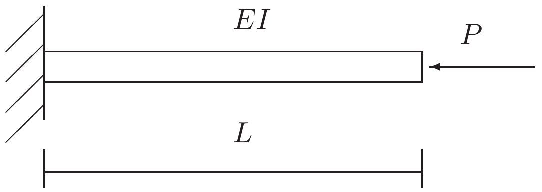
Stability of structures

4. exercise – continuous systems, column buckling

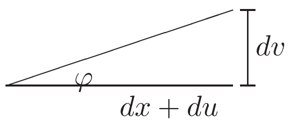
Problem 1. Derive the Euler equations of the cantilever beam shown below. Assume inextensible beam and small deflections. Solve the equations and determine the eigenmodes and show that the eigenmodes are orthogonal.



Solution. The total potential energy functional is

$$\Pi(v) = \frac{1}{2}\int_0^L \left[EI(v'')^2 - P(v')^2\right] dx,$$

where the horizontal deflection Δ under the load P can be determined as



$$\begin{aligned}
\varphi &= v' \ (v' \text{ small}) \\
du &= dx(1-\cos\varphi) \\
&\approx dx\left[1-\left(1-\frac{1}{2}(v')^2\right)\right] = \frac{1}{2}(v')^2 dx \\
\Delta &= \int du = \frac{1}{2}\int (v')^2 dx
\end{aligned}$$

The Euler equations are obtained from the stationarity condition of the functional

$$\delta\Pi = \Pi_{,v}\delta v = \int_0^L (EIv''\delta v'' - Pv'\delta v')dx = 0,$$

where δv is the variation of the deflection, i.e. an arbitrary function satisfying the homogenious kinematical boundary conditions $v(0) = v'(0) = 0$. After integration by parts we get the term δv as a common factor inside the integral

$$\begin{aligned}
\delta\Pi &= \Big|_0^L EIv''\delta v' - \int_0^L (EIv'')'\delta v' dx - \Big|_0^L Pv'\delta v + \int_0^L \delta v dx \\
&= \Big|_0^L EIv''\delta v' - \Big|_0^L (EIv'')'\delta v - \Big|_0^L Pv'\delta v + \int_0^L \left[(EIv'')'' + (Pv')'\right]\delta v dx
\end{aligned}$$

At the lower limit $\delta v(0) = \delta v'(0) = 0$, and taking into account the definitions of the moment and shear force: $M = -EIv''$ sekä $Q = -(EIv'')'$, we get

$$\delta\Pi = -M(L)\delta v'(L) + [Q(L) - Pv'(L)]\delta v(L) + \int_0^L \left[(EIv'')'' + (Pv')'\right]\delta v dx = 0,$$

since δv is arbitrary function satisfying the boundary conditions $v(0) = v'(0) = 0$, thus the following equations have to be satisfied

$$(EIv'')'' + (Pv')' = 0 \qquad x \in (0, L) \text{ (Euler equation)}$$

$$\left.\begin{aligned} M(L) &= 0 \\ Q(L) - Pv'(L) &= 0 \end{aligned}\right\} \quad \text{natural boundary conditions}$$

$$\left.\begin{aligned} v(0) &= 0 \\ v'(0) &= 0 \end{aligned}\right\} \quad \text{essential boundary conditions}$$

If the bending stiffness EI and the compressive force P are constants in the domain, we get a homogeneous differential equation with constant coefficients

$$\begin{aligned} EIv^{(4)} + Pv'' &= 0 \\ \Rightarrow EIv'' + Pv &= Cx + D, \ (C, D \text{ constants}) \\ \Rightarrow v &= A \sin kx + B \cos kx + Cx + D, \ k = \sqrt{\frac{P}{EI}} \end{aligned}$$

The derivatives are

$$\begin{aligned} v' &= Ak \cos kx - Bk \sin kx + C \\ v'' &= -Ak^2 \sin kx - Bk^2 \cos kx \\ v''' &= -Ak^3 \cos kx + Bk^3 \sin kx \end{aligned}$$

$$\begin{aligned} v(0) = 0 &\ \Rightarrow\ B + D = 0 \\ v'(0) = 0 &\ \Rightarrow\ Ak + C = 0 \\ v''(L) = 0 &\ \Rightarrow\ A \sin kL + B \cos kL = 0 \end{aligned}$$

$$\begin{aligned} &-EIv'''(L) - Pv'(L) = 0 \ \Rightarrow \\ &\qquad -EI(-Ak^3 \cos kL + Bk^3 \sin kL) - P(Ak \cos kL - Bk \sin kL + C) = 0 \quad (1) \end{aligned}$$

$$P = \lambda \frac{EI}{L^2} \ \Rightarrow\ k = \frac{\sqrt{\lambda}}{L}$$

It follows from equation (1) that $A = 0 \ \Rightarrow\ C = 0 \ \Rightarrow\ B \cos kL = 0 \ \Rightarrow\ B = 0$ or $\cos kL = 0$. If $B = 0 \ \Rightarrow\ v \equiv 0$ it yields a trivial solution, hence we should have

$$\cos kL = 0 \ \Rightarrow\ kL = \frac{\pi}{2} + n\pi, \ n = 0, 1, 2, ...$$

$$\Rightarrow \lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2,$$

and the lowest buckling load is

$$\lambda_0 = \left(\frac{\pi}{2}\right)^2 \ \Rightarrow\ P_{\text{cr}} = \frac{\pi^2}{4}\frac{EI}{L^2}$$

