H1

1.1	a) P(X=1, Y=1)	b) p^2
1.2	a) (6 nCr 3)*0.33*$(1-0.3)^3$ ≈0.185	b) (6 nCr 5)*0.3^5*0.7+(6 nCr 6)*0.3^6*0.7^0≈0.01
1.3	p=0.65 n=16 P(X=7) ≈0,0442	
1.4	P(Z=1)=P(Z=-1)=0.15 P(Z=2)=P(Z=-2)=0.1 P(Z=5)=P(Z=-5)=0.25	
1.5	a) P(Y=0)=(20 nCr 10)*0.5^{10}*0.5^{10}≈0.1762	b)1-P(X≤14) ≈0.0207 (R:llä: 1-pbinom(14,20,0.5))
1.6	0.5<p<1	
1.7	a) 20*$p^3(1-p)^3$	b) p=0.5
1.8	a) $1-(1-p)^{50}$	b) B-näytteiden jakauma: Bin(20,$1-(1-p)^{50}$)

H2

2.1	E(X)=2 Var(X)=4/5 P(X=1)=2/5 P(X=2)=1/5 P(X=3)=2/5	
2.2	$M_x(t)=(1-p+e^tp)^n$ -> E(X)= $M_x(0)'$=np ja $E(X^2)=M_x(0)''=np+np^2(n-1)$	
2.3	a) $1-(1-3p(1-p))^{n-1}=1-(1-3pq)^{n-1}$	b) ratkaistaan yhtälö $1-(1-3p(1-p))^n≥0.95$ -> n=3
2.4	$P(W_b=b+i)=((b+i-1)$ nCr $(b-1))*p^b*(1-p)^i$, missä i ∈ {0,1,..,a-1} Jos a=b=4 ja p=0.6, i=0,1,2,3 ja näiden todennäköisyyksien summa on 0.710. R:llä pbinom(q=3, size=4, prob=6)	
2.5	a) ääretön c) merk. V=voittosumma, $E(V)=1-(38/37)^{S+1}$	b) $1-(19/37)^{S+1}$
2.6	W_5~NHGeo(5, N, 25/N) -> $P(W_5≤50)$ tämän todennäköisyysfunktiosta $E(W_5)=5(N+1)/26$ N=500 -> $P(W_5≤50)$ ≈0.0926 ja $E(W_5)$ ≈ 96.34	
2.7	W_{10}~NHGeo(10,60,40/60) -> $E(W_{10})$ ≈14.88 P('kannattajat ovat enemmistönä')=$P(W_{10}≤19)$ ≈0.9677	
2.8	Saa johdettua purkamalla binomikertoimet: esim. (a nCr x)=$a^{(x)}$/x!	

H3

3.1 a) $P(N(1/4)\geq 2)=1-3.5*e^{-2.5} \approx 0.7127$ b) $P(N(1/6)=0)=e^{-10/6} \approx 0.189$

3.2 $(12 \text{ nCr } 2)*(e^{-5})^2*(1-e^5)^{10} \approx 0.0028$

3.3 OC-käyrän arvon annetuilla p-arvoilla
0.99265929 0.92555126 0.78175216 0.60148551 0.42851323 0.03815646

3.4 a) 0.99100445 0.92157358 0.77910159 0.60234169 0.43248796 0.04095166
b) 0.99092014 0.92118651 0.77872291 0.60251972 0.43347012 0.04238011

3.5 a) $M'(t)=\theta/(1-t\theta)^2$ -> $E(X)=\theta$ b)Ratkaistaan yhtälö 0.5=F(3) -> $\theta= 3/\ln 2 \approx 4.328$

