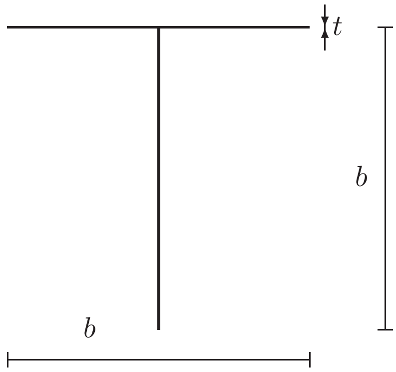
Stability of structures

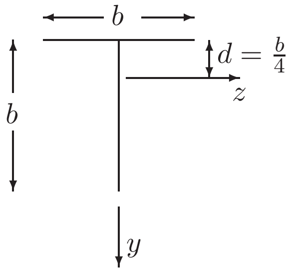
6. exercise – torsional and lateral buckling

Problem 1. Determine the critical load P_{cr} for a centrally compressed clamped beam. The cross-section is shown below and $b = 10t$, $\nu = 0$. Determine the critical load as a function of the length.



Solution. The differential equations for torsional buckling for a column are

$$\begin{cases} EI_z v^{(4)} + P(v'' + z_v\varphi'') = 0 \\ EI_y w^{(4)} + P(w'' - y_v\varphi'') = 0 \\ EI_\omega \varphi^{(4)} - GI_t\varphi'' + P(z_v v'' - y_v w'' + r^2\varphi'') = 0 \end{cases}$$



For a T-beam we have

$$\begin{aligned} &z_v = 0, \quad EI_\omega = 0 \\ &y_v = -b/4 \\ &I_y \approx \frac{b^3 t}{12},\ I_z = \frac{5}{24}b^3 t \\ &I_t \approx \frac{2}{3}t^3 b,\ r^2 = \frac{I_p}{A} + y_v^2 + z_v^2 = \frac{5}{24}b^2 \end{aligned}$$

The equations simplify now to the form

$$\begin{aligned} &EI_z v^{(4)} + Pv'' = 0 \\ &EI_y w^{(4)} + P[w'' - y_v\varphi''] = 0 \\ &-GI_t\varphi'' + P[-y_v w'' + r^2\varphi''] = 0 \end{aligned}$$

The upper equation, i.e. the displacement in y-direction uncouples from the displacement in the z-direction and from the twist-rotation, thus the buckling in $y-$direction gives the load

$$P_y = 4\pi^2\frac{EI_z}{L^2}$$

Function which satisfy the boundary conditions are

$$\begin{aligned} w &= B\left(1 - \cos 2\frac{n\pi x}{L}\right) \\ \varphi &= C\left(1 - \cos 2\frac{n\pi x}{L}\right) \end{aligned}$$

Let's denote $P = \lambda G I_t r^{-2}$

$$\frac{GI_t}{r^2}\begin{bmatrix} \alpha - \lambda & y_v\lambda \\ y_v\lambda & r^2(1-\lambda) \end{bmatrix}\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\alpha = 4\pi^2(r/L)^2 EI_y/GI_t$. In order to have a non-trivial solution for A, B the determinant has to vanish. The critical value for the λ parameter can be found by solving the characteristic polynomial

$$\left[1 - \left(\frac{y_v}{r}\right)^2\right]\lambda^2 - (1+\alpha)\lambda + \alpha = 0$$

If we denote $I_y = I$, then $I_z = \frac{5}{2}I$ and $I_t = \frac{2}{25}I$. If $\nu = 0$ then $G = E/2$ and $GI_t = \frac{1}{25}EI$. Also $(y_v/r)^2 = \frac{3}{10}$, thus the characteristic polynomial has the form

$$\lambda^2 - \frac{10}{7}(1+\alpha)\lambda + \frac{10}{7}\alpha = 0$$

where $\alpha = \frac{125}{6}\pi^2(b/L)^2$. The smaller toot is

$$\lambda_1 = \tfrac{5}{7}(1+\alpha)\left(1 - \sqrt{1 - \frac{14\alpha}{5(1+\alpha)^2}}\right)$$

