On some inequalities associated with ordinary least squares and the Kantorovich inequality

George P. H. Styan

McGill University
Montréal, Québec, Canada

Key Words and Phrases: Efficiency of ordinary least squares, Kantorovich inequality, characteristic roots, hat matrix.

In this paper we study three different measures of the efficiency of the ordinary least squares estimator in the general linear model $E(\mathbf{y}) = \mathbf{X}\beta$, $Var(\mathbf{y}) = \sigma^2\mathbf{V}$. We concentrate on the special case where $\mathbf{X}$ has rank one and where $\mathbf{V}$ is positive definite. The inequalities are illustrated with two examples.

1. INTRODUCTION

In this paper we will study three different measures of the efficiency of the ordinary least squares (OLS) estimation procedure in the usual general linear model:

$$E(\mathbf{y}) = \mathbf{X}\beta, \quad Var(\mathbf{y}) = \sigma^2\mathbf{V}, \tag{1}$$

where $\mathbf{X}$ is $n \times q$ of rank $r \leq q$ and $\mathbf{V}$ is nonnegative definite of rank $s \leq n$. We suppose that both $\mathbf{X}$ and $\mathbf{V}$ are fixed and known and we are concerned with the estimation of the $q \times 1$ vector β of regression coefficients. The variance σ^2 is unknown but plays no role in this paper; we will, therefore, with-

George P. H. Styan is an Associate Professor in the Department of Mathematics, McGill University, Burnside Hall, 805 ouest, rue Sherbrooke, Montréal, Québec, Canada H3A 2K6.

out loss of generality set $\sigma^2 = 1$. The linear estimator **By** is said to be best linear unbiased (BLU) for $\mathbf{X}\beta$ whenever

$$Var(\mathbf{Ay}) - Var(\mathbf{By}) \geq \mathbf{0}, \tag{2}$$

i.e., nonnegative definite, for all unbiased estimators **Ay** of $\mathbf{X}\beta$.

The ordinary least squares estimator $\mathbf{X}\hat{\beta}$ of $\mathbf{X}\beta$ may be written as:

$$\mathbf{X}\hat{\beta} = \hat{\mathbf{y}} = \mathbf{Hy}, \tag{3}$$

where

$$\mathbf{H} = \mathbf{X}(\mathbf{X'X})^{-}\mathbf{X'} = \mathbf{XX}^{+} \tag{4}$$

is the unique symmetric idempotent »hat» matrix whose columns span the column space of **X**. In (4) the $q \times q$ matrix $(\mathbf{X'X})^{-}$ is a generalized inverse of **X'X** satisfying $\mathbf{X'X}(\mathbf{X'X})^{-}\mathbf{X'X} = \mathbf{X'X}$, while $\mathbf{X}^{+}$ is the Moore-Penrose generalized inverse of **X**.

In a recent paper Alalouf and Styan (1983) surveyed the necessary and sufficient conditions for the OLSE $\mathbf{X}\hat{\beta}$ to be the BLUE of $\mathbf{X}\beta$. One of these conditions, due to Zyskind (1967), is that

$$\mathbf{HV} = \mathbf{VH}. \tag{5}$$

This suggests measuring the efficiency of OLS by the norm of the commutator

$$\mathbf{K} = \mathbf{HV} - \mathbf{VH}, \tag{6}$$

as suggested by Bloomfield and Watson (1975); see also Styan and Zlobec (1982). It is straightforward to show that

$$k = \tfrac{1}{2}\mathrm{tr}\mathbf{K'K} = -\tfrac{1}{2}\mathrm{tr}\mathbf{K}^2 = \mathrm{tr}(\mathbf{HV}^2) - \mathrm{tr}(\mathbf{HV})^2 \geq 0. \tag{7}$$

Bloomfield and Watson (1975) proved that when $r \leq [\frac{1}{2}n]$, the largest integer less than or equal to $\frac{1}{2}n$, then

$$k \leq \tfrac{1}{4}\sum_{i=1}^{r}(\lambda_i - \lambda_{n-i+1})^2, \tag{8}$$

where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \tag{9}$$

are the characteristic roots of the covariance matrix **V**. The inequality (8) also holds when **V** is indefinite (but symmetric), since **K** is unchanged by replacing **V** by $\mathbf{V} + a\mathbf{I}$, for any scalar a.