3.6 a) Tarkistettava, että f(x)>0 kun x ∈(0,4) ja että $\int_0^4 f(x)dx = 1$

b) F(x)=

	0,	, x≤0
	$\frac{x^2}{88}(x+3/2)$	, 0<x<4
	1	, x≥ 4

c) F(3)-F(2)=53/176 ≈ 0.30

3.7 F(x)=

0	, x≤0
$x/2$	, 0≤x≤1
½	, 1<x<2
½(x-1)	, 2≤x≤3
1	, x>3

3.8 c=3/8 ja P(0.5≤X≤1)= 5/32

H4

4.1

$Z \sim N(0,1)$

$Y = e^Z$

$Z = \ln y = g(y)$

$g'(y) = \frac{1}{y}$

$f_Y(y) = f_Z(\ln y)\frac{1}{y} \quad , y > 0$

4.2 a) $E(|X-\mu|)=\sqrt{\frac{2}{\pi}}\,\sigma$

b) $\gamma 2 = E(Z^4) = M_z^{(4)}(0) = 3$

4.3 a) $M_{X1+X2}(t)=\left(\frac{1}{1-\theta t}\right)^2$

b) $M_{Sn}(t)=\left(\frac{1}{1-\theta t}\right)^n$

c) $S=\frac{Sn-E(Sn)}{\sqrt{Var(Sn)}}=\frac{Sn-n\theta}{\sqrt{n}\theta}$ -> $M_S(t)=E(e^{tS})=e^{-\sqrt{n}t}\frac{1}{(1-\frac{t}{\sqrt{n}})^n}$

4.4 $F_Y(y)=F_x(\log\frac{1}{1-y})=y$ -> $f_Y(y)=F_Y'(y)=1$ -> Y~Tas(0,1)

4.5 a) $f_Y(y)=\frac{1}{12}y^{-2/3}$ b) $f_Z(z)=\frac{1}{8}z^{-3/4}$

4.6

$X \sim N(100,14^2)$

$P(89{,}5 < X < 97{,}5)$

$Z=\frac{X-100}{14}\sim N(0{,}1)$

$P(-0{,}75 < Z < -0{,}179)$

$\Phi(-0{,}179)-\Phi(-0{,}75)$

$=(1-\Phi(0{,}179))-(1-\Phi(0{,}75))$

$=(1-0{,}5714)-(1-0{,}7734)$

$=0{,}202$

$P(20\,\textit{satunnaisesti valitusta naisesta}\ 5\ \textit{luokasta}\ M)$

$=\frac{20!}{5!15!}\cdot 0{,}202^5\cdot(1-0{,}202)^{15}$

$=0{,}177$

4.7

$M(t)=(1-7t)^{-1} \quad , \ t<\frac{1}{7}$

$E(X)=7$

$Var(X)=49$

$f(x)=\frac{1}{7}e^{-\frac{x}{7}}$

$F(x)=1-e^{-\frac{x}{7}}$

4.8

$X \sim N(7,4)$

$P(15{,}364 \leq (X-7)^2 \leq 20{,}096)$

$Z=\frac{X-7}{\sqrt{4}} \sim N(0{,}1)$

$P(15{,}364 \leq (X-7)^2 \leq 20{,}096)$

$P(15{,}364 \leq 4Z^2 \leq 20{,}096)$

$P(3{,}841 \leq Z^2 \leq 5{,}024)$

$2*P(1{,}960 \leq Z \leq 2{,}241)$

$2*(\Phi(2{,}241)-\Phi(1{,}960))=2*(0{,}9875-0{,}9750)$

$=0{,}0250$

H5

5.1 Y = 2X + 1 ~ Gamma(1,θ) = Exp(θ)

$f(y)=\frac{1}{2\theta}e^{-(y-1)/2\theta}$

$M(t)=E(e^{t(2X+1)})=\frac{e^t}{1-2\theta t}$

5.2 a) R: pchisq(32.01,23)-pchisq(14.85,23)= 0.7999933

b) a= qchisq(0.025,23), b= qchisq((1-0.025),23) -> a = 11.68855, b = 38.07563

c) Khi(23)=Gamma(23/2, 2) -> E($\sqrt{X}$) $=\frac{Gamma(\frac{23}{2}+\frac{1}{2})\sqrt{2}}{Gamma(\frac{23}{2})} \approx 4{,}744$

5.3 X ~ Khi2(24)=Gamma(12,2)

a) E(X) =24, Var(X) =48

b) pchisq(42.98,24)-pchisq(15.66,24) = 0.8899...

