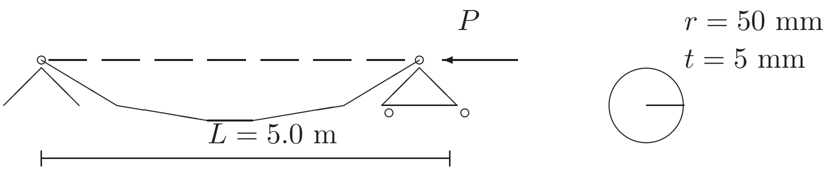
Stability of structures

5. exercise – beam-columns, inelastic buckling

Problem 1. A beam with circular cross-section has an initial deflection $v_0(x) = v_0 \sin(\pi x/L)$. What is the safety factor with respect to the yield limit if the compressive load has the value $P = 50$ kN? The yield stress is $\sigma_y = 220$ MPa and the Young's modulus $E = 210$ GPa. The amplitude of the initial deflection is $v_0 = L/1000$.



Solution. The bending moment distribution due to the compressive force is

$$M(x) + P[v(x) + v_0(x)] = 0$$

$$\Rightarrow v''(x) + k^2 v(x) = -k^2 v_0(x), \text{ where } k^2 = \frac{P}{EI} \tag{1}$$

Let's find the particular solution of the differential equation above.

$$v_y(x) = A \sin\left(\frac{\pi x}{L}\right)$$

Substituting the trial function above into equation 1

$$\Rightarrow \quad \left(-\frac{\pi^2}{L^2} + k^2\right) A \sin\frac{\pi x}{L} = -k^2 v_0 \sin\frac{\pi x}{L}$$

$$\Rightarrow \quad A = -\frac{k^2 v_0}{k^2 - \frac{\pi^2}{L^2}}$$

The solution is the sum of the general solution of the homogeneous equation and the particular solution

$$v(x) = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 - \frac{k^2 v_0}{k^2 - \frac{\pi^2}{L^2}} \sin\frac{\pi x}{L}$$

Boundary conditions:

$$\begin{aligned} v(0) &= C_2 + C_4 = 0 & \\ v''(0) &= k^2 C_2 = 0 & \Rightarrow C_2 = 0 \Rightarrow C_4 = 0 \\ v''(L) &= k^2 C_1 \sin kL = 0 & \Rightarrow C_1 = 0(*) \\ v(L) &= C_3 L = 0 & \end{aligned}$$

At $(*)$ the solution $kL = n\pi$ is not valid, since the equation must hold on for all values of k:

$$\Rightarrow v(x) = -\frac{k^2 v_0}{k^2 - \frac{\pi^2}{L^2}} \sin\frac{\pi x}{L},$$

and the bending moment has the expression

$$M(x) = -EIv''(x) = -\frac{EIk^2 v_0 \pi^2}{k^2 L^2 - \pi^2} \sin\frac{\pi x}{L}$$

The largest bending moment is at the middle ($k^2 = P/EI$):

$$M\left(\frac{L}{2}\right) = -\frac{P v_0 \pi^2}{\frac{PL^2}{EI} - \pi^2}$$

The buckling load for an ideal straight column is $P_E = \pi^2 EI/L^2$, the bending moment can be expressed as

$$M\left(\frac{L}{2}\right) = -\frac{Pv_0}{\frac{P}{P_E} - 1}$$

The bending moment $M(L/2)$ approaches to infinity when $P \to P_E$! The stresses at the middle of the beam in the outmost fibers are

$$\sigma = -\frac{P}{A} \pm \frac{M}{W} = -P\left(\frac{1}{A} \pm \frac{1}{\frac{P}{P_E} - 1}\frac{v_0}{W}\right) \tag{2}$$

Taking the cross-section dimensions into account

$$\begin{array}{llll} A & = \pi(50^2 - 45^2) & = 1492\text{mm}^2 \\ I & = \frac{\pi}{4}(50^4 - 45^4) & = 1.688 \cdot 10^6\text{mm}^4 \\ W & = \frac{1}{50\,\text{mm}^2} I & = 33760\text{mm}^3 \\ k^2 & = 1.41 \cdot 10^{-3} & , \text{when } P = 50\,\text{kN} \end{array}$$