When **V** is positive definite, the BLUE of $\mathbf{X}\beta$ is the generalized least squares (GLS) estimator

$$\mathbf{X}(\mathbf{X'V}^{-1}\mathbf{X})^{-}\mathbf{X'V}^{-1}\mathbf{y} = \mathbf{X}\beta^{*}, \tag{10}$$

say, and $\mathbf{X}\beta^*$ has covariance matrix

$$\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}' = \mathbf{H}(\mathbf{H}\mathbf{V}^{-1}\mathbf{H})^{-}\mathbf{H}, \tag{11}$$

while the OLSE of $\mathbf{X}\beta$ has covariance matrix

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \mathbf{HVH}. \tag{12}$$

Puntanen (1982) suggested measuring the efficiency of OLS by the difference matrix

$$\mathbf{D} = \mathbf{HVH} - \mathbf{H}(\mathbf{H}\mathbf{V}^{-1}\mathbf{H})^{-}\mathbf{H} \geq \mathbf{0}, \tag{13}$$

and its trace

$$d = \mathrm{tr}\mathbf{HV} - \mathrm{tr}(\mathbf{H}\mathbf{V}^{-1}\mathbf{H})^{-}\mathbf{H} \geq 0. \tag{14}$$

When both $\mathbf{X}$ and $\mathbf{V}$ are of full rank, then the »usual» measure of efficiency of OLS is the ratio of the generalized variances of the OLS and the BLU estimators, i.e.,

$$f = \frac{\det[Var(\beta^*)]}{\det[Var(\hat{\beta})]} = \frac{[\det(\mathbf{X}'\mathbf{X})]^2}{\det(\mathbf{X}'\mathbf{V}\mathbf{X}) \cdot \det(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})} \leq 1. \tag{15}$$

Bloomfield and Watson (1975) showed that when $r = q \leq [\frac{1}{2}n]$, then

$$f \geq \prod_{i=1}^{q} \frac{4\lambda_i\lambda_{n-i+1}}{(\lambda_i + \lambda_{n-i+1})^2} \tag{16}$$

and that equality holds in (16) if and only if equality holds in (8).

Our purpose in this paper is to consider the simple case when $\mathbf{X}$ has rank 1, and to offer short (and possibly) new proofs of (8) and (16) in this special case. Moreover we will also prove that when $\mathbf{X}$ has rank 1, then

$$d \leq (\sqrt{\lambda_1} - \sqrt{\lambda_n})^2. \tag{17}$$

The inequality (17) appears to be new, and an upper bound for d when $\mathbf{X}$ has rank greater than or equal to 1 does not seem to be available. Furthermore, equality in (17) does *not* hold when equality holds in either (16) or (8). When $q = 1$ the inequality (16) reduces to the Kantorovich inequality, cf. e.g., Marcus and Minc (1964, pp. 110, 117).

2. RESULTS

When $\mathrm{rank}(\mathbf{X}) = 1 = \mathrm{rank}(\mathbf{H})$ there exists an $n \times 1$ vector $\mathbf{h}$ so that

$$\mathbf{H} = \mathbf{hh}', \quad \mathbf{h}'\mathbf{h} = 1. \tag{18}$$

When $q = 1$, i.e., $\mathbf{X}$ has just one column, then we may write $\mathbf{X} = \mathbf{x} \neq \mathbf{0}$, and $\mathbf{h} = \mathbf{x}/(\mathbf{x}'\mathbf{x})^{\frac{1}{2}}$. Substituting into (7), (14), and (15), yields, respectively, the formulas for k, f, and d, in (19), (20), and (21), below.

THEOREM 1. *Let the* $n \times 1$ *vector* $\mathbf{h}$ *satisfy* $\mathbf{h}'\mathbf{h} = 1$ *and let the* $n \times n$ *matrix* $\mathbf{V}$ *be symmetric, but not necessarily nonnegative definite, and let the characteristic roots of* $\mathbf{V}$ *be denoted by* $\lambda_1 \geq \cdots \geq \lambda_n$. *Then*

$$k = \mathbf{h}'\mathbf{V}^2\mathbf{h} - (\mathbf{h}'\mathbf{V}\mathbf{h})^2 \leq \tfrac{1}{4}(\lambda_1 - \lambda_n)^2. \tag{19}$$

If $\mathbf{V}$ *is positive definite then*

$$f = \frac{1}{\mathbf{h}'\mathbf{V}\mathbf{h}\ \mathbf{h}'\mathbf{V}^{-1}\mathbf{h}} \geq \frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2}, \tag{20}$$

and

$$d = \mathbf{h}'\mathbf{V}\mathbf{h} - \frac{1}{\mathbf{h}'\mathbf{V}^{-1}\mathbf{h}} \leq (\sqrt{\lambda_1} - \sqrt{\lambda_n})^2. \tag{21}$$

