \Phi\left(-\frac{2{,}5}{14}\right) - \Phi\left(-\frac{10{,}5}{14}\right) = \Phi(-0.1786) - \Phi(-0{,}75) \approx 0{,}2025 = p$

T_n, että 20:stä sat. valitusta 5 kuuluu luokkaan M

$\binom{20}{5} p^5 (1-p)^{15} = 0{,}177$

(5.6.) $X \sim N(7, 4)$ kts. Lause 6.7 s. 172

$P(15.364 \le (x-7)^2 \le 20{,}096) = P\left(\frac{15.364}{4} \le \frac{(x-7)^2}{4} \le \frac{20{,}096}{4}\right)$

$= F_{\chi_1^2}\left(\frac{20{,}096}{4}\right) - F_{\chi_1^2}\left(\frac{15.364}{4}\right) \qquad \frac{(x-7)^2}{4} \sim \chi_1^2$

R:llä

$=$ pchisq(20.096/4, 1) − pchisq(15.364/4, 1)

≈ 0.025

(5.7) $X \sim \chi^2(23)$

a) $P(14.85 < X < 32.01)$

R:llä: pchisq(32.01, 23) − pchisq(14.85, 23) = 0.80

b) $P(a < X < b) = 0.95$ ja $P(a < X) = 0.025$

R:llä

qchisq(0.025, 23) = 11,69 — joten $a = 11{,}69$

qchisq(0.95 + 0.025, 23) = 38.08 joten $b = 38{,}08$

c) Lause 6.6 kohta 3.

$$E(X^c) = \frac{\Gamma(\alpha + c)\beta^c}{\Gamma(\alpha)}$$

$X \sim \text{Gamma}(\alpha, \beta)$ Erityistap.
$X \sim \chi^2(r)$, kun $\alpha = \frac{r}{2}$, $\beta = 2$

Nyt siis

$X \sim \text{Gamma}(\frac{23}{2}, 2)$

$$E(X^{\frac{1}{2}}) = \frac{\Gamma(\frac{23}{2} + \frac{1}{2})2^{\frac{1}{2}}}{\Gamma(\frac{23}{2})} = \frac{\Gamma(12)2^{\frac{1}{2}}}{\Gamma(\frac{23}{2})}$$

$$= \frac{\text{gamma}(12)\sqrt{2}}{\text{gamma}(\frac{23}{2})} = 4{,}744$$

(5.8) $M(t) = (1 - 2t)^{-12}$ eli $X \sim \text{Gamma}(12, 2)$ ts. $X \sim \chi^2(24)$

6.6.1
6.6.2

a) $E(X) = 24$

$\text{Var}(X) = 2 \cdot 24 = 48$

b) $P(15.66 < X < 42.98)$ R:llä = pchisq(42.98, 24) − pchisq(15.66, 24)

= 0,89