Matemaattisen tilastotieteen perusteet   Harj. 1   Vk. 44   2011

(1.1) $P(X=0) = 1 - P(X=1)$ ja $E(X) = 3\,Var(X)$

Merk. $P(X=1) = p$, $P(X=0) = 1-p$

$E(X) = 0\cdot(1-p) + 1\cdot p = p$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 0^2\cdot(1-p) + 1^2\cdot p = p$

$Var(X) = p - p^2 = p(1-p)$

$E(X) = 3\cdot Var(X)$

$p = 3p(1-p)$

$\therefore\ 3p(1-p) - p = 0 \iff p[3(1-p) - 1] = 0$

$\iff p[2-3p] = 0$

$\iff p = 0$ tai $p = \frac{2}{3}$

eli $P(X=0) = 1-0$ tai

$P(X=0) = 1 - \frac{2}{3} = \frac{1}{3}$

(1.2) X:n arvoalue $S_X = \{-1, 0, 1\}$

$f(-1) = \frac{(|-1|+1)^2}{9} = \frac{4}{9}$

$f(0) = \frac{1}{9}$

$f(1) = \frac{4}{9}$ $\sum_{x\in S_X} f(x) = 1$

$E(X) = -1\cdot\frac{4}{9} + 0\cdot\frac{1}{9} + 1\cdot\frac{4}{9} = 0$

$E(X^2) = (-1)^2\cdot\frac{4}{9} + 0^2\cdot\frac{1}{9} + 1^2\cdot\frac{4}{9} = \frac{8}{9}$ $\approx 0{,}888$

$E(3X^2 - 2X + 4) = 3E(X^2) - 2E(X) + 4$

$= 3\cdot\frac{8}{9} + 4 = \frac{8}{3} + 4 = \frac{8}{3} + \frac{12}{3} = \frac{20}{3} = 6\frac{2}{3}$ $\approx 6{,}667$

(1.3.) $g(b) = E[(X-b)^2] = E[X^2 - 2bX + b^2] = E(X^2) - 2bE(X) + b^2$

Derivoidaan

$g'(b) = -2E(X) + 2b$ merk. $E(X) = \mu$

Derivaatan nollakohta

$g'(b) = 0$, kun $2b = 2\mu$ eli $b = \mu$

Onko kyseessä minimi?

$g''(b) = 2 > 0 \Rightarrow$ on minimi

<u>Vastaus</u>: $\underline{\underline{b = \mu}}$

1.4. a) $P(X=1) = P(X^{-1}(1)) = P(\omega_1) = \frac{1}{3}$

$P(Y=1) = P(Y^{-1}(1)) = P(\omega_3) = \frac{1}{3}$

$P(Z=1) = P(Z^{-1}(1)) = P(\omega_2) = \frac{1}{3}$

$\therefore \quad P(X=1) = P(Y=1) = P(Z=1) = \frac{1}{3}$

Samalla tavalla

$P(X=2) = P(Y=2) = P(Z=2) = \frac{1}{3}$ ja

$P(X=3) = P(Y=3) = P(Z=3) = \frac{1}{3}$

b) $(X+Y)(\omega_1) = X(\omega_1) + Y(\omega_1) = 1+2 = 3$

$(X+Y)(\omega_2) = X(\omega_2) + Y(\omega_2) = 2+3 = 5$

$(X+Y)(\omega_3) = X(\omega_3) + Y(\omega_3) = 3+1 = 4$

$\therefore P(X+Y=3) = P(X+Y=4) = P(X+Y=5) = \frac{1}{3}$

$(Y+Z)(\omega_1) = Y(\omega_1) + Z(\omega_1) = 2+3 = 5$

$(Y+Z)(\omega_2) = Y(\omega_2) + Z(\omega_2) = 3+1 = 4$

$(Y+Z)(\omega_3) = Y(\omega_3) + Z(\omega_3) = 1+3 = 3$

$P(Y+Z=3) = P(Y+Z=4) = P(Y+Z=5) = \frac{1}{3}$

$\sqrt{(X^2(\omega_1) + Y^2(\omega_1))Z(\omega_1)} = \sqrt{(1^2+2^2)\cdot 3} = \sqrt{15}$

$\sqrt{(X^2(\omega_2) + Y^2(\omega_2))Z(\omega_2)} = \sqrt{(2^2+3^2)\cdot 1} = \sqrt{13}$

$\sqrt{(X^2(\omega_3) + Y^2(\omega_3))Z(\omega_3)} = \sqrt{(3^2+1^2)\cdot 2} = \sqrt{20}$

$P(\sqrt{(X^2+Y^2)Z} = \sqrt{15}) = P(\sqrt{(X^2+Y^2)Z} = \sqrt{13}) = P(\sqrt{(X^2+Y^2)Z} = \sqrt{20}) = \frac{1}{3}$

1.5. $P(\{X=i\} \cap \{Y=j\}) = P(X=i, Y=j) \overset{\text{merk.}}{=} p_{ij}, \quad i,j \in \{0,1\}$

$P(X=1) = P(Y=1) = p$

a) $E(XY) = 0\cdot 0\, p_{00} + 0\cdot 1\, p_{01} + 1\cdot 0\, p_{10} + 1\cdot 1\, p_{11} = \underline{\underline{p_{11}}}$

Ei saada numeerista arvoa tentävässä annettujen tietojen perusteella

b) Kun $X \perp\!\!\!\perp Y$, niin $p_{11} = P(X=1, Y=1) \overset{\perp\!\!\!\perp}{=} P(X=1)P(Y=1) = p^2$

$p = 0.75 \Rightarrow \quad E(XY) = \underline{\underline{0{,}75^2}}$ s.93

$Bin(n,p)$
$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$

1.6. X ykkösten lkm viestissä

$X = X_1 + \dots + X_6$, $\underbrace{X_i}_{\text{viestin i. numero}} \sim Ber(0.3)$

$X \sim Bin(6; 0.3)$

(dbinom(3, 6, 0.3) = 0.18522)

a) $P(X=3) = \binom{6}{3} 0.3^3 0.7^3 = 0.18522$ "3 ykköstä"

b) $P(X \geq 5) = P(X=5) + P(X=6)$ "vähemmän kuin 2 nollaa eli 5 tai 6 ykköstä"

$= \binom{6}{5} 0.3^5 0.7 + \binom{6}{6} 0.3^6 = 0.010935$

(dbinom(5,6,0.3)+dbinom(6,6,0.3) = 0.010935)

tai

$P(X \geq 5) = 1 - P(X<5) = 1 - P(X \leq 4)$

$= 1 - F_X(4)$ (1- pbinom(4,6,0.3) = 0.010935)

1.7. $P(Z=z) = P(-Z=z)$ kaikilla z

$P(Z=1) = P(-Z=1) = P(Z=-1) = 0.15$

$P(Z=2) = P(Z=-2) = 0.1$

$P(Z=5) = P(Z=-5) = 0.25$

Niin $S_Z = \{-5,-2,-1,1,2,5\}$ ja yllä $P(Z=z) = f_Z(z)$, $z \in S_Z$

1.8. $M(t) = E(e^{tX}) = \sum_{x=1}^{10} p_x e^{tx}$, s.99

missä $p_x = P(X=x)$, $x = 1,2,\dots,10$

$p_x = \frac{1}{10}$, $x = 1,2,\dots,10$

$P(1 \leq X \leq 2) = P(X=1) + P(X=2) = \frac{2}{10}$