Proof. There exists an $n \times n$ orthogonal matrix $\mathbf{Q}$ so that $\mathbf{Q}'\mathbf{V}\mathbf{Q} = \mathbf{D}$ is diagonal with $\lambda_1, \ldots, \lambda_n$, the characteristic roots of $\mathbf{V}$, on the diagonal. Let $\mathbf{Q}'\mathbf{h} = \mathbf{a}$. Then $\mathbf{a}'\mathbf{a} = 1$ and

$$\begin{aligned} k &= \mathbf{a}'\mathbf{D}^2\mathbf{a} - (\mathbf{a}'\mathbf{D}\mathbf{a})^2 \\ &= \mathbf{a}'\mathbf{D}^2\mathbf{a} - [\mathbf{a}'\mathbf{D}\mathbf{a} - \tfrac{1}{2}(\lambda_1 + \lambda_n)]^2 + \tfrac{1}{4}(\lambda_1 + \lambda_n)^2 - (\lambda_1 + \lambda_n)\mathbf{a}'\mathbf{D}\mathbf{a} \\ &\leq \tfrac{1}{4}(\lambda_1 + \lambda_n)^2 + \mathbf{a}'\mathbf{D}^2\mathbf{a} - (\lambda_1 + \lambda_n)\mathbf{a}'\mathbf{D}\mathbf{a} \\ &= \tfrac{1}{4}(\lambda_1 - \lambda_n)^2 + \lambda_1\lambda_n + \mathbf{a}'\mathbf{D}^2\mathbf{a} - (\lambda_1 + \lambda_n)\mathbf{a}'\mathbf{D}\mathbf{a} \\ &= \tfrac{1}{4}(\lambda_1 - \lambda_n)^2 - \mathbf{a}'[\mathrm{dg}\{(\lambda_1 - \lambda_i)(\lambda_i - \lambda_n)\}]\mathbf{a} \\ &\leq \tfrac{1}{4}(\lambda_1 - \lambda_n)^2, \end{aligned} \tag{22}$$

where $\mathrm{dg}\{\cdot\}$ denotes the diagonal matrix with diagonal elements as given inside the braces. This establishes (19).

We now assume that $\mathbf{V} > \mathbf{0}$, and so $\lambda_n > 0$. Then (20) is equivalent to

$$\lambda_1\lambda_n/f = \lambda_1\lambda_n\mathbf{a}'\mathbf{D}\mathbf{a} \cdot \mathbf{a}'\mathbf{D}^{-1}\mathbf{a} \leq \tfrac{1}{4}(\lambda_1 + \lambda_n)^2. \tag{23}$$

To establish (23) we use the inequality, due to Marshall and Olkin (1964),

$$\lambda_1\lambda_n\mathbf{a}'\mathbf{D}^{-1}\mathbf{a} \leq \lambda_1 + \lambda_n - \mathbf{a}'\mathbf{D}\mathbf{a}, \tag{24}$$

which is true since it is equivalent to

$$\begin{aligned} &\mathbf{a}'(\mathrm{dg}\{\lambda_1 + \lambda_n - \lambda_i - \lambda_i^{-1}\lambda_1\lambda_n\})\mathbf{a} \\ &= \mathbf{a}'[\mathrm{dg}\{(1 - \lambda_i^{-1}\lambda_n)(\lambda_1 - \lambda_i)\}]\mathbf{a} \geq 0. \end{aligned} \tag{25}$$

Hence

$$\begin{aligned} \lambda_1\lambda_n/f = \lambda_1\lambda_n\mathbf{a}'\mathbf{D}\mathbf{a} \cdot \mathbf{a}'\mathbf{D}^{-1}\mathbf{a} &\leq \mathbf{a}'\mathbf{D}\mathbf{a}(\lambda_1 + \lambda_n - \mathbf{a}'\mathbf{D}\mathbf{a}) \\ &= \tfrac{1}{4}(\lambda_1 + \lambda_n)^2 - [\mathbf{a}'\mathbf{D}\mathbf{a} - \tfrac{1}{2}(\lambda_1 + \lambda_n)]^2 \\ &\leq \tfrac{1}{4}(\lambda_1 + \lambda_n)^2, \end{aligned} \tag{26}$$

which establishes (23), and hence also (20).

