TILTA1B Matemaattisen tilastotieteen perusteet
Syksy 2009
Ratkaisut harjoitus 5

1. $f_X(1) = \sum_{y=0}^{2} f(1,y) = f(1,0) + f(1,1) + f(1,2) = \frac{1}{12} + \frac{3}{12} + 0 = \frac{1}{3}$

$E(Y|X=1) = \sum_{y=0}^{2} y f_{Y|X}(y|X=1) = \frac{1}{f_x(1)} \sum_{y=0}^{2} y f_{X,Y}(1,y) = \frac{1}{3}(0 \cdot f(1,0) + 1 \cdot f(1,1) + 2 \cdot f(1,2)) = \frac{1}{3}\left(1 \cdot \frac{1+2}{12} + 2 \cdot 0\right) = \frac{3}{4}$

$E(Y^2|X=1) = \frac{1}{3} 1^2 \cdot f(1,1) = \frac{1+2}{4} = \frac{3}{4}$

$Var(Y|X=1) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{16}$

2.

$E(X) = \sum_{x=0}^{2} x \cdot f_X(x) = 0 \cdot f_X(0) + 1 \cdot f_X(1) + 2 \cdot f_X(2) = 0 + 1 \cdot [f_{X,Y}(1,0) + f_{X,Y}(1,1)] + 2 \cdot 0 = \frac{2}{3}$

$E(Y) = \sum_{y=0}^{2} y \cdot f_Y(y) = \frac{13}{12}$

$E(XY) = 0 \cdot P(XY=0) + 1 \cdot P(XY=1) = P(XY=1) = f(1,1) = \frac{1}{4}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = -\frac{17}{36}$

$E(X^2) = \sum_{x=0}^{2} x^2 \cdot f_X(x) = 1, Var(X) = E(X^2) - E(X)^2 \approx 0{,}556$

$E(Y^2) = \sum_{y=0}^{2} y^2 \cdot f_Y(y) = \frac{21}{12}, Var(Y) = E(Y^2) - E(Y)^2 \approx 0{,}576$

$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \approx -0{,}83$

3. $\frac{X+Y}{2} \sim N(\mu, \frac{3}{4}\sigma^2)$

$P(-1{,}5\sigma \le \frac{X+Y}{2} - \mu \le 1{,}5\sigma) = P(-\frac{1{,}5}{\sqrt{3/4}} \le Z \le \frac{1{,}5}{\sqrt{3/4}}) = P(-\sqrt{3} \le Z \le \sqrt{3}) = 2\Phi(\sqrt{3}) - 1 \approx 0{,}917$

4. $\overline{Z} \sim N(0, \frac{1}{3})$

$P(|\overline{Z}| \le a) = P(-a \le \overline{Z} \le a) = 2\Phi(a\sqrt{3}) - 1 = 0.95 \Leftrightarrow a\sqrt{3} = 1.96 \Leftrightarrow a \approx 1{,}13$

$U \sim Khi2(3)$

$P(U \ge b) = 1 - P(U < b) = 0.975 \Leftrightarrow u \approx 9{,}35$

R:llä:

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> qchisq(0.975,3)
[1] 9.348404
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5. $z = 7 - x - y$

$$f_X(x) = \sum_{y=0}^{x-7} f_{X,Y}(x,y) = \frac{\binom{5}{x}}{\binom{23}{7}} \sum_{y=0}^{x-7} \binom{10}{y}\binom{8}{7-x-y} = \frac{\binom{5}{x}\binom{18}{7-x}}{\binom{23}{7}}$$

6. $f_X(x) = \int_x^1 8xy\,dy = 4x - 4x^3$, $f_Y(y) = \int_0^y 8xy\,dx = 4y^3$

$$f(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{8xy}{4y^3} = \tfrac{2x}{y^2}, 0 < x < y$$

$$f(y|x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{8xy}{4x - x^3} = \tfrac{2y}{1-x^2}, x < y < 1$$

$$E(X|Y = y) = \int_0^y x f(x|y)dx = \tfrac{2}{3}y$$

$$E(Y|X = x) = \int_x^1 y f(y|x)dy = \tfrac{2}{3}\tfrac{1-x^3}{1-x^2}$$

7. $\int_0^1\int_0^1 c(x+2y)dxdy = c\int_0^1 \Big/_0^1(\tfrac{1}{2}x^2 + 2xy)dy = c\int_0^1 (\tfrac{1}{2} + 2y)dy = \tfrac{3}{2}c = 1 \Leftrightarrow c = \tfrac{2}{3}$

8.

a) $P(0 \le X \le \tfrac{1}{2}) = \int_0^{\frac{1}{2}}\int_{x^2}^1 \tfrac{3}{2}dydx = \int_0^{\frac{1}{2}} \tfrac{3}{2}(1 - x^2)dx = \tfrac{11}{16}$

b) $P(\tfrac{1}{2} \le Y \le 1) = \int_{\frac{1}{2}}^1\int_0^{\sqrt{y}} \tfrac{3}{2}dxdy = \int_0^{\frac{1}{2}} \tfrac{3}{2}\sqrt{y}dy = 1 - \sqrt{\tfrac{1}{2}}^3 \approx 0{,}646$

c) $f(x)f(y) = \tfrac{3}{2}(1 - x^2)\cdot\tfrac{3}{2}\sqrt{y} \ne f(x,y)$,
siis X ja Y eivät ole riippumattomat