From equation 2 we get

$$\sigma = -33.5 \pm 15.4 \text{ MPa}$$

Let's solve the compressive force value P, when the outmost fibers at the mid-section attains the yield point σ_y. From the equation 2 we get

$$\begin{array}{lll} & \sigma_y = -P\left(\frac{1}{A} + \frac{1}{\frac{P}{P_E}-1}\frac{v_0}{W}\right) & \\ \Rightarrow & \left(\frac{P}{P_E} - 1\right)\left(\sigma_y - \frac{P}{A}\right) - \frac{v_0}{W}P & = 0 \\ \Rightarrow & P^2 - \left(\sigma_m A + P_E + \frac{P_E A v_0}{W}\right)P + \sigma_y P_E A & = 0 \end{array}$$

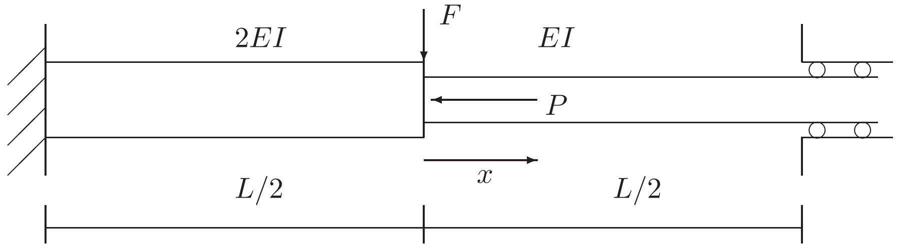
Substituting the dimensions, gives

$$P^2 - 509.5P + 45945 = 0 \Rightarrow \begin{cases} P_1 = & 117.1 \text{ kN} \\ P_2 = & 392.4 \text{ kN} \end{cases}$$

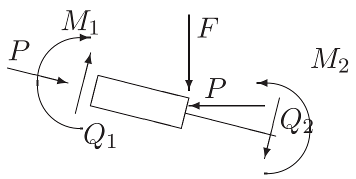
Safety factor with respect to the yield is thus

$$n = \frac{117.1}{50} = 2.34$$

Problem 2. Determine the bending moment distribution at the load levels $P/P_E = 0.25$, 0.50 and 0.75, where P_E is the critical load of the buckling problem. Determine also the expressions of the support moments at both ends and the bending moment in the midspan as a function of the compressive force.



Osalla 1 $v_1^{(4)} + k^2 v_1'' = 0$, missä $k^2 = \frac{P}{2EI}$
2 $v_2^{(4)} = 0$



$$\begin{aligned}
\text{BC}: \quad & v_1\left(-\tfrac{L}{2}\right) = v_1'\left(-\tfrac{L}{2}\right) = 0 \\
& v_2\left(\tfrac{L}{2}\right) = v_2'\left(\tfrac{L}{2}\right) = 0 \\
& v_1(0) = v_2(0) \\
& v_1'(0) = v_2'(0) \\
& M_1(0) = M_2(0) \\
& Q_1(0) = Q_2(0) + Pv_2'(0) + F
\end{aligned}$$

Solution for the homogeneous differential equations are:

$$\begin{aligned}
v_1 &= C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 \\
v_1' &= C_1 k \cos kx - C_2 k \sin kx + C_3 \\
v_1'' &= -C_1 k^2 \sin kx - C_2 k^2 \cos kx \\
v_1''' &= -C_1 k^3 \cos kx + C_2 k^3 \sin kx \\
v_2 &= C_5 x^3 + C_6 x^2 + C_7 x + C_8 \\
v_2' &= 3C_5 x^2 + 2C_6 x + C_7 \\
v_2'' &= 6C_5 x + 2C_6 \\
v_2''' &= 6C_5
\end{aligned}$$