With $\mathbf{V} = \mathbf{QDQ'}$ and $\mathbf{a} = \mathbf{Q'h}$ we see that (21) may be written as

$$d = \mathbf{a'Da} - \frac{1}{\mathbf{a'D}^{-1}\mathbf{a}} \leq (\sqrt{\lambda_1} - \sqrt{\lambda_n})^2. \tag{27}$$

To prove (27) we use (24) to obtain

$$d \leq \mathbf{a'Da} - \lambda_1\lambda_n(\lambda_1 + \lambda_n - \mathbf{a'Da})^{-1} = \lambda_1 + \lambda_n - z - \lambda_1\lambda_n z^{-1}, \tag{28}$$

where

$$z = \lambda_1 + \lambda_n - \mathbf{a'Da} = \mathbf{a}'(\mathrm{dg}\{\lambda_1 - \lambda_i + \lambda_n\})\mathbf{a} > 0. \tag{29}$$

Hence

$$\begin{aligned} d &\leq \lambda_1 + \lambda_n - z^{-1}(z^2 + \lambda_1\lambda_n) \\ &= \lambda_1 + \lambda_n - z^{-1}(z - \sqrt{\lambda_1\lambda_n})^2 - 2\sqrt{\lambda_1\lambda_n} \\ &\leq \lambda_1 + \lambda_n - 2\sqrt{\lambda_1\lambda_n} = (\sqrt{\lambda_1} - \sqrt{\lambda_n})^2, \end{aligned} \tag{30}$$

and the proof is complete. (QED)

When $\mathbf{V}$ is not a scalar matrix, i.e., when $\lambda_1 \neq \lambda_n$, then it turns out that equality holds in (19) if and only if equality holds in (20). Equality in (21), however, we will see can never hold simultaneously with equality in (19) and (20). We establish these results in the following

THEOREM 2. *Let* $\mathbf{h}$ *and* $\mathbf{V}$ *be defined as in Theorem 1 and suppose that the characteristic root* λ_1 *has multiplicity s and* λ_n *multiplicity t; let the columns of the* $n \times s$ *matrix* $\mathbf{Q}_s$ *and of the* $n \times t$ *matrix* $\mathbf{Q}_t$ *be the corresponding orthonormalized characteristic vectors. Let* $\mathbf{a}_s = \mathbf{Q}_s'\mathbf{h}$ *and* $\mathbf{a}_t = \mathbf{Q}_t'\mathbf{h}$. *If* $\mathbf{V}$ *is not a scalar matrix, i.e.,* $\lambda_1 \neq \lambda_n$, *then equality holds in* (19) *if and only if equality holds in* (20) *if and only if both*

$$\mathbf{h} = \mathbf{Q}_s\mathbf{a}_s + \mathbf{Q}_t\mathbf{a}_t \tag{31}$$

and

$$\mathbf{a}_s'\mathbf{a}_s = \mathbf{a}_t'\mathbf{a}_t = \tfrac{1}{2}. \tag{32}$$

If $\mathbf{V}$ *is not a scalar matrix then equality holds in* (21) *if and only if*

$$\mathbf{a}_s'\mathbf{a}_s = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}} \text{ and } \mathbf{a}_t'\mathbf{a}_t = \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}} \tag{33}$$

and (31) *hold.*

Proof. Equality in (19) holds if and only if equality holds in both the third line and in the last line of (22). Equality in the last line of (22) holds if and only if all the elements in the $n \times 1$ vector $\mathbf{a}$ are zero except for the first s and the last t, i.e., if and only if

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_s \\ 0 \\ \mathbf{a}_t \end{pmatrix} \text{ if } s + t < n \text{ or } \mathbf{a} = \begin{pmatrix} \mathbf{a}_s \\ \mathbf{a}_t \end{pmatrix} \text{ if } s + t = n. \tag{34}$$

Let $\mathbf{Q}$ be an $n \times n$ orthogonal matrix that diagonalizes $\mathbf{V}$ as in the proof of Theorem 1. Then we may choose $\mathbf{Q}$ so that its first s columns are the columns of $\mathbf{Q}_s$ and the last t columns of $\mathbf{Q}$ are the columns of $\mathbf{Q}_t$. Since $\mathbf{h} = \mathbf{Qa}$ it follows at once that (34) and (31) are equivalent.

Equality holds in the third line of (22) if and only if

$$\mathbf{a'Da} = \tfrac{1}{2}(\lambda_1 + \lambda_n). \tag{35}$$

Hence (35) and (34) hold simultaneously if and only if (32) and (31) hold simultaneously.

