TILTA1B Matematisen tilastotieteen perusteet
Syksy 2009
Ratkaisut harjoitus 6

1. $X|Y=13 \sim N\left(-3+\frac{\frac{3}{5}\cdot\sqrt{25}}{\sqrt{9}}(13-10), 25\cdot(1-(\frac{3}{5})^2)\right)$ (ks. lause 7.9, kohta 3)

$X|Y=13 \sim N(0,16)$

$Y|X=2 \sim N\left(10+\frac{\frac{3}{5}\cdot\sqrt{9}}{\sqrt{25}}(2+3), 9\cdot(1-(\frac{3}{5})^2)\right)$

$Y|X=2 \sim N(11{,}8; 5{,}76)$

a) $P(-5<X<5|Y=13)=P(-\frac{5}{4}<Z<\frac{5}{4})=2\Phi(\frac{5}{4})-1\approx 0{,}788$

b) $P(7<Y<16)=P(-1<Z<2)=\Phi(2)-\Phi(-1)\approx 0{,}819$

c) $P(7<Y<16|X=2)=P(-2<Z<\frac{7}{4})=\Phi(\frac{7}{4})-\Phi(-2)\approx 0{,}937$

2. $Y|X=178 \sim N\left(152+\frac{0{,}45\cdot 6{,}9}{7{,}6}(178-180); 6{,}9^2\cdot(1-0{,}45^2)\right)$

$Y|X=178 \sim N(151{,}18; 37{,}97)$

$P(X \geq 150|X=178)=1-P(Z\geq -0{,}191)=1-\Phi(-0{,}191)\approx 0{,}576$

3. $f(\alpha)=Var(\alpha X+Y)=\alpha^2 Var(X)+Var(Y)+2\alpha Cov(X,Y)=\alpha^2\sigma_1^2+\sigma_2^2+2\alpha\rho\sigma_1\sigma_2$

$f'(\alpha)=2\alpha\sigma_1^2+2\rho\sigma_1\sigma_2=0$

$\Leftrightarrow \alpha=-\rho\frac{\sigma_2}{\sigma_1}$

(minimi, koska $f''(\alpha)=2\sigma_1^2>0$)

4. $E(X+2Y)=0$

$Var(X+2Y)=Var(X)+2^2Var(Y)+2\cdot 2Cov(X,Y)=Var(X)+4Var(Y)+4Cor(X,Y)\sqrt{Var(X)Var(Y)}=5+4Cor(X,Y)$

a) $X+2Y \sim N(0,5)$

$P(X+2Y\leq 3)=\Phi(\frac{3}{\sqrt{5}})\approx 0{,}910$

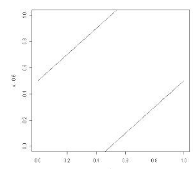
b) $X+2Y \sim N(0,7)$

$P(X+2Y\leq 3)=\Phi(\frac{3}{\sqrt{7}})\approx 0{,}872$

5.

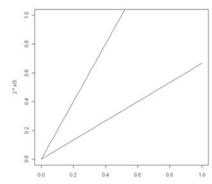
a) $P(|X-Y|\leq\frac{1}{2})=P(-\frac{1}{2}\leq X-Y\leq\frac{1}{2})=P(X-\frac{1}{2}\leq Y\leq X+\frac{1}{2})=1-\frac{1}{8}-\frac{1}{8}=\frac{3}{4}$

(Todennäköisyys saadaan laskemalla suorien $y=x-\frac{1}{2}$ ja $y=x+\frac{1}{2}$ väliin jäävä pinta-ala, kun $0\leq x\leq 1$ ja $0\leq y\leq 1$)



b) $P(|\frac{X}{Y}-1|\leq\frac{1}{2})=P(-\frac{1}{2}\leq\frac{X}{Y}-1\leq\frac{1}{2})=P(\frac{2}{3}X\leq Y\leq 2X)=1-\frac{1}{3}-\frac{1}{4}=\frac{5}{12}$

(Todennäköisyys saadaan laskemalla suorien $y=\frac{2}{3}x$ ja $y=2x$ väliin jäävä pinta-ala, kun $0\leq x\leq 1$ ja $0\leq y\leq 1$)



6.

$$E(Y_1) = E(X_1 + X_2) = 3$$
$$Var(Y_1) = Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2) = 4$$
$$E(Y_2) = E(2X_1 - 3X_2) = -4$$
$$Var(Y_2) = Var(2X_1 - 3X_2) = 2^2 Var(X_1) + (-3)^2 Var(X_2) + 2\cdot(-3)\cdot 2Cov(X_1, X_2) = 91$$
$$Cov(Y_1, Y_2) = Cov(X_1 + X_2, 2X_1 - 3X_2) = 2Cov(X_1, X_1) + 2Cov(X_2, X_1) - 3Cov(X_1, 3X_2) - 3Cov(X_2, X_2) = -17$$

Lauseen 7.10 perusteella $(X_1, X_2) \sim N_2(\cdot,\cdot)$, joten $aY_1 + bY_2 = (a+2b)X_1 + (a-3b)X_2 \sim N_1(\cdot,\cdot)$ ja edelleen $(Y_1, Y_2) \sim N_2(\cdot,\cdot)$.

Siis

$$(Y_1, Y_2) \sim N\left((3,-4); \begin{pmatrix} 4 & -17 \\ -17 & 91 \end{pmatrix}\right)$$

7.

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

(ks. lause 6.4)

$$Cov(X_1, \sum_{i=1}^{n} X_i) = Cov(X_1, X_1) + \sum_{i=2}^{n} Cov(X_1, X_i) = Var(X_1) + (n-1)\cdot 0 = \sigma^2$$

$$(X_1, \sum_{i=1}^{n} X_i) \sim N\left((\mu, n\mu); \begin{pmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & n\sigma^2 \end{pmatrix}\right)$$

8. Merk. $U = X + Y$ ja $U = X - Y$

$$f_{U,V}(u,v) = \tfrac{1}{2} f_{x,y}\left(\tfrac{u+v}{2}, \tfrac{u-v}{2}\right)$$

(ks. kohta 7.6.5 s. 205 ja 7.6.12 s.209)

$$= \frac{1}{2\cdot 2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\tfrac{u+v}{2}\right)^2 - 2\rho\,\tfrac{u+v}{2}\cdot\tfrac{u-v}{2} + \left(\tfrac{u-v}{2}\right)^2\right)\right]$$

$$= \frac{1}{2\pi\sqrt{2(1+\rho)\cdot 2(1-\rho)}} \exp\left[-\frac{1}{2}\left(\tfrac{u^2}{2(1+\rho)} + \tfrac{v^2}{2(1-\rho)}\right)\right]$$

Siis $(U,V) \sim N_2\left(0, 0, 2(1+\rho), 2(1-\rho), 0\right)$.

U ja V ovat riippumattomat (kohta 7.3.1 s. 198), sillä

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi\cdot 2(1+\rho)}} \exp\left[-\frac{1}{2}\left(\tfrac{u^2}{2(1+\rho)}\right)\right] \frac{1}{\sqrt{2\pi\cdot 2(1-\rho)}} \exp\left[-\frac{1}{2}\left(\tfrac{v^2}{2(1-\rho)}\right)\right] = f_U(u) f_V(v)$$