TOP 10

1♠

$$\mathbf{P}_{(\mathbf{A}\,:\,\mathbf{B})} = \mathbf{P}_{\mathbf{A}} + \mathbf{P}_{(\mathbf{I}-\mathbf{P}_{\mathbf{A}})\mathbf{B}}$$

2♠

$\mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{B}}$ is orthogonal projector

$\Leftrightarrow\ \mathbf{A}'\mathbf{B} = \mathbf{0}$,

in which case

$$\mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{B}} = \mathbf{P}_{\mathcal{C}(\mathbf{A}\,:\,\mathbf{B})}$$

3♠

$\mathbf{P}_{\mathbf{A}} - \mathbf{P}_{\mathbf{B}}$ is orthogonal projector

$\Leftrightarrow\ \mathbf{P}_{\mathbf{A}}\mathbf{P}_{\mathbf{B}} = \mathbf{P}_{\mathbf{B}}\mathbf{P}_{\mathbf{A}} = \mathbf{P}_{\mathbf{B}}$,

in which case

$$\mathbf{P}_{\mathbf{A}} - \mathbf{P}_{\mathbf{B}} = \mathbf{P}_{\mathcal{C}(\mathbf{A})\cap\mathcal{C}(\mathbf{B})^{\perp}}$$

Application of **1♠**

$$\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}' = \mathbf{P}_{\mathbf{X}}$$
$$= \mathbf{P}_{\mathbf{X}_1} + \mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}$$
$$\mathbf{M}_1 = \mathbf{I} - \mathbf{P}_{\mathbf{X}_1},\ \mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$$

$$\mathbf{H} - \mathbf{P}_{\mathbf{X}_1} = \mathbf{P}_{\mathbf{M}_1\mathbf{X}_2} = \mathbf{P}_{C(\mathbf{X})\cap C(\mathbf{M}_1)}$$
$$= \mathbf{M}_1\mathbf{X}_2(\mathbf{X}';_2\,\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}';_2\,\mathbf{M}_1$$

$$\mathbf{M} = \mathbf{I} - \mathbf{H}$$
$$= \mathbf{I} - (\mathbf{P}_{\mathbf{X}_1} + \mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}) = \mathbf{M}_1 - \mathbf{P}_{\mathbf{M}_1\mathbf{X}_2}$$

4♠

$\mathbf{LAY} = \mathbf{MAY}$ & $r(\mathbf{AY}) = r(\mathbf{A})$

$\Rightarrow \mathbf{LA} = \mathbf{MA}$

5♠

$$r(\mathbf{AB}) = r(\mathbf{A}) - \dim C(\mathbf{A}') \cap C(\mathbf{B})^{Ö}$$

$$r(\mathbf{A} : \mathbf{B}) = r(\mathbf{A}) + r[(\mathbf{I} - \mathbf{P}_{\mathbf{A}})\mathbf{B}]$$

6♠

$\mathbf{AB}^{-}\mathbf{C}$ is invariant w.r.t. the choice of $\mathbf{B}^{-}$

$\Leftrightarrow C(\mathbf{C})\ 1\ C(\mathbf{B})$ & $C(\mathbf{A}')\ 1\ C(\mathbf{B}')$

Application of **4♠**

ALWAYS:

$$\mathbf{A'VA(A'VA)^{-}A'VA = A'VA}. \qquad (1)$$

IF

$$r(\mathbf{A'VA}) = r(\mathbf{VA'}), \qquad (2)$$

then

$$\mathbf{VA(A'VA)^{-}A'VA = VA} \qquad (3)$$

(2) is true e.g. if $\mathbf{V}$ is nnd.

IF $r(\mathbf{A'VA}) = r(\mathbf{A})$, then

$$\mathbf{A(A'VA)^{-}A'VA = A}. \qquad (3)$$

Application of **5♠**

$\mathrm{r}(\mathbf{X}) = \mathrm{r}(\mathbf{1} : \mathbf{X}_0)$
$= 1 + \mathrm{r}(\mathbf{X}_0) - \dim[\mathcal{C}(\mathbf{1}) \cap \mathcal{C}(\mathbf{X}_0)]$
$= 1 + \mathrm{r}[(\mathbf{I} - \mathbf{J})\mathbf{X}_0]$
$= 1 + \mathrm{r}(\mathbf{T}_1)$
$= 1 + \mathrm{r}(\mathbf{R}_1)$
and hence

$|\mathbf{R}_1| \neq 0 \quad \Leftrightarrow \quad \mathrm{r}(\mathbf{X}) = k + 1$
$\Leftrightarrow \quad \mathrm{r}(\mathbf{X}_0) = k \;\; \& \;\; \mathbf{1} \notin \mathcal{C}(\mathbf{X}_0)$
$\Leftrightarrow \quad |\mathbf{X}'\mathbf{X}| \neq 0$

$\mathbf{X} = (\mathbf{1} : \mathbf{X}_0)$, $\mathbf{R}_1 = \mathrm{kor}(\mathbf{X}_0)$

