TILTA1B Matemaattisen tilastotieteen perusteet
Syksy 2009
Ratkaisut harjoitus 3

1.

a)

R:llä:

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> curve(2*(1-x),0,1)
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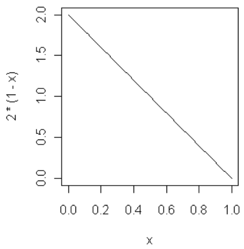

b) $F_X(x) = \int_0^x f_X(t)dt = \begin{cases} 0, x < 0 \\ 2x - x^2, 0 \le x \le 1 \\ 1, x > 1 \end{cases}$

c) $P(0 \le X \le \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \frac{3}{4}$

$P(\frac{1}{4} \le X \le \frac{3}{4}) = F(\frac{3}{4}) - F(\frac{1}{4}) = \frac{1}{2}$

$P(X \ge \frac{3}{4}) = 1 - F(\frac{3}{4}) = \frac{1}{16}$

2.

a) $f_X(x) \ge 0 \forall x \in \Re$

$\int_0^1 f_X(x)dx = \int_0^1 xe^x dx = \Big/_0^1 xe^x - \int_0^1 e^x dx = 1$

b)

$E(X) = \int_0^1 x \cdot f_X(x)dx = \int_0^1 x^2 e^x dx = \Big/_0^1 x^2 e^x - 2\int_0^1 xe^x dx = e - 2$

$E(X^2) = \int_0^1 x^2 \cdot f_X(x)dx = \int_0^1 x^3 e^x dx = \Big/_0^1 x^3 e^x - 3\int_0^1 x^2 e^x dx = e - 3 \cdot (e-2) = 6 - 2e$

$Var(X) = E(X^2) - E(X)^2 = 6 - 2e - (e-2)^2 = -e^2 + 2e + 2$

3.

a) $\int_1^\infty f_X(x)dx = \int_1^\infty ax^{-3}dx = -\frac{1}{2} a \Big/_1^\infty x^{-2} = \frac{1}{2} a = 1$

$\Leftrightarrow a = 2$

b) $E(X) = \int_1^\infty x \cdot f_X(x)dx = \int_1^\infty 2x^{-2}dx = 2$

c) $E(X^2) = \int_1^\infty x^2 \cdot f_X(x)dx = \int_1^\infty \frac{2}{x}dx = \Big/_1^\infty 2\ln x = 2 \cdot \lim_{x\to\infty} \ln x = \infty$

4.

$\int_0^x f_n(t)dt = \int_0^x ndt = nx$

$F_n(x) = \begin{cases} 0, x \le 0 \\ nx, 0 < x < \frac{1}{n} \\ 1, x \ge \frac{1}{n} \end{cases}$

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Esim. > curve(5*x,0,1/5)
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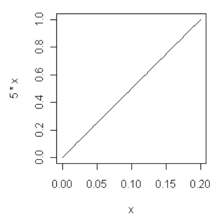

5.

$X \sim Tas(4{,}5)$

$E(X) = \frac{4+5}{2} = 4{,}5$

$Var(X) = \frac{(5-4)^2}{12} = \frac{1}{12}$

$P(4{,}2 \le X \le 4{,}7) = \int_{4,2}^{4,7} f(x)dx = \int_{4,2}^{4,7} dx = 4{,}7 - 4{,}2 = 0{,}5$

6. $x \in [-2,2]$

$y \in [-8,8]$

$G_Y(y) = P(Y \le y) = P(X^3 \le y)$

$= P(X \le \sqrt[3]{y}) = \begin{cases} 0, y < -8 \\ \frac{\sqrt[3]{y}+2}{4}, -8 \le y \le 8 \\ 1, y > 8 \end{cases}$

$g_Y(y) = G_Y'(y) = \begin{cases} \frac{1}{12\sqrt[3]{y^2}}, -8 \le y \le 8 \\ 0, muualla \end{cases}$

$v \in [0,16]$

$G_V(v) = P(V \le v) = P(X^4 \le v) = P(-\sqrt[4]{v} \le X \le \sqrt[4]{v})$

$= \begin{cases} 0, v < 0 \\ \int_{-\sqrt[4]{v}}^{\sqrt[4]{v}} \frac{1}{4} dx, 0 \le v \le 16 \\ 1, v > 16 \end{cases} = \begin{cases} 0, v < 0 \\ \frac{1}{2}\sqrt[4]{v}, 0 \le v \le 16 \\ 1, v > 16 \end{cases}$

$g_V(v) = G_V'(v) = \begin{cases} \frac{1}{8\sqrt[4]{v^3}}, 0 \le v \le 16 \\ 0, muualla \end{cases}$

Y:n tiheysfunktio voidaan määrittää myös Lauseen 6.5 (s.163) ja V:n tiheysfunktio kaavan 6.5.3 (s. 164) avulla:

$g(y) = \sqrt[3]{y}$

$f_Y(y) = f_x(g(y))|g'(y)| = \frac{1}{4} \cdot \left| \frac{1}{3\sqrt[3]{y^2}} \right| = \frac{1}{12\sqrt[3]{y^2}}, y \in [-8,8]$

(0 muualla)

$g_1(v) = \sqrt[4]{v}, x \ge 0$

$g_2(v) = -\sqrt[4]{v}, x < 0$

$f_V(v) = f_x(g_1(v))|g_1'(v)| + f_x(g_2(v))|g_2'(v)| = 2 \cdot \frac{1}{4} \cdot \frac{1}{4\sqrt[4]{v^3}} = \frac{1}{8\sqrt[4]{v^3}}, 0 \le v \le 16$

(0 muualla)

7.

$y > 0$

$G_Y(y) = P(Y \le y) = P(X^{\frac{3}{2}} \le y) = P(X \le y^{\frac{2}{3}})$

$= \int_0^{y^{\frac{2}{3}}} e^{-x} = \begin{cases} 0, y \le 0 \\ 1 - e^{-y^{2/3}}, y > 0 \end{cases}$

$g_Y(y) = G_Y'(y) = \begin{cases} \frac{2}{3\sqrt[3]{y}} e^{-y^{\frac{2}{3}}}, y > 0 \\ 0, muualla \end{cases}$

$0 < v < 1$

$G_V(v) = P(V \le v) = P(e^{-X} \le v) = P(X \ge -\ln v)$

$= 1 - P(X < \ln v) = 1 - \int_0^{-\ln v} e^{-x} dx = e^{\ln v} = \begin{cases} v, 0 < v < 1 \\ 0, muualla \end{cases}$

$g_V(v) = \begin{cases} 1, 0 < v < 1 \\ 0, muualla \end{cases}$

8.

$Y = \frac{1}{X}$

$G_Y(y) = P(Y \le y) = P(\frac{1}{X} \le y) = P(X \ge \frac{1}{y}) = 1 - F(\frac{1}{y})$

$g_Y(y) = G_Y'(y) = -\frac{1}{y^2} \cdot \left(-f_X(\frac{1}{y})\right) = \frac{1}{y^2} \cdot \frac{1}{\pi(1+1/y^2)} = \frac{1}{\pi(1+y^2)}$

$f(x) = f(-x), \forall x \in \Re$ joten f on origon suhteen symmetrinen