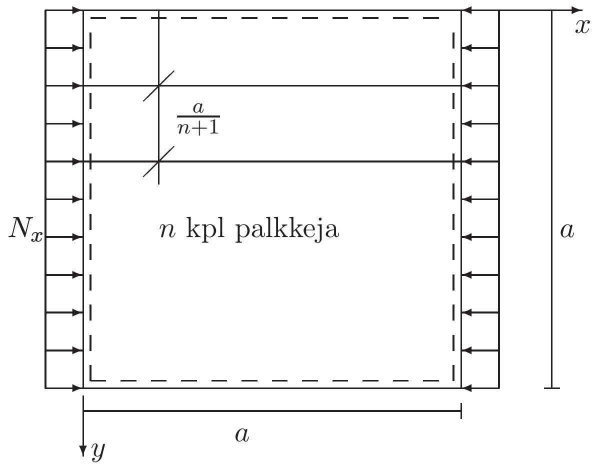
Stability of structures

9. exercise – buckling of plates

Problem 1. A square plate is stiffened by equidistant beams of rectangular cross-section in the loading direction. How many stiffeners are required to obtain a buckling load N_x at least the value $10\frac{\pi^2 D}{a^2}$. Thickness of the plate is h, which is also the width of the beam. The height of the beams is $\alpha h = 4h$. The material is isotropic with Poisson's ratio 0.3. Use the energy method and a one-parametric trial function for the deflection $w(x, y)$. The plate is simply supported and the torsional stiffness of the beams need not to be taken into account. $h = a/40$, where a is the side-length of the plate.



Solution. Let us use the following trial function to the deflection

$$w(x, y) = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

Expression for the total potential energy of the plate is

$$\Delta\Pi = \Delta U + \Delta V = \Delta U_{\text{plate}} + \Delta U_{\text{beams}} + \Delta V_{\text{plate}} + \Delta V_{\text{beams}}$$

$$\Delta U_{\text{plate}} = \frac{D}{2}\int_A (\Delta w)^2 dA$$

$$\Delta w = w_{,xx} + w_{,yy}, \text{ and } w_{,xx} = -w_0\frac{\pi^2}{a^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} = w_{,yy}$$

$$\Rightarrow \Delta U_{\text{plate}} = \frac{D}{2}\frac{\pi^4}{a^2}w_0^2$$

$$\Delta V_{\text{plate}} = -\frac{N_x}{2}\int_A w_{,x}^2 dA = -\frac{N_x}{2}\frac{\pi^2}{4}w_0^2$$

$$\Delta U_{\text{beams}} = \sum_{i=1}^{n}\frac{EI}{2}\int_0^a w_{,xx}^2 dx = \frac{EI}{4}\frac{\pi^4}{a^3}w_0^2\sum\sin^2\frac{\pi i}{n+1}$$

$$
\begin{aligned}
\Delta V_{\text{beams}} &= -\sum_{i=1}^{n} \frac{\sigma_x h \alpha h}{2} \int_0^a w_{,x}^2 dx, \text{ where } \sigma_x h = N_x \\
&= -\frac{N_x}{4} \alpha \frac{h}{a} w_0^2 \pi^2 \sum \sin^2 \frac{\pi i}{n+1} \\
\Rightarrow \Delta\Pi &= \left[\frac{D}{2} \frac{\pi^4}{a^2} - \frac{N_x}{2} \frac{\pi^2}{4} + \frac{EI}{4} \frac{\pi^4}{a^3} \sum \sin^2 \frac{\pi i}{n+1} - \frac{N_x}{4} \alpha \frac{h}{a} \pi^2 \sum \sin^2 \frac{\pi i}{n+1} \right] w_0^2
\end{aligned}
$$

When computing the ΔV_{palkit} term, it is assumed that the load N_x is equally distributed for the cross-sectional area of the beam. Using the notation

$$
N_x = \lambda \frac{\pi^2 E h^3}{12 a^2}, \; I = \frac{\alpha^3 h^4}{12}, \; D = \frac{E h^3}{12}, \text{ when } \nu = 0
$$

In the example case $\alpha = 4$ and $h = a/40$.

$$
\Rightarrow \Delta\Pi = \frac{E h^3}{24} \frac{\pi^4}{a^2} w_0^2 \left[1 + \alpha^3 \frac{h}{2a} \sum_{i=1}^{n} \sin^2 \frac{\pi i}{n+1} - \lambda \left(\frac{1}{4} + \alpha \frac{h}{2a} \sum_{i=1}^{n} \sin^2 \frac{\pi i}{n+1} \right) \right]
$$