Equality holds in (20) if and only if equality holds in the first line and in the third line of (26). Equality in the first line of (26) holds if and only if equality holds throughout (25) and this is so if and only if (34) holds. Equality in the third line of (26) holds if and only if (35) holds, and thus equality holds in (20) if and only if equality holds in (19).

Equality holds in (21) if and only if equality holds in the first line and in the third line of (30). Equality holds in the first line of (30) if and only if equality holds throughout (28) and this is so if and only if (34) holds. Equality holds in the third line of (30) if and only if

$$z = \lambda_1 + \lambda_n - \mathbf{a'Da} = \sqrt{\lambda_1\lambda_n}. \tag{36}$$

It follows then that equality holds throughout (30) if and only if both (33) and (31) hold simultaneously. (QED)

When $\lambda_1 \neq \lambda_n$ we see that equality cannot hold simultaneously in (32) and in (33), and hence equality in (21) cannot hold simultaneously with equality in (19) and (20). When the multiplicities $s = t = 1$, i.e., when both λ_1 and λ_n are simple characteristic roots, the vectors $\mathbf{a}_s$ and $\mathbf{a}_t$ become scalars and Theorem 2 simplifies to the following

COROLLARY. *Let the characteristic roots λ_1 and λ_n in Theorem 1 both be simple, i.e., $s = t = 1$, and let $\mathbf{q}_1$ and $\mathbf{q}_n$ be corresponding normalized characteristic vectors. Then equality holds in* (19) *if and only if equality holds in* (20) *if and only if*

$$\mathbf{h} = \frac{1}{\sqrt{2}}\mathbf{q}_1 \pm \frac{1}{\sqrt{2}}\mathbf{q}_n, \tag{37}$$

and equality holds in (21) *if and only if*

$$\mathbf{h} = \left[\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}\right]^{\frac{1}{2}} \mathbf{q}_1 \pm \left[\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_1} + \sqrt{\lambda_n}}\right]^{\frac{1}{2}} \mathbf{q}_n. \tag{38}$$

When $s \geq 1$ *and* $t \geq 1$ *then* (37) *is sufficient for equality in* (19) *and* (20), *while* (38) *is sufficient for equality in* (21).

In the next section we illustrate these results with two examples.

3. EXAMPLES

To illustrate our results we first consider what happens to k, f, and d when (31), or equivalently (34), holds. In this event we write

$$b^2 = \mathbf{a}_s'\mathbf{a}_s\,; \quad 1 - b^2 = \mathbf{a}_t'\mathbf{a}_t. \tag{39}$$

Straightforward algebra yields:

$$k = b^2(1 - b^2)(\lambda_1 - \lambda_n)^2 \leq \tfrac{1}{4}(\lambda_1 - \lambda_n)^2, \tag{40}$$

$$f = \frac{1}{1 + b^2(1 - b^2)[(\lambda_1 - \lambda_n)^2/\lambda_1\lambda_n]} \geq \frac{4\lambda_1\lambda_n}{(\lambda_1 + \lambda_n)^2}, \tag{41}$$

$$d = \frac{b^2(1 - b^2)(\lambda_1 - \lambda_n)^2}{\lambda_1 - b^2(\lambda_1 - \lambda_n)} = \frac{k}{\lambda_1 - b^2(\lambda_1 - \lambda_n)} \leq (\sqrt{\lambda_1} - \sqrt{\lambda_n})^2. \tag{42}$$

For our first example let us consider the »intraclass correlation» matrix

$$\mathbf{V} = (1 - \varrho)\mathbf{I} + \varrho\mathbf{ee}' = \begin{pmatrix} 1 & \varrho & \cdots & \varrho & \varrho \\ \varrho & 1 & \cdots & \varrho & \varrho \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \varrho & \varrho & \cdots & 1 & \varrho \\ \varrho & \varrho & \cdots & \varrho & 1 \end{pmatrix}, \tag{43}$$

where $\mathbf{e}$ is the $n \times 1$ column vector of ones. We will suppose that $\varrho > 0$. Then

$$\left.\begin{array}{ll} \lambda_1 = 1 + \varrho(n - 1), & \text{with multiplicity } s = 1 \\ \lambda_n = 1 - \varrho, & \text{with multiplicity } t = n - 1. \end{array}\right\} \tag{44}$$