Taking the boundary conditions into account

$$\begin{aligned}
Q_1(0) &= Q_2(0) + Pv_2'(0) \\
-2EIv_1'''(0) &= -EIv_2'''(0) + Pv_2'(0) + F \\
2C_1 k^3 &= -6C_5 + 2k^2 C_7 + \frac{F}{EI} \\
C_5 &= -\frac{1}{3}k^3 C_1 + \frac{1}{3}k^2 C_7 + \frac{F}{6EI}
\end{aligned}$$

$$\begin{aligned}
M_1(0) &= M_2(0) \\
-2EIv_1''(0) &= -EIv_2''(0) \\
2C_2 k^2 &= -2C_6 \;\Rightarrow\; C_6 = -k^2 C_2
\end{aligned}$$

$$\begin{aligned}
v_1'(0) &= v_2'(0) \\
C_1 k + C_3 &= C_7 \Rightarrow C_5 = -\frac{1}{3}k^3 C_1 + \frac{1}{3}k^2 (C_1 k + C_3) + \frac{F}{6EI} = \frac{1}{3}k^2 C_3 + \frac{F}{6EI}
\end{aligned}$$

$$
\begin{aligned}
v_1(0) &= v_2(0)\\
C_2 + C_4 &= C_8\\
v_1\left(-\frac{L}{2}\right) &= 0 \Rightarrow C_4 = C_1 \sin\frac{kL}{2} - C_2\cos\frac{kL}{2} + C_3\frac{L}{2}\\
v_1'\left(-\frac{L}{2}\right) &= 0 \Rightarrow C_3 = -k(C_1\cos\frac{kL}{2} + C_2\sin\frac{kL}{2})\\
v_2\left(\frac{L}{2}\right) &= 0 \Rightarrow \left(\frac{1}{3}k^2C_3 + \frac{F}{6EI}\right)\left(\frac{L}{2}\right)^3 - k^2C_2\left(\frac{L}{2}\right)^2 + (C_1k + C_3)\frac{L}{2} + C_2 + C_4 = 0\\
&\Rightarrow \left[\frac{kL}{2} - kL\cos\frac{kL}{2}\left(1+\frac{1}{24}(kL)^2\right) + \sin\frac{kL}{2}\right]C_1\\
&\quad + \left[1 - \frac{1}{4}(kL)^2 - \cos\frac{kL}{2} - kL\sin\frac{kL}{2}\left(1+\frac{1}{24}(kL)^2\right)\right]C_2 = -\frac{FL^3}{48EI}\\
v_2'\left(\frac{L}{2}\right) &= 0 \Rightarrow \left(k^2C_3 + \frac{F}{2EI}\right)\left(\frac{L}{2}\right)^2 - 2k^2C_2\frac{L}{2} + C_1k + C_3 = 0\\
&\Rightarrow \left[1 - \left(1+\frac{1}{4}(kL)^2\right)\cos\frac{kL}{2}\right]kC_1 + \left[-kL - \left(1+\frac{1}{4}(kL)^2\right)\sin\frac{kL}{2}\right]kC_2 = -\frac{FL^2}{8EI}
\end{aligned}
$$

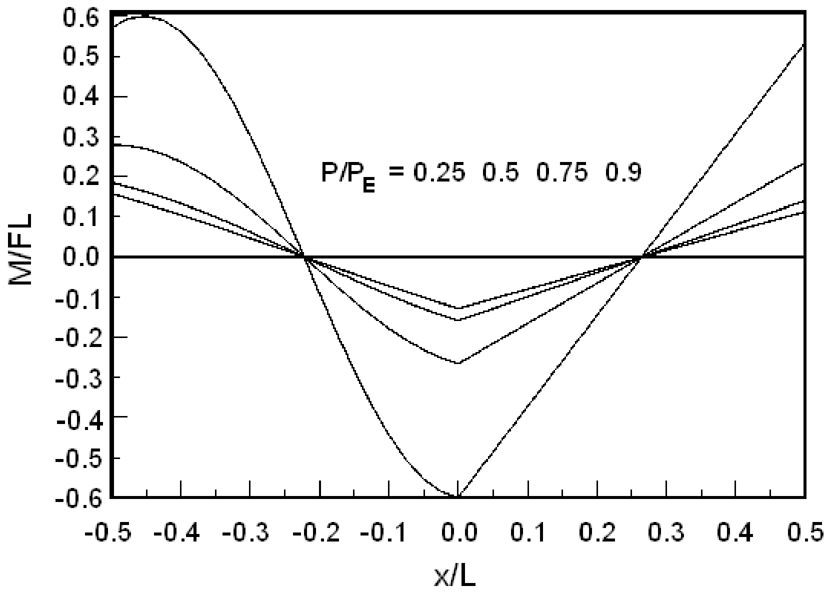
The expressions for the bending moments are