5.4 $Z_i \sim N(0,1) \quad , i=1,2$

a) $P(|Z_i|<1) \quad ,i=1,2$

$= P(-1 < Z_i < 1)$

$= \Phi(1) - \Phi(-1) = 0.682689$

b) $E(Z_1+Z_2) = 0+0=0$

$Var(Z_1+Z_2) = 1^2+1^2 = 2$

$Z_1+Z_2 \sim N(0,2)$

$P(|Z_1+Z_2|<1)$

$= P(-1 < Z_1+Z_2 < 1)$

$= P(\frac{-1-0}{\sqrt{2}} < \frac{Z_1+Z_2-0}{\sqrt{2}} < \frac{1-0}{\sqrt{2}})$

$= \Phi(\frac{1}{\sqrt{2}}) - \Phi(-\frac{1}{\sqrt{2}}) = 0{,}5205$

$\vec{Z} = (Z_1+Z_2)/2$

$E(\vec{Z}) = \frac{1}{2}E(Z_1+Z_2) = \frac{1}{2}(E(Z_1)+E(Z_2)) = 0$

$Var(\vec{Z}) = (\frac{1}{2})^2 Var(Z_1+Z_2) = (\frac{1}{2})^2(Var(Z_1)+Var(Z_2)) = (\frac{1}{2})^2 \cdot 2 = \frac{1}{2}$

$\vec{Z} \sim N(0,\frac{1}{2})$

$P(|\vec{Z}|<1)$

$= P(-1 < \vec{Z} < 1)$

$= P(\frac{-1-0}{\sqrt{1/2}} < \frac{\vec{Z}-0}{\sqrt{1/2}} < \frac{1-0}{\sqrt{1/2}})$

$= \Phi(\frac{1}{\sqrt{1/2}}) - \Phi(-\frac{1}{\sqrt{1/2}}) = 0{,}8427$

c) $Z_1^2 + Z_2^2 \sim Khi2(2)$

$P(Z_1^2+Z_2^2 < 1)$

=pchisq(1,2)= 0.3934693

5.5

a) $E(X) = 0*1/2 + 1*1/3 + 2*1/6 = 2/3$
$E(Y) = 0*1/4 + 1*5/12 + 2*1/3 = 13/12$

b) $E(Y \mid X = 1) = E(Y \mid 1) = \sum_{y=0}^{2} yf(y \mid 1)$

$$f(y \mid x) = \sum_{y=0}^{2} y \frac{f(1,y)}{f_X(1)} = \sum_{y=0}^{2} y \frac{f(1,y)}{1/3} = 3/4$$

c) $E(XY) = 1*1*f(1,1) = 1/4$
Cov(X,Y) = E(XY) – E(X)E(Y)
$= \frac{1}{4} - \frac{2}{3} \cdot \frac{13}{12} = -\frac{17}{36}$

5.6

f(x,y)	y=0	y=1	$f_X(x)$
x=0	$p_{00} = 1/8$	$p_{01} = 2/8$	3/8
x=1	$p_{10} = 3/8$	$p_{11} = 2/8$	5/8
$f_Y(x)$	4/8	4/8	1

a) Jotta X ja Y riippumattomia, niin P(X = x, Y = y) = P(X = x)P(Y = y) kaikilla $x \in \{0,1\}$ ja $y \in \{0,1\}$

P(X = 1, Y = 1) = $p_{11} = 2/8$
P(X = 1) = $p_1 = 5/8$
P(Y = 1) = $p_2 = 4/8$
P(X = 1) P(Y = 1) = 5/16
Eli P(X = 1, Y = 1) ≠ P(X = 1) P(Y = 1), joten satunnaismuuttujat X ja Y eivät ole riippumattomat.