And so $s + t = n$, and thus both (31) and (34) hold. Furthermore we note that $\mathbf{e}$ is a characteristic vector of V corresponding to $\lambda_1 = 1 + \varrho(n - 1)$, and any vector orthogonal to $\mathbf{e}$ (i.e., with components summing to 1) is a characteristic vector corresponding to $\lambda_n = 1 - \varrho$. Let us consider, therefore,

$$\mathbf{X} = \mathbf{x} = (1 + c, 1 - c, 1, 1, \ldots, 1)'. \tag{45}$$

Hence

$$b^2 = n/(n + 2c^2), \quad \lambda_1 - \lambda_n = n\varrho, \tag{46}$$

and

$$k = 2n^3c^2\varrho^2/(n + 2c^2)^2 \leq \tfrac{1}{4}n^2\varrho^2, \tag{47}$$

$$f = \frac{1}{1 + \dfrac{k}{(1-\varrho)[1+\varrho(n-1)]}} \geq \frac{4(1-\varrho)[1+\varrho(n-1)]}{[n\varrho + 2(1-\varrho)]^2}, \quad (48)$$

$$d = \frac{k}{1 - \varrho + c\sqrt{2k/n}} \leq (\sqrt{1+\varrho(n-1)} - \sqrt{1-\varrho})^2. \quad (49)$$

The bounds in (47) and (48) are attained simultaneously at $c^2 = \frac{1}{2}n$, while the upper bound in (49) is attained if and only if $c^2 = \frac{1}{2}n\sqrt{\lambda_n/\lambda_1} < \frac{1}{2}n$.

When $n = 8$ and $\varrho = \frac{1}{2}$ we obtain

$$k = \frac{64c^2}{(c^2+4)^2} \leq 4\,; \quad f = \frac{9(c^2+4)^2}{(9c^2+4)(c^2+36)} \geq \frac{9}{25} = 0.36\,; \quad (50)$$

the bounds are attained when $c^2 = \frac{1}{2}n = 4$. Furthermore,

$$d = \frac{128c^2}{(9c^2+4)(c^2+4)} \leq 2\,; \quad (51)$$

the upper bound in (51) is attained when $c^2 = 4/3$.

As a second example, suppose that n is even and equal to $2m$, say, and that

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{pmatrix} \text{ with } \mathbf{V}_i = (1-\varrho_i)\mathbf{I} + \varrho_i \mathbf{ee}' : m \times m\,;\ i = 1,2. \quad (52)$$

If $\varrho_1 > 0$ and $\varrho_2 < 0$ then the largest characteristic root of $\mathbf{V}_1$ will be $1 + \varrho_1(m-1)$, and the smallest characteristic root of $\mathbf{V}_2$ will be $1 + \varrho_2(m-1)$. These two roots will be, respectively, the largest and smallest roots of V, provided

$$m - 1 > \varrho_1/-\varrho_2 > 1/(m-1) \quad (53)$$

and this can only happen if $m \geq 3$. So let $m = 3$, and $\varrho_1 = \frac{1}{4}$ and $\varrho_2 = -\frac{1}{4}$. Then (53) is satisfied and both the largest and smallest characteristic roots of V will be simple.

Now let

$$\mathbf{X} = \mathbf{x} = \begin{bmatrix} \mathbf{e} \\ c\mathbf{e} \end{bmatrix} \begin{matrix} m \times 1 \\ m \times 1 \end{matrix} \text{ so that } \mathbf{h} = \begin{bmatrix} \mathbf{e} \\ c\mathbf{e} \end{bmatrix} / \sqrt{m(c^2+1)}. \quad (54)$$

Thus $b^2 = 1/(c^2+1)$, and so with $m = 3$, and $\varrho_1 = +\frac{1}{4}$ and $\varrho_2 = -\frac{1}{4}$,

$$k = \frac{c^2}{(c^2+1)^2} \leq \frac{1}{4}\,; \quad f = \frac{3(c^2+1)^2}{(3c^2+1)(c^2+3)} \geq \frac{3}{4}\,; \quad (55)$$

the bounds in (55) are attained when $c^2 = 1$.

Furthermore,

$$d = \frac{2c^2}{(c^2 + 1)(3c^2 + 1)} \leq 2 - \sqrt{3}; \tag{56}$$

the upper bound in (56) is attained when $c^2 = 1/\sqrt{3}$.

ACKNOWLEDGEMENTS

The author is most grateful to Peter Bloomfield, Simo Puntanen, Keith Worsley and Sanjo Zlobec for helpful discussions. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant Number A7274, and by the Gouvernement du Québec, Programme de formation de chercheurs et d'action concertée, subvention no. EQ-961.