The equilibrium equations from the condition $\delta\Pi = 0 \Rightarrow w_0 = 0$, and the critical point is characterized by

$$
\begin{aligned}
\delta^2 \Pi &= 0 \Rightarrow \frac{\partial^2 \Pi}{\partial w_0^2} = 0 \\
\Rightarrow \lambda &= \frac{1 + \alpha^3 \frac{h}{2a} \sum \sin^2 \frac{\pi i}{n+1}}{\frac{1}{4} + \alpha \frac{h}{2a} \sum \sin^2 \frac{\pi i}{n+1}} \geq 10
\end{aligned}
$$

Substituting $\alpha = 4$ ja $h = a/40$ and trying different n's:

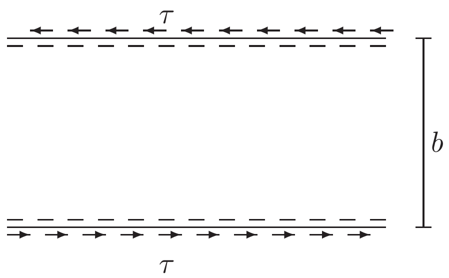
$$
\begin{aligned}
n = 1 \quad &\Rightarrow \quad \sum \sin^2 \frac{\pi i}{n+1} = 1 \Rightarrow \lambda = \frac{1 + \frac{4}{5}}{\frac{1}{4} + \frac{1}{20}} = 6 \\
n = 2 \quad & \quad \sum \sin^2 \frac{\pi i}{n+1} = 2 \cdot \frac{3}{4} = \frac{3}{2} \Rightarrow \lambda = \frac{1 + \frac{4}{5}\frac{3}{2}}{\frac{1}{4} + \frac{1}{20}\frac{3}{2}} \approx 6.8 \\
n = 5 \quad & \quad \sum \sin^2 \frac{\pi i}{n+1} = 3 \Rightarrow \lambda = 8.5 \\
n = 9 \quad & \quad \sum \sin^2 \frac{\pi i}{n+1} = 5 \Rightarrow \lambda = 10
\end{aligned}
$$

Nine stiffeners will be sufficient.

Problem 2. Determine τ_{cr} for an infinite plate strip using a trial function

$$w(x, y) = A \sin(\pi y/b) \sin[\pi(x - \alpha y)/s]$$

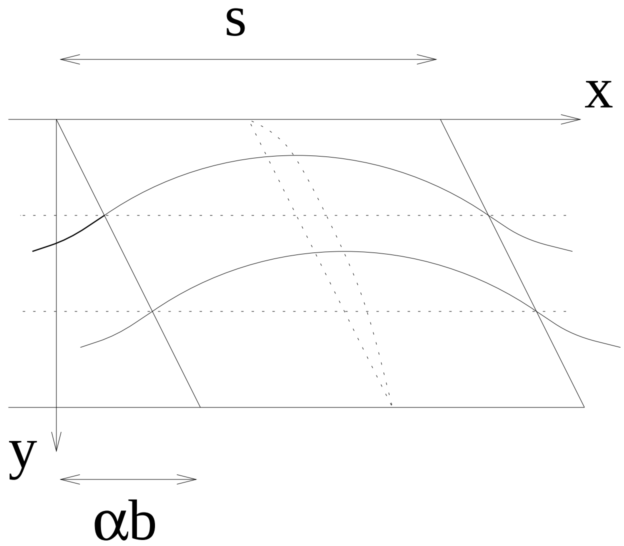
where s is the half wavelength of the buckling mode. The plate is simply supported and it's bending stiffness is D. How large is the error in comparison to the analytical solution $\tau_{\text{cr}} = 5.35\pi^2 D/b^2 t$ (t is the thickness of the plate)?



Solution. Using the trial function

$$w(x, y) = A \sin\frac{\pi y}{b} \sin\frac{\pi}{s}(x - \alpha y)$$

where s is the half wavelength in x-axis direction. Deflection vanish ($w = 0$) at lines $x = \alpha y$ and $x = \alpha y + s$ in addition to the boundaries.



The expression for the total potential energy is

$$\Delta\Pi = \frac{D}{2} \int_A (\Delta w)^2 dA + N_{xy} \int_A w_{,x} w_{,y} dA$$