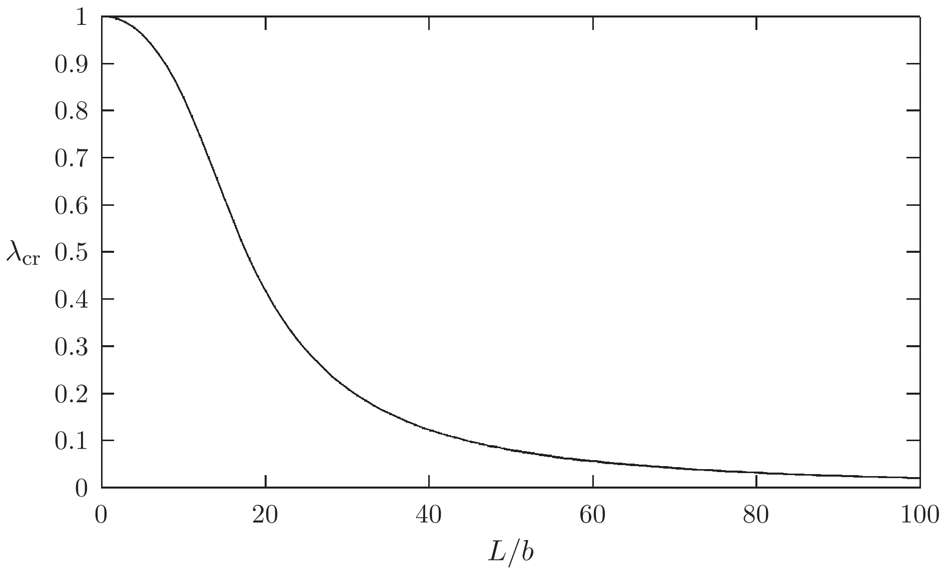
Note, that $\lambda_1 \leq 1$. The buckling load is now the minimum from

$$P_y = 4\pi^2\frac{EI_z}{L^2} = 250\pi^2\left(\frac{r}{L}\right)^2\frac{GI_t}{r^2} = \frac{625}{12}\pi^2\left(\frac{b}{L}\right)^2\frac{GI_t}{r^2} \qquad P_{z,\phi,1} = \lambda_1\frac{GI_t}{r^2}$$

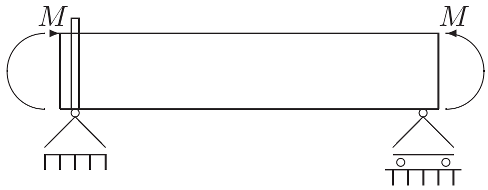
Note that, if the torsional mode is prevented the buckling load in $z-$direction is

$$P_z = 4\pi^2\frac{EI_y}{L^2} = \alpha\frac{GI_t}{r^2} = \frac{125}{6}\pi^2\left(\frac{b}{L}\right)^2\frac{GI_t}{r^2} = \frac{2}{5}P_y > P_{z,\phi,1}$$

The critical load parameter $\lambda_{\text{cr}} = \lambda_1$ is shown below as a function of the slenderness (L/b)



Problem 2. Determine the critical lateral buckling moment M_{cr} for the beam shown below. The support on the rhs side prevents vertical and lateral displacements but the cross-section can rotate about the support. The cross-section is rectangular with dimensions $b \times h$ where $h \gg b$.



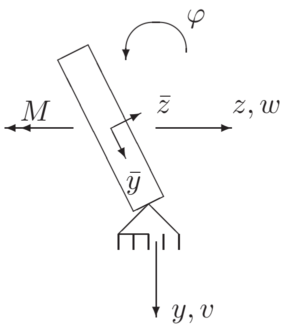
Solution. The differential equations takes now the form

$$\left\{ \begin{array}{rcl} EI_y w^{(4)} - M_z^0 \varphi'' & = & 0 \\ -GI_t \varphi'' - M_z^0 w'' & = & 0 \end{array} \right. \tag{1}$$

Boundary conditions on the lhs support

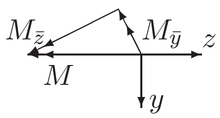
$$w(0) = 0,\ w''(0) = 0,\ \varphi(0) = 0$$