REFERENCES

Alalouf, I. S., and Styan, George P. H. (1983). Characterizations of the conditions for the ordinary least squares estimator to be best linear unbiased. *Topics in Applied Statistics (Proceedings of the Canadian Conference on Applied Statistics, Montréal, 1981)*, Marcel Dekker, New York, in press. [McGill University Department of Mathematics Report No. 83—3.]

Bloomfield, Peter, and Watson, Geoffrey S. (1975). The inefficiency of least squares. *Biometrika*, 62, 121—128.

Marcus, Marvin, and Minc, Henryk (1964). *A Survey of Matrix Theory and Matrix Inequalities.* Allyn and Bacon, Boston. [Reprinted: 1969, Prindle, Weber & Schmidt, Boston.]

Marshall, Albert W., and Olkin, Ingram (1964). Reversal of the Lyapunov, Hölder, and Minkowski inequalities and other extensions of the Kantorovich inequality. *J. Math. Anal. Appl.*, 8, 503—514.

Puntanen, Simo (1982). Personal communication.

Styan, George P. H., and Zlobec, Sanjo (1982). An inequality connected with the efficiency of the least-squares estimator. (Abstract) *Inst. Math. Statist. Bull.*, 11, 192.

Zyskind, George (1967). On canonical forms, non-negative covariance matrices and best and simple least squares linear estimators in linear models. *Ann. Math. Statist.*, 38, 1092—1109.

Acta
Universitatis
Tamperensis
SER. A VOL. 153

Department of Mathematical Sciences, Statistics

FESTSCHRIFT

FOR

EINO HAIKALA ON HIS SEVENTIETH BIRTHDAY

UNIVERSITY OF TAMPERE
TAMPERE 1983

Preface

This Festschrift was compiled to mark the seventieth birthday of *Eino Haikala*. Contributors include many of his former colleagues and students. The editors wish to thank the writers for their participation and the City of Tampere, the Tampere University Foundation and the Tampere University Publications Committee for financial support.

Tampere, May 12th, 1983

The Editors

Contents

FESTSCHRIFT for Eino Haikala to mark his seventieth birthday

Eino Haikala

70 years on May 12th, 1983

Eino Haikala was born in Hanhijärvi, Lappee on May 12th, 1913, third child of Onni Christian Haikala, farmer, and his wife Irene Wilhelmina, née Helmann. He was married in 1948 to Marja Emilia Inkeri Henttu, daughter of Councillor Evald Henttu and Ingrid Emilia Maria née Berkan; the couple have four sons, Eino Kristian (1949), Veikko Evald (1950), Ilkka Juhani (1952) and Klaus Mikael (1955).

Eino Haikala matriculated from Lappeenranta coeducational in 1932 and took his first degree at the University of Helsinki in 1943. Like others of his generation he had to leave off studies with the outbreak of war, but resumed during the lull in hostilities and received his degree papers literally in the trenches. He majored in economics, with finance theory and politics as subsidiary subjects. His studies were not, however, confined to these but extended to mathematics, physics and astronomy, with languages, in particular French, and cultural and musical interests to complete the range. When theoretical statistics became an academic subject at Helsinki, Eino Haikala, as the first student in Finland, took a major in it in 1949.

He completed his military service, his »unofficial visit to the Soviet Union», as commander of an artillery battery, with a number of woundings. He was promoted captain on June 22nd, 1942. Decorations included three Crosses of Freedom with sword and commemoration medals from both the Winter War and its sequel.

Eino Haikala took his doctorate in 1956 with an admirable econometric study of agricultural conditions and the cobweb theory.

Professor Haikala was engaged as an economic statistics researcher in the Pellervo Society Marketing Research Institue 1950—53 and directed the Institute 1953—55. From 1955 to 1961 he was director of the Kyösti Haataja Research Bureau.

Eino Haikala's career as a teacher commenced with a teaching assistantship in Economics in the Helsinki School of Technology from 1947 to 1951. This was followed by a lengthy period (1953—70) teaching mainly economics in the Faculty of Agriculture and Forestry, sciences in which he acquired considerable practical and theoretical expertise. His knowledge was to prove extremely useful for example in selections for the Chair in Agricultural Economy and in discharge of opponency at a dissertation on agricultural policy. For the year 1961—62 Eino Haikala was Acting Professor of Economics at the University of Oulu and simultaneously Acting Associate Professor of Statistics first in the School of Social Sciences and subsequently in the Faculty of Social Sciences in the University of Tampere, this up to November 1st, 1965, and thereafter jointly to June 30th, 1967. He held an Acting Professorship in Statistics from August 31st, 1965 to January 1st, 1970, when he was appointed Professor of Statistics in the Faculty of Economics and Administration in Tampere University. From this Chair he retired on August 31st, 1976. Eino Haikala's life's work was acknowledged with the award of a Commandership in the Order of the Lion of Finland on December 6th, 1976.