Let's integrate a slice between the lines $y = 0,\ y = b,\ x = \alpha y$ and $x = \alpha y + s$, i.e. the area of one half-wavelength:

$$
\begin{aligned}
w_{,x} &= A\frac{\pi}{s}\sin\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\\
w_{,xx} &= -A\frac{\pi^2}{s^2}\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y)\\
w_{,y} &= A\frac{\pi}{b}\cos\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y) - A\alpha\frac{\pi}{s}\sin\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\\
w_{,yy} &= -A\frac{\pi^2}{b^2}\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y) - A\alpha\frac{\pi}{b}\frac{\pi}{s}\cos\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\\
&\quad -A\alpha\frac{\pi}{s}\frac{\pi}{b}\cos\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y) - A\alpha^2\frac{\pi^2}{s^2}\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y)\\
\Delta w &= w_{,xx} + w_{,yy} = -A\left[\frac{\pi^2}{s^2} + \frac{\pi^2}{b^2} + \alpha^2\frac{\pi^2}{s^2}\right]\sin\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y)\\
&\quad -2A\alpha\frac{\pi}{b}\frac{\pi}{s}\cos\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\\
w_{,x}w_{,y} &= A^2\frac{\pi}{s}\sin\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\left[\frac{\pi}{b}\cos\frac{\pi y}{b}\sin\frac{\pi}{s}(x-\alpha y) - \alpha\frac{\pi}{s}\sin\frac{\pi y}{b}\cos\frac{\pi}{s}(x-\alpha y)\right]
\end{aligned}
$$

Change of variables

$$
\left\{\begin{array}{c} x = t + \alpha r \\ y = r \end{array}\right. \Rightarrow \frac{\partial(x,y)}{\partial(t,r)} = \begin{bmatrix} x_t & y_t \\ x_r & y_r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}
$$

Since det[$\partial(x,y)/\partial(t,r)$] = 1, the scale is preserved.

$$
\int_0^b \int_{x-\alpha y}^{x-\alpha y+s} (\Delta w)^2 dxdy = \int_0^b\int_0^s (\Delta w)^2 dtdr = A^2\left[\left(\frac{\pi^2}{b^2} + \frac{\pi^2}{s^2} + \alpha\frac{\pi^2}{s^2}\right)^2 + 4\alpha^2\frac{\pi^4}{(bs)^2}\right]\frac{b}{2}\frac{s}{2}
$$

$$
\int_0^b\int_0^s w_{,x}w_{,y}dtdr = -A^2\int\int \alpha\frac{\pi^2}{s^2}\sin^2\frac{\pi r}{b}\cos^2\frac{\pi t}{s}dtdr = -A^2\alpha\frac{\pi^2}{4}\frac{b}{s}
$$

$$
\begin{aligned}
\Rightarrow \frac{\partial^2}{\partial A^2}\Delta\Pi &= 2\frac{D}{2}\left[\left(\frac{\pi^2}{s^2}(1+\alpha^2) + \frac{\pi^2}{b^2}\right)^2\frac{bs}{4} + \alpha^2\frac{\pi^4}{bs}\right] - 2\alpha\frac{\pi^2}{4}\frac{b}{s}N_{xy} = 0\\
&\Rightarrow N_{xy} = \frac{\pi^2 D}{2\alpha b^2}\left[2 + 6\alpha^2 + \frac{s^2}{b^2} + \frac{b^2}{s^2}(1+\alpha^2)^2\right]\\
&\Rightarrow \tau = \frac{\pi^2 D}{2\alpha b^2 t}\left[2 + 6\alpha^2 + \frac{s^2}{b^2} + \frac{b^2}{s^2}(1+\alpha^2)^2\right]
\end{aligned}
$$

The expression of the shear stress still contains two free parameters α and s. The minimum is obtained when τ is minimized with respect to these two paramaters:

$$
\tau = \frac{\pi^2 D}{2b^2 t}\left[\frac{2}{\alpha} + 6\alpha + \frac{s^2}{b^2\alpha} + \frac{b^2}{s^2}\frac{(1+\alpha^2)^2}{\alpha}\right] = \frac{\pi^2 D}{2b^2 t}f(\alpha, s)
$$

$$
\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{2s}{\alpha b^2} + \frac{(1+\alpha^2)^2}{\alpha}\frac{(-2b^2)}{s^3} = 0 \Rightarrow \frac{s}{b} = \sqrt{1+\alpha^2} \\
&\Rightarrow \tilde{f} = \frac{2}{\alpha} + 6\alpha + 2\frac{1+\alpha}{\alpha} \\
\frac{\partial \tilde{f}}{\partial \alpha} &= -\frac{2}{\alpha^2} + 6 + 2\frac{2\alpha^2 - (1+\alpha^2)}{\alpha^2} = 0 \Rightarrow \alpha = \pm\frac{1}{\sqrt{2}} \Rightarrow \frac{s}{b} = \sqrt{\frac{3}{2}} \\
&\Rightarrow \tau_{\text{cr}} = 4\sqrt{2}\frac{\pi^2 D}{b^2 t} \approx 5.66\frac{\pi^2 D}{b^2 t}
\end{aligned}
$$

The difference to the analytical value 5.35 $\frac{\pi^2 D}{b^2 t}$, is thus 5.8 %.