$$
\begin{cases}
M_1(x) = 2EIk^2(C_1\sin kx + C_2\cos kx) & \text{when } -\frac{L}{2} \le x \le 0\\
M_2(x) = -EI(6C_5x + 2C_6) & \text{when } 0 < x \le \frac{L}{2}
\end{cases}
$$

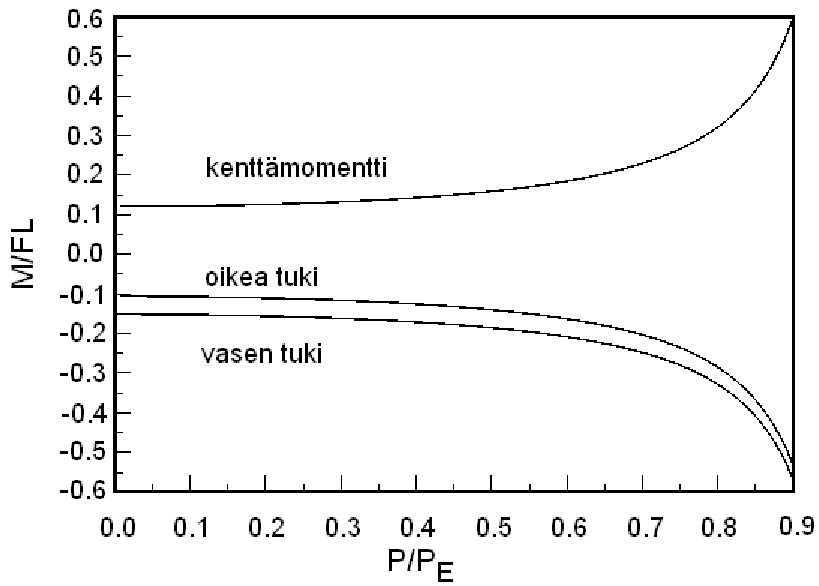
The coefficients C_1 and C_2 can be solved from the equation system below

$$
\left\{
\begin{aligned}
[\cdot]\,C_1 + [\cdot]\,C_2 &= -\tfrac{FL^3}{48EI}\\
[\cdot]\,kC_1 + [\cdot]\,kC_2 &= -\tfrac{FL^2}{8EI}
\end{aligned}
\right\}
$$

The coefficients C_5 and C_6 have already been solved as a functions of C_1 and C_2.