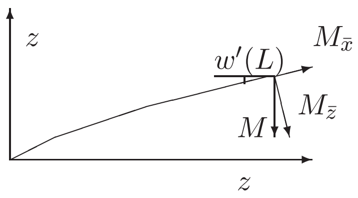
The rhs boundary conditions are slightly more complicated



The kinematical constraint at the center of gravity of the cross-section is

$$w(L) = -\frac{h}{2}\varphi(L)$$





Let's divide the external moment M into componets parallel to the deformed coordinate axis

$$\begin{array}{rcl} M_{\bar{z}} & \approx & M \\ M_{\bar{y}} & = & EI_y w'' \approx -\varphi(L) M \\ M_{\bar{x}} & = & -w'(L) M \end{array}$$

The boundary conditions are

$$\begin{array}{rl} w(0) = 0 & w(L) = -\dfrac{h}{2}\varphi(L) \\ w''(0) = 0 & -EI_y w''(L) = -\varphi(L) M \\ \varphi(0) = 0 & GI_t \varphi'(L) = -w'(L) M \end{array}$$

Substituting equation (1_2) into equation (1_1) saadaan

$$w^{(4)} + k^2 w'' \ = \ 0,\ k^2 = \frac{M^2}{EI_y GI_t}$$

$$\Rightarrow w = A\sin kx + B\cos kx + Cx + D$$

From boundary conditions we get

$$w(0) = w''(0) = 0 \;\Rightarrow\; D = B = 0$$
$$\Rightarrow w = A\sin kx + Cx$$

From the differential equation (1_2) we can deduce that φ is of similar form

$$\Rightarrow \varphi = E\sin kx + Fx$$
$$\Rightarrow -GI_t k^2 E\sin kx - Mk^2 A\sin kx = 0$$
$$\Rightarrow E = -\tfrac{M}{GI_t}A$$

Let's substitute the boundary conditions into these trial functions

$$\begin{aligned}
GI_t\varphi'(L) = -w'(L)M \quad &\Rightarrow \quad -GI_t(k\frac{M}{GI_t}A\cos kx - F) = -(Ak\cos kx + C)M \\
&\Rightarrow F = -\frac{M}{GI_t}C \\
w(L) = -\frac{h}{2}\varphi(L) \quad &\Rightarrow \quad A\sin kL + CL = \frac{h}{2}\frac{M}{GI_t}(A\sin kL + CL) \\
&\Rightarrow \left(1 - \frac{Mh}{2GI_t}\right)A\sin kL + \left(1 - \frac{Mh}{2GI_t}\right)CL = 0 \\
-EI_y w''(L) = -\varphi(L)M \quad &\Rightarrow \quad EI_y k^2 A\sin kL = \frac{M}{GI_t}(A\sin kL + CL) \\
&\Rightarrow \left(EI_y k^2 - \frac{M^2}{GI_t}\right)A\sin kL + -\frac{M}{GI_t}C = 0
\end{aligned}$$

Since $k^2 = M^2/EI_yGI_t$ it follows from equation (2) $-(M/GI_t)C = 0 \Rightarrow C = 0$. From equation (2) we obtain

$$\left[\left(1 - \frac{Mh}{2GI_t}\right)\sin kL\right] = 0$$

The critical moment is then

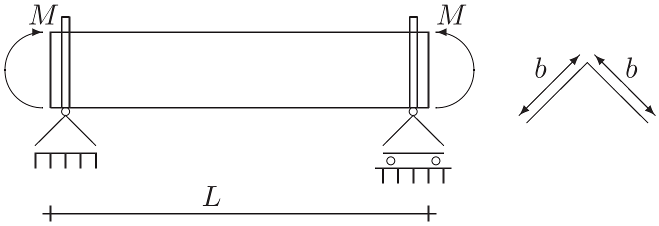
$$M_{\text{cr}} = \min\left\{\frac{2GI_t}{h}\;,\;\pi\frac{\sqrt{EI_yGI_t}}{L}\right\}$$

The eigenmodes are

$$\begin{aligned}
w(x) &= A\sin kx \\
\varphi(x) &= -\frac{M}{GI_t}A\sin kx
\end{aligned}$$

Note! if $M_{\text{cr}} = \frac{2GI_t}{h} \Rightarrow kL \neq \pi \Rightarrow w(L), \varphi(L) \neq 0$. If $M_{\text{cr}} = \frac{2GI_t}{h} \Rightarrow kL = \pi$.