Eino Haikala's career as a teacher in Tampere University fell in a period of vigorous economic expansion. The student body swelled many-fold in a short time, among other things with the establishment of new faculties; teaching resources in the field of statistics, on the other hand, were but slowly incremented. Under Professor Haikala this »difficult» subject was to become a popular choice. He is indeed remembered by generations of students as an inspiring and creative instructor who could undestand an encourage a student regardless of a possibly incomplete grounding in mathematics. His popularity as a lecturer is perhaps best reflected in the size of his audiences at the Tampere and Jyväskylä Summer Universities. At Jyväskylä the seating the main auditorium proved inadequate for the occasion.

In the earlier stages of Eino Haikala's teaching and researches the then School of Social Sciences had no actual Department of Statistics, let alone a research tradition; everything had to built up from scratch. This proved possible in that for example positions as they were created could be filled by outstanding students who, as their own studies progressed, could take over from Professor Haikala first intermediate and eventually advanced courses. In the early 1970's the first Licentiate papers were completed, by the middle of that decade the first Doctorates. The Department developed surprisingly quickly without significant assistance from without. Professor Haikala's own researches since the 50's have concentrated on econometry and its applications, particularly to problems of agriculture. His work has earned recognition and has in many cases been in advance of the times. In the latter 1960's his interest focussed largely on mathematical statistics and allied fields. Aside from this, however, he has produced a considerable range of high-standard but eminently readable newspaper and magazine articles on economic and statistical problems.

In addition to his academic duties Professor Haikala has served as member and chairman in the administration of various departments in the University, among them the Research Institute, the Institute for Extension Studies and the Institute for Folk Studies.

Outside the University Professor Haikala has maintained two particular fields of active interest: on the one hand socioeconomic problems, with membership on a number of committees and participation for example in the work of the EFTA delegation in 1960—61, and on the other the economic aspects of agriculture, forestry and market gardening, with research and experimental activities these have entailed.

In his time Eino Haikala was actively involved in the work of the Agrarian/Central Party, representing above all the views of the small-farmer; he was candidate for chairmanship of the Party Council for Mikkeli in 1971. When in 1973 a moderate faction broke from the Finnish Rural Party and founded the Party for Finnish National Unity, Eino Haikala was invited to become its first chairman. Political history will remember him as the Party's candidate for Finland's Presidency in the 1978 presidential elections.

Professor Haikala has also occupied many positions of trust and responsibility in organizations concerned with agriculture, for example on the board of the Pellervo Society, the administrative council of the Cooperative Bank of Greater Helsinki, the board of the Central Association of Small-Farmers, the Central Committee of the Association for Population Settlement and the board of the Federation of Agricultural Associations.

In 1967 Eino and Marja Haikala bought a farm at Säijä in Lempäälä, where before long they had erected a nurseries covering some 2000 square metres. Their breeding experiments soon produced results; unusually productive long-season strawberry strains, blackberries, shrubbery plants, world record lilies. His expert knowledge Eino Haikala has passed on as an inspiration to others for example in popular gardening courses arranged by the Summer Universities.

Since 1970 Professor Haikala has been Rector of Tampere Summer University, in which cultural capacity he has among other things presided as personable host at Rector's Coffee. Many an international and Finnish artist and teacher on the first-rate Summer University music courses has enjoyed the hospitality of the Rector and his lady in the uniquely pleasurable and inspiring milieu of Säijä. Others too, invited either by the Department of Statistics or as personal friends, have visited the Haikala home, among them Professors Fred. C. Andrews, Paul E. Kustaanheimo, K. V. Laurikainen, Peter Naeve, Klaus Schmidt, George P. H. Styan, Leo Törnquist and many more. Eino Haikala's broad cultural perspective, his inimitable humour and facility for anecdote have made these occasions always both enjoyable and memorable experiences.

This same »Säijä Institution» has also in its time accommodated preparation of the Department's projected teaching and research programmes and important matters of Department administration.