<u>Application of **6♠**</u>

$$\mathbf{K}'\int \text{ is estimable}$$

$$\Leftrightarrow$$

$$\mathbf{K}'\int;\hat{\ } = \mathbf{K}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y} \text{ is unique for every choice of } (\mathbf{X}'\mathbf{X})^{-}$$

$$\Leftrightarrow$$

$$\mathcal{C}(\mathbf{K}) \subset \mathcal{C}(\mathbf{X}')$$

7♠ The inverse of pd mtx **A**

$\mathbf{A} = \mathbf{L}'\mathbf{L}$

$= \left(;\mathbf{L}';_1\mathbf{L}_1;\mathbf{L}';_1\mathbf{L}_2;\mathbf{L}';_2\mathbf{L}_1;\mathbf{L}';_2\mathbf{L}_2 \right)$

$= \left(;\mathbf{A}_{11};\mathbf{A}_{12};\mathbf{A}_{21};\mathbf{A}_{22} \right)$

$\mathbf{A}^{-1} =$

$\left((\mathbf{L}';_1\mathbf{Q}_2\mathbf{L}_1)^{-1};\#;\#;(\mathbf{L}';_2\mathbf{Q}_1\mathbf{L}_2)^{-1} \right)$

$= \left(;\mathbf{A}^{11};\mathbf{A}^{12};\mathbf{A}^{21};\mathbf{A}^{22} \right)$,

$\mathbf{L}';_2\, \mathbf{Q}_1\mathbf{L}_2 = \mathbf{A}_{22\bullet 1}$

$\mathbf{Q}_i = \mathbf{I} - \mathbf{P}_{\mathbf{L}_i}$

$\mathbf{A}_{22\bullet 1} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}^{-1};_{11}\, \mathbf{A}_{12}$

= the Schur complement of $\mathbf{A}_{11}$ in $\mathbf{A}$

$\mathbf{A}^{22}$ can be expressed also as

$(\mathbf{L}';_2\ \mathbf{L}_2)^{-1}$ +

$(\mathbf{L}';_2\ \mathbf{L}_2)^{-1}\mathbf{L}';_2\ \mathbf{L}_1(\mathbf{L}';_1\ \mathbf{Q}_2\mathbf{L}_1)^{-1}\mathbf{L}';_1$

$\mathbf{L}_2(\mathbf{L}';_2\ \mathbf{L}_2)^{-1}$

8♠

$\mathbf{A}' = \mathbf{A} = \Big(\;;\mathbf{B};\mathbf{C};\mathbf{C}';\mathbf{D} \quad \Big) =$
$\Big(\;;\mathbf{A}_{11};\mathbf{A}_{12};\mathbf{A}_{21};\mathbf{A}_{22} \quad \Big)$

$\mathbf{A} \geq \mathbf{0}$ $\Leftrightarrow$ iff (a), (b), and (c) hold:

(a) $\mathbf{D} \geq \mathbf{0}$
(b) $\mathrm{C}(\mathbf{C}')$ 1 $\mathrm{C}(\mathbf{D})$
(c) $\mathbf{B} - \mathbf{C}\mathbf{D}^{-}\mathbf{C}' \geq \mathbf{0}$

$\mathbf{A} \geq \mathbf{0}$ $\Leftrightarrow$ iff (d), (e), and (f) hold:

(d) $\mathbf{B} \geq \mathbf{0}$
(e) $\mathrm{C}(\mathbf{C})$ 1 $\mathrm{C}(\mathbf{B})$
(f) $\mathbf{D} - \mathbf{C}'\mathbf{B}^{-}\mathbf{C} \geq \mathbf{0}$

9♠

Let **A** and **B** be symmetric and nnd.Then:

$$\mathbf{A} \le \mathbf{B} \iff C(\mathbf{A})\ 1\ C(\mathbf{B}) \ \ \& \ \ \mathrm{ch}_1(\mathbf{A}\mathbf{B}^-) \le 1$$

Problem: Given a nonnegative definite **V**, find the scalar a such that

$$a\mathbf{1}\mathbf{1}' \le \mathbf{V} \qquad (*)$$

and $a\mathbf{1}\mathbf{1}'$ is “as close to **V**” as possible.

Soln: (*) holds iff $\mathbf{1} \in C(\mathbf{V})$ and $\mathrm{ch}_1(\mathbf{1}\mathbf{1}'\mathbf{V}^-) = \mathbf{1}'\mathbf{V}^-\mathbf{1} \le 1/a$, and hence the requested a is $a;^{\sim} = 1/\mathbf{1}'\mathbf{V}^-\mathbf{1}$.

Note: $a;^{\sim}$ = Var[BLUE(a)] under {**y**, $\mathbf{1}a$, **V**}.

10♠

The following statements are equivalent:

(a) $\mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{B}) = \{\mathbf{0}\}$

(b) $\left(;\mathbf{A}';\mathbf{B}' \right)(\mathbf{AA}' + \mathbf{BB}')^{-}(\mathbf{AA}' : \mathbf{BB}')$

$$= \left(;\mathbf{A}';\mathbf{0};\mathbf{0};\mathbf{B}' \right)$$

(c) $\mathbf{A}'(\mathbf{AA}' + \mathbf{BB}')^{-}\mathbf{AA}' = \mathbf{A}'$

(d) $\mathbf{A}'(\mathbf{AA}' + \mathbf{BB}')^{-}\mathbf{B} = \mathbf{0}$

(e) $(\mathbf{AA}' + \mathbf{BB}')^{-}$ is a g-inv. of $\mathbf{AA}'$

(f) $\mathbf{A}'(\mathbf{AA}' + \mathbf{BB}')^{-}\mathbf{A} = \mathbf{P}_{\mathbf{A}'}$