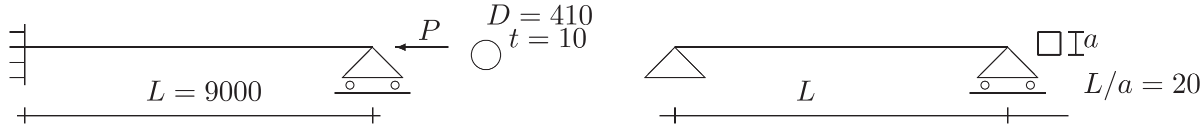


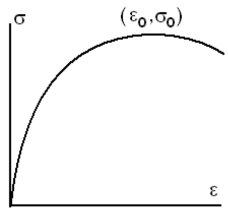
P/P_E = 0.25 0.5 0.75 0.9
M/FL
x/L



kenttämomentti
oikea tuki
vasen tuki
M/FL
P/P_E

Problem 3. The buckling length of a uniform straight column is L_n. The stress-strain curve of the material is quadratic ($\sigma = A\varepsilon^2 + B\varepsilon + C$), which has an apex at $\sigma_0 = 392$ MPa, $\varepsilon_0 = 0.002$. Determine the expression for the tangent modulus $E_\mathrm{t}(\sigma)$ and show that the critical load according to the tangent modulus theory is $P_\mathrm{cr} = 2\sigma_0 A(\sqrt{K+1})/K$, where $K = (\varepsilon_0 L_n^2 A/\pi^2 I)^2$.
Calculate the value of the critical load for the two columns shown below. Measures shown in mm.





Solution. The stress-strain relationship is

$$\sigma = \sigma(\epsilon) = A\epsilon^2 + B\epsilon + C$$

- when $\epsilon = 0 \Rightarrow \sigma = 0 \Rightarrow C = 0$
- when $\epsilon = \epsilon_0 \Rightarrow \sigma = \sigma_0$
- when $\epsilon = \epsilon_0,\ \frac{d\sigma}{d\epsilon} = 0$

$$\left.\frac{d\sigma}{d\epsilon}\right|_{\epsilon=\epsilon_0} = 2A\epsilon_0 + B = 0 \Rightarrow B = -2A\epsilon_0$$

$$\sigma_0 = A\epsilon_0^2 + B\epsilon_0 = -A\epsilon_0^2 \Rightarrow A = -\frac{\sigma_0}{\epsilon_0^2}$$

$$\Rightarrow \sigma = -\frac{\sigma_0}{\epsilon_0^2}\epsilon^2 + 2\frac{\sigma_0}{\epsilon_0}\epsilon \Rightarrow \frac{d\sigma}{d\epsilon} = E_t = 2\frac{\sigma_0}{\epsilon_0}\left(1 - \frac{\epsilon}{\epsilon_0}\right)$$

Let's denote

$$\left.\frac{d\sigma}{d\epsilon}\right|_{\epsilon=0} = 2\frac{\sigma_0}{\epsilon_0} = E \Rightarrow E_t = E\left(1 - \frac{\epsilon}{\epsilon_0}\right)$$

Solving $\epsilon = \epsilon(\sigma)$

$$\begin{aligned}\frac{\sigma}{\sigma_0} &= -\left(\frac{\epsilon}{\epsilon_0}\right)^2 + 2\frac{\epsilon}{\epsilon_0} \Rightarrow \frac{\epsilon}{\epsilon_0} = 1 \pm \sqrt{1 - \frac{\sigma}{\sigma_0}} \\ \Rightarrow E_t &= E\sqrt{1 - \frac{\sigma}{\sigma_0}}\ , \text{ when } \epsilon < \epsilon_0\end{aligned}$$

The critical load according to the tangent modulus theory

$$\begin{aligned}P_\mathrm{cr} &= \frac{\pi^2 E_t I}{L_n^2}\ ,\ \sigma_{kr} = \frac{P_\mathrm{cr}}{A},\ \text{merk. } \alpha = \frac{\pi^2 I}{L_n^2} \\ \Rightarrow P_{kr} &= -\frac{\alpha^2 E^2}{2\sigma_0 A} \pm \sqrt{\frac{\alpha^4 E^4}{4\sigma_0^2 A^2} + \alpha^2 E^2} = \frac{\alpha^2 E^2}{2\sigma_0 A}\left(\sqrt{1 + \frac{4\sigma_0^2 A^2}{\alpha^2 E^2}} - 1\right) \\ &= \frac{2\sigma_0 A}{K}\left(\sqrt{1+K} - 1\right),\ K = \left(\frac{\epsilon_0 L_n^2 A}{\pi^2 I}\right)^2\end{aligned}$$