b) Cov(X,Y) = E(XY) – E(X)E(Y)
$= p_{11} - p_1 \cdot p_2 = \frac{2}{8} - \frac{5}{8} \cdot \frac{4}{8} = -\frac{1}{16}$

5.7 a)(X,Y) noudattaa moniulotteista hypergeometrista jakaumaa ja f(x,y)=$\frac{\binom{4}{x}\binom{4}{y}\binom{44}{5-x-y}}{\binom{52}{5}}$

b) f(x|y=2) = $\frac{\binom{4}{x}\binom{44}{3-x}}{\binom{48}{3}}$, x ∈ $\{0,1,2,3\}$

5.8 f(x) = 2x, 0≤x≤1

H6

6.1 a) $F_{X,Y}(0.6,0.8) = xy = 0.6 \cdot 0.8 = 0.48$

b) P(0.25 ≤ X ≤ 0.75, 0.1 ≤ Y ≤ 0.75)

=F(0.75,0.75)-F(0.75,0.1)-F(0.25,0.75)+F(0.25,0.1)

=0.325

6.2

$$E(X) = \int_0^1 \int_0^x 8yx^2 dydx = \int_0^1 4x^4 dx = \frac{4}{5}$$

$$E(X^2) = \int_0^1 \int_0^x 8yx^3 dydx = \int_0^1 4x^5 dx = \frac{2}{3}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{4}{6} - (\frac{4}{5})^2 = \frac{2}{75}$$

6.3

$$f_X(x) = \int_0^x 8xy\ dy = 4x^3 \qquad ,0 \le x \le 1$$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2} \qquad ,0 \le y \le x, 0 \le x \le 1$$

$$E(Y | X = x) = E(Y|x) = \int_0^x y\frac{2y}{x^2}\ dy = \frac{2x^3}{3x^2} = \frac{2}{3}x$$

$$Var(Y | X = x) = \frac{1}{18}x^2$$

6.4

$$P(Y < \frac{1}{2}) = \frac{7}{16}$$

$$P(Y < \frac{1}{2} | X = \frac{1}{2})$$

$$= \int_0^{\frac{1}{2}} f(y|\frac{1}{2})dy = \int_0^{\frac{1}{2}} \frac{2y}{(1/2)^2}dy = \int_0^{\frac{1}{2}} 8y\,dy = 1$$

6.5

(0.0.1) = $E(Y|x) = \frac{1}{2} + \frac{1}{2}x$

(0.0.2) = $E(Y \mid x) = \mu_Y + \rho \dfrac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$$\mu_X = \frac{1}{3}$$

$$\mu_Y = \frac{2}{3}$$

Luentomateriaalin luvusta 7, s. 192:

$$\sigma_X = \sigma_Y$$

$$\rho = \frac{1}{2}$$

$$0.0.2 = \frac{2}{3} + \frac{1}{2}(x - \frac{1}{3}) = \frac{2}{3} - \frac{1}{6} + \frac{1}{2}x = \frac{1}{2} + \frac{1}{2}x = 0.0.1$$

6.6 a) E(Y|X)=-0.48+0.9x b) 0.1586 c) 0.5249

6.7 a) $f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{3}} e^{-\frac{2}{3}((x-1)^2-(x-1)\left(\frac{y-1}{2}\right)+(\frac{y-1}{2})^2)}$

b) 0.5

6.8 Todistuksen voi pohjata esimerkiksi siihen, että $U \sim N(\mu_x + \mu_y, 2(\sigma^2 - \frac{\sigma_{xy}}{\sigma^2})$ ja $V \sim N(\mu_x - \mu_y, 2(\sigma^2 - \frac{\sigma_{xy}}{\sigma^2})$, ja U,V:n yhteisjakauma on kaksiulotteinen normaalijakauma, jossa Cov(U,V)=0.