Problem 3. Determine the critical moment M_{cr} for the beam shown below, the proportions are $b = 10t, L = 20b, \nu = 1/3$. What is the result if M is negative?



Solution. The cross-sectional constants are

$$I_t = \frac{2}{3}t^3 b,\ I_y = \frac{1}{3}tb^3,\ I_z = \frac{1}{12}tb^3,\ y_v = -\frac{\sqrt{2}}{4}b,\ z_v = 0$$

$$\beta_z = \frac{1}{I_z}\int y(y^2+z^2)dA - 2y_v,\ \int y^3 dA = 0,\ \int yz^2 dA = 2\frac{bt}{6}\frac{\sqrt{2}}{4}b\frac{1}{2}b^2 = \frac{\sqrt{2}}{24}tb^4 \Rightarrow \beta_z = \sqrt{2}b$$

The differential equations for the lateral/torsional buckling are

$$\begin{cases} EI_y w^{(4)} - M\varphi'' = 0 \\ -GI_t\varphi'' - Mw'' - \beta_z M\varphi'' = 0 \quad \Rightarrow \quad \varphi'' = -\dfrac{M}{GI_t + \beta_z M}w'' \end{cases}$$

$$w^{(4)} + \frac{M^2}{EI_y(GI_t + \beta_z M)}w'' = 0$$

The general solution is

$$w = A\sin kx + B\cos kx + Cx + D \qquad \text{where} \qquad k^2 = \frac{M^2}{EI_y(GI_t+\beta_z M)}$$

The boundary conditions are

$$\begin{aligned} w(0) &= 0 \ \Rightarrow \ B + D = 0 \\ w''(0) &= 0 \qquad B = 0 \\ w(L) &= 0 \qquad A\sin kL + CL = 0 \\ w''(L) &= 0 \qquad Ak^2 \sin kL = 0 \quad \Rightarrow \quad kL = n\pi, \end{aligned}$$

The lowest buckling load is obtained when $n = 1$, hence

$$M^2 - \beta_z\frac{\pi^2}{L^2}EI_y M - EI_y GI_t\frac{\pi^2}{L^2} = 0$$

denoting $M = \lambda\sqrt{EI_y GI_t}/L$ and $EI_y = \alpha^2 GI_t$

$$\lambda^2 - \pi^2\alpha\frac{\beta_z}{L} - \lambda - \pi^2 = 0$$

The roots are

$$\lambda = \frac{\pi^2}{2}\alpha\left(\frac{\beta_z}{L}\right)\left(1 \pm \sqrt{1 + \frac{4}{\pi^2\alpha^2}\left(\frac{L}{\beta_z}\right)^2}\right)$$

Substituting $\beta_z = \sqrt{2}\,b,\ L = 20\,b,\ \alpha^2 = 4000/3$, gives the result

$$\lambda = 2.62\pi^2 \quad \vee \quad \lambda = -0.04\pi^2$$

Let's check if the expression for k^2 is positive for negative λ values, i.e. if it holds $GI_t + \beta_z M > 0$.

$$GI_t + \beta_z\lambda\frac{\sqrt{EI_y GI_t}}{L} = GI_t\left(1 + 2\frac{\sqrt{5}}{\sqrt{3}}\lambda\right) = -0.02$$

Therefore the trial function for w is wrong for a negative moment. In this case

$$\begin{aligned} &w'''' - k^2 w'' = 0,\ \text{where } k^2 = -\frac{M^2}{EI_y(GI_t+\beta_z M)} \\ &w(x) = A\sinh kx + B\cosh kx + Cx + D \end{aligned}$$

From boundary conditions we get $B = D = 0$ and

$$\begin{pmatrix} A\sinh kL + CL = 0 \\ Ak^2\sinh kL = 0 \end{pmatrix} \Rightarrow A = C = 0 \vee k = 0$$

Since $k \neq 0$ the beam does not buckle laterally. However, the flanges can buckle in a plate-like mode.