In the example cases

1. $A \approx \pi(D-t)t = 1.257 \cdot 10^4 \text{ mm}^2,\ I \approx \pi/8(D-t)^3 t = 2.513 \cdot 10^8 \text{ mm}^4,\ L = 9.0 \text{ m} \Rightarrow L_n = 0.699L = 6.291 \text{ m} \Rightarrow K = 0.1608 \Rightarrow P_\mathrm{cr} = 4.742$ kN
2. solid cross-section $A = a^2,\ I = 1/12\, a^4,\ L = 20a \Rightarrow K = 0.947 \Rightarrow \sigma_\mathrm{cr} = P_\mathrm{cr}/A = 327$ MPa

Problem 4. Determine the dependence of the critical stress σ_{cr} on the slenderness $\lambda = L_n/i$ (where L_n is the buckling length and $i = \sqrt{I/A}$ is the radius of gyration of the cross-section) for a uniform centrally compressed straight column. The tangent modulus E_{t} has the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon} = E_{\text{t}} = E\frac{\sigma_{\text{y}} - \sigma}{\sigma_{\text{y}} - c\sigma},$$

where σ_{y} is the yield stress and c is an additional material constant. Draw the figure showing the critical buckling stress as a function of the slenderness in a $(\sigma_{\text{cr}}/\sigma_{\text{y}}) - \lambda$ -coordinate system with $(\sigma_{\text{cr}}/\sigma_{\text{y}}) \in [0, 1], \lambda \in [0, 200]$ Use the value $c = 0,9$ and ratios $E/\sigma_{\text{y}} = 500$ (steel) and $E/\sigma_{\text{y}} = 200$ (aluminium, pinewood). Draw also in the same figure the elastic buckling stress.

Solution. According to the tangent modulus theory the critical load is obtained from

$$P_{\text{cr}} = \frac{\pi^2 E_t I}{L_n^2},$$

where

$$E_t = E\frac{\sigma_y - \sigma}{\sigma_y - c\sigma} \Rightarrow \sigma_{\text{cr}} = \frac{\pi^2 E \frac{\sigma_y - \sigma}{\sigma_y - c\sigma} I}{L_n^2 A}$$

Using notations $i = \sqrt{I/A},\ \lambda = L_n/i$

$$\begin{aligned}
\sigma_{\text{cr}} &= \frac{\pi^2 E i^2}{L_n^2}\frac{\sigma_y - \sigma_{\text{cr}}}{\sigma_y - c\sigma_{\text{cr}}} \\
&\Rightarrow \lambda^2\sigma_{\text{cr}} = \pi^2 E\frac{\sigma_y - \sigma_{\text{cr}}}{\sigma_y - c\sigma_{\text{cr}}} \\
&\Rightarrow \lambda^2\frac{\sigma_{\text{cr}}}{\sigma_y}\left(1 - c\frac{\sigma_{\text{cr}}}{\sigma_y}\right) = \pi^2\frac{E}{\sigma_y}\left(1 - \frac{\sigma_{\text{cr}}}{\sigma_y}\right) \\
&\Rightarrow \lambda^2 c\left(\frac{\sigma_{\text{cr}}}{\sigma_y}\right)^2 - \left(\lambda^2 + \pi^2\frac{E}{\sigma_y}\right)\left(\frac{\sigma_{\text{cr}}}{\sigma_y}\right) + \pi^2\frac{E}{\sigma_y} = 0 \\
&\Rightarrow \frac{\sigma_{\text{cr}}}{\sigma_y} = \frac{1}{2\lambda^2 c}\left(\lambda^2 + \pi^2\frac{E}{\sigma_y} - \sqrt{\left(\lambda^2 + \pi^2\frac{E}{\sigma_y}\right)^2 - 4\lambda^2 c\pi^2\frac{E}{\sigma_y}}\right)
\end{aligned}$$

Let's draw the figure using the values $c = 0.9$ and $E/\sigma_y = 500$ and 200. The elastic critical load is

$$P_{\text{cr}} = \frac{\pi^2 EI}{L_n^2} \Rightarrow \left(\frac{\sigma_{\text{cr}}}{\sigma_y}\right) = \left(\frac{E}{\sigma_y}\right)\frac{\pi^2}{\lambda^2}$$