Matemaattisen tilastotieteen perusteet   Harj. 2   vk 45 2011

2.1 Kruunan todennäköisyys p

Klaavan todennäköisyys $1-p=q$

$A_i \equiv$ 'i. ja (i+1). heitto kruuna' $P(i.\text{ kr}) = p$ $P(i \text{ ja } i+1 \text{ kr}) = p^2$

Indikaattorimuuttuja

$$I_{A_i} = \begin{cases} 1, & \text{kun } A_i \text{ sattuu} \\ 0, & \text{muutoin} \end{cases} \qquad E(I_{A_i}) = 0 \cdot P(A_i)^c + 1 \cdot P(A_i)$$

Olk. sm X toistojen lkm n:n pituisessa sarjassa

$$X = I_{A_1} + \dots + I_{A_{n-1}}$$

$$E(X) = E(I_{A_1} + \dots + I_{A_{n-1}}) = \sum_{i=1}^{n-1} E(I_{A_i}) = \sum_{i=1}^{n-1} P(A_i) = \sum_{i=1}^{n-1} p^2 = \underline{\underline{(n-1)p^2}}$$

2.2 sm X arvojoukko $S_X = \{0,1,2,\dots\}$

Merk. $p_n = P(X=n)$

$$E(X) = \sum_{n=0}^{\infty} n \cdot P(X=n) = \sum_{n=1}^{\infty} n \cdot p_n$$

$$\begin{aligned} = p_1 + p_2 + p_3 + p_4 + \dots &\quad P(X>0) \\ + p_2 + p_3 + p_4 + \dots &\quad P(X>1) \\ + p_3 + p_4 + \dots &\quad P(X>2) \\ + p_4 + \dots &\quad P(X>4) \\ \vdots &\quad \vdots \end{aligned}$$

$$= \sum_{n=0}^{\infty} P(X>n)$$

2.3. p = onnistumisen todennäköisyys

X = ensimmäiseen onnistumiseen tarvittavien kokeiden lkm

$X \sim Geo(p)$, $P(X=k) = (1-p)^{k-1} p$, $k = 1,2,\dots$

a) $$P(X>n) = \sum_{k=n+1}^{\infty} P(X=k) = \sum_{k=n+1}^{\infty} q^{k-1} p \quad , \text{missä } q = 1-p$$

$$= q^{(n+1)-1} p + q^{(n+2)-1} p + \dots = q^n p + q^{n+1} p + \dots$$

$$= pq^n \underbrace{(1 + q + q^2 + \dots)}_{\frac{1}{1-q} \text{ Geom. sarja}} = pq^n \cdot \frac{1}{\underbrace{1-q}_{p}} = \underline{\underline{q^n}}$$

b) $$E(X) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} = \underline{\underline{\frac{1}{p}}}$$

(2.4.) X = synnytyskerran järj. nro, jonka jälkeen on ensimmäisen kerran kumpaakin sukupuolta oleva lapsi

Pojan todennäköisyys on p, tytön $1-p$

$$P(X>n) = P(\text{'n ensimmäistä lasta joko poikia tai tyttöjä'})$$
$$= p^n + (1-p)^n, \quad n=2,3,\dots$$
$$P(X>0)=1, \quad P(X>1)=1$$

$$\therefore E(X) = \sum_{n=0}^{\infty} P(X>n) = 1+1+\sum_{n=2}^{\infty} P(X>n)$$
$$= 2 + \sum_{n=2}^{\infty}\left[p^n + (1-p)^n\right] = 2 + \sum_{n=2}^{\infty} p^n + \sum_{n=2}^{\infty}(1-p)^n$$
$$= 2 + \sum_{n=1}^{\infty} p^2 p^{n-1} + \sum_{n=1}^{\infty}(1-p)^2(1-p)^{n-1} = 2 + \frac{p^2}{1-p} + \frac{(1-p)^2}{1-(1-p)}$$
$$= 2 + \frac{p^2}{1-p} + \frac{(1-p)^2}{p} = 2 + \frac{p^3}{p(1-p)} + \frac{(1-p)^3}{p(1-p)} = 2 + \frac{p^3 + (1-p)^3}{p(1-p)}$$
$$= 2 + \frac{\cancel{p^3} + 1^3 - 3\cdot 1^2 p + 3\cdot 1\cdot p^2 - \cancel{p^3}}{p(1-p)} = 2 + \frac{3p^2 - 3p + 1}{p(1-p)} = 2 + \frac{-3p(1-p)+1}{p(1-p)}$$
$$= 2 - 3 + \frac{1}{p(1-p)} = \frac{1}{p(1-p)} - 1$$

Kun $p = \frac{1}{2}$, $E(X) = \dfrac{1}{\frac{1}{2}(1-\frac{1}{2})} - 1 = \underline{\underline{3}}$

(2.5.) sm X = ykkösten lkm kahden nopan heitossa
Y = kuutosten lkm

X \ Y	0	1	2	
0	16	8	1	25
1	8	2	0	10
2	1	0	0	1
	25	10	1	36

$X=0$ ja $Y=0$ lkm   1 noppa 4 tavalla 2 noppa 4 tavalla
$4\cdot 4 = 16$
$X=0$ ja $Y=1$ lkm   1 tai 2 noppa 6 ja toinen 2, 3, 4 tai 5
$\Rightarrow 1\cdot 4 + 4\cdot 1 = 8$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$E(XY) = 0\cdot 0\cdot \frac{16}{36} + 0\cdot 1\cdot \frac{8}{36} + 1\cdot 0\cdot \frac{8}{36} + 0\cdot 2\cdot \frac{1}{36} + 2\cdot 0\cdot \frac{1}{36} + 1\cdot 1\cdot \frac{2}{36} + 0 + 0 + 0 = \frac{2}{36} = \frac{1}{18}$$
$$E(X) = E(Y) = 0\cdot \frac{25}{36} + 1\cdot \frac{10}{36} + 2\cdot \frac{1}{36} = \frac{10+2}{36} = \frac{12}{36} = \frac{1}{3}$$
$$Cov(X,Y) = \frac{1}{18} - \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{18} - \frac{2}{18} = -\frac{1}{18}$$

$$\boxed{\rho_{XY} = Cor(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$
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$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$
$$Var(X) = Var(Y) = E(X^2) - \left(\tfrac{1}{3}\right)^2$$
$$E(X^2) = 0^2\cdot\frac{25}{36} + 1^2\cdot\frac{10}{36} + 2^2\cdot\frac{1}{36} = \frac{10+4}{36} = \frac{14}{36} = \frac{7}{18}$$
$$Var(X) = \frac{7}{18} - \frac{2}{18} = \frac{5}{18}$$
$$Cor(X,Y) = -\frac{\frac{1}{18}}{\frac{5}{18}} = \underline{\underline{-\frac{1}{5}}}$$

(2.6.) sm X momenttifunktio on $M(t) = e^t \frac{2}{5} + e^{2t}\frac{1}{5} + e^{3t}\frac{2}{5}$

Alaluku 4.6.2

$M'(t) = e^t \frac{2}{5} + 2e^{2t}\cdot\frac{1}{5} + 3e^{3t}\cdot\frac{2}{5}$

$M'(0) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = \frac{10}{5} = 2$ ts. $E(X) = 2$

$M''(t) = e^t\cdot\frac{2}{5} + 4e^{2t}\cdot\frac{1}{5} + 9\cdot e^{3t}\cdot\frac{2}{5}$

$M''(0) = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} = \frac{24}{5} = 4\frac{4}{5}$ ts. $E(X^2) = 4\frac{4}{5}$

$Var(X) = E(X^2) - [E(X)]^2 = 4\frac{4}{5} - 2^2 = \frac{4}{5}$

$M(0) = \sum f(x_i)$

$M(0) = \frac{2}{5} + \frac{1}{5} + \frac{2}{5}$; $S_X = \{1, 2, 3\}$ $\quad (M(t) = E(e^{tX}))$

$f(1) = \frac{2}{5}$, $f(2) = \frac{1}{5}$, $f(3) = \frac{2}{5}$

(2.7.) sm X arvojoukko $S_X = \{-1, 0, 1\}$ Merk. $P(X=-1) = p_1$

$P(X=0) = p_2$

$P(X=1) = p_3$

$E(X) = -1\cdot p_1 + 0\cdot p_2 + 1\cdot p_3 = 0$

$\Rightarrow p_3 - p_1 = 0 \Rightarrow p_1 = p_3 = p$

$Var(X) = E(X^2) - \underbrace{E(X)}_{0}^2 = E(X^2) = (-1)^2\cdot p + 0^2\cdot p_2 + 1^2\cdot p = 2p$

$\max_p Var(X) = \max_p (2\cdot p) = 1$, kun $p = \frac{1}{2}$

$(f(-1) = \frac{1}{2} \quad f(0) = 0 \quad f(1) = \frac{1}{2})$

(2.8.) $X_i = \begin{cases} 1, & \text{kun } R \\ 0, & \text{kun } L \end{cases}$, $i = 1, \ldots, 10$

kun R, niin 1€ voitto

kun L, niin −1€ tappio

$P(X_i = 1) = \frac{1}{2}$

$aX_i + b = 1$, kun $X_i = 1$

$aX_i + b = -1$, kun $X_i = 0 \Rightarrow a\cdot 0 + b = -1 \Rightarrow b = -1$

kun $X_i = 1$ ja $b = -1$

$a\cdot 1 - 1 = 1 \Rightarrow a = 2$

$E(X_i) = 1\cdot\frac{1}{2} = \frac{1}{2}$

$Var(X_i) = E(X_i^2) - \underbrace{E(X_i)^2}$

$= 1^2\cdot\frac{1}{2} - \frac{1}{4} = \underline{\underline{\frac{1}{4}}}$

$Y_i = 2X_i - 1$, voitto/tappio i heitossa

$Y = \sum_{i=1}^{10} Y_i$

$E(Y) = \sum_{i=1}^{10} E(Y_i) = \sum_{i=1}^{10} (2E(X_i) - 1) = 2\sum_{i=1}^{10} E(X_i) - 10 = 2\cdot 10\cdot\frac{1}{2} - 10 = \underline{\underline{0}}$

$Var(Y) = Var(2\sum_{i=1}^{10} X_i - 10) = 2^2\cdot Var(\sum_{i=1}^{10} X_i) = 4\cdot\sum_{i=1}^{10} Var(X_i) = 4\cdot 10\cdot\frac{1}{4} = \underline{\underline{10}}$