The eigenmode corresponding to the eigenvalue λ_n is

$$v_n = B(\cos k_n x - 1), \quad k_n = \frac{1}{L}\left(\frac{\pi}{2} + n\pi\right)$$

It was asked to give the normalized eigenmodes. For that we should define how this normalization should be done. It is usual to use the energy norm

$$||v_n||_E^2 = \int_0^L EI(v_n'')^2 dx.$$

The energy orthogonality thus means

$$\int_0^L EIv_n''v_m''dx = 0, \text{ kun } n \neq m.$$

Lets normalize the eigenmodes v_n such, that $||v_n||_E = E_1$, where E_1 is the energy unit and $[E_1] = \sqrt{\text{Nm}}$.

$$\begin{aligned} v_n'' &= -Bk_n^2 \cos k_n x \\ &\Rightarrow \quad E_1^2 = EIB^2k_n^4 \int_0^L \cos^2 k_n x dx \end{aligned}$$

Lets change variables such, that

$$y = k_n x, \; dx = \frac{1}{k_n} dy \text{ rajat } \begin{cases} x = 0 & \Rightarrow \quad y = 0 \\ x = L & \Rightarrow \quad y = \frac{\pi}{2} + n\pi \end{cases}$$

$$\begin{aligned} \Rightarrow \quad & E_1^2 = EIB^2k_n^3 \int_0^{\frac{\pi}{2}+n\pi} \cos^2 y dy = EIB^2k_n^3 \frac{1}{2}\left(\frac{\pi}{2} + n\pi\right) \\ \Rightarrow \quad & B^2 = \frac{2E_1^2}{EIk_n^3\left(\frac{\pi}{2}+n\pi\right)} = \frac{2E_1^2}{EI\frac{\pi^4}{16}(1+2n)^4} \\ \Rightarrow \quad & B = \frac{4\sqrt{2}L^{3/2}E_1}{\pi^2(1+2n)^2\sqrt{EI}} \end{aligned}$$

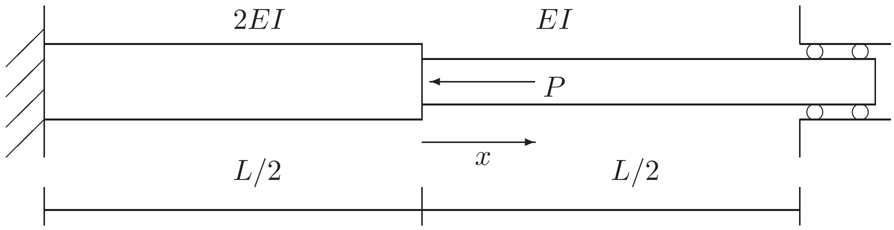
The energy orthonormal eigenfunctions are thus

$$v_n(x) = B_n(\cos k_n x - 1), \quad B_n = \frac{4\sqrt{2}}{\pi^2(1+2n)^2} \frac{E_1 L^{3/2}}{\sqrt{EI}}$$

Orthogonality:

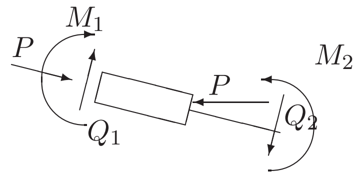
$$\begin{aligned} \int_0^L EIv_n''v_m''dx &= 0, \text{ kun } n \neq m. \\ \int_0^L v_n''v_m''dx &= \int_0^L \cos\left[\frac{\pi}{2}(1+2n)\frac{x}{L}\right] \cos\left[\frac{\pi}{2}(1+2m)\frac{x}{L}\right] \left(\text{merk. } y = \frac{\pi}{2}\frac{x}{L}, \; dx = \frac{2L}{\pi}dy\right) \\ &= \frac{2L}{\pi}\int_0^L \cos[(1+2n)y]\cos[(1+2m)y]dy = 0, \text{ kun } n \neq m. \end{aligned}$$

Problem 2. For the structure shown below, determine P_{cr} starting from the differential equation.



Solution.

In part 1 1 $v_1^{(4)} + k^2 v_1'' = 0$, where $k^2 = P/2EI$.
2 $v_2^{(4)} = 0$



$$
\begin{aligned}
\text{BC}: \quad & v_1\left(-\tfrac{L}{2}\right) = v_1'\left(-\tfrac{L}{2}\right) = v_2\left(\tfrac{L}{2}\right) = v_2'\left(\tfrac{L}{2}\right) = 0 \\
& v_1(0) = v_2(0) \\
& v_1'(0) = v_2'(0) \\
& M_1(0) = M_2(0) \\
& Q_1(0) = Q_2(0) + P v_2'(0)
\end{aligned}
$$

Solutions for the homogenious equations are

$$
\begin{aligned}
v_1 &= C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 \\
v_1' &= C_1 k \cos kx - C_2 k \sin kx + C_3 \\
v_1'' &= -C_1 k^2 \sin kx - C_2 k^2 \cos kx \\
v_1''' &= -C_1 k^3 \cos kx + C_2 k^3 \sin kx \\
v_2 &= C_5 x^3 + C_6 x^2 + C_7 x + C_8 \\
v_2' &= 3C_5 x^2 + 2C_6 x + C_7 \\
v_2'' &= 6C_5 x + 2C_6 \\
v_2''' &= 6C_5
\end{aligned}
$$

Taking the boundary conditions into account

$$
\begin{aligned}
Q_1(0) &= Q_2(0) + P v_2'(0) \\
-2EI v_1'''(0) &= -EI v_2'''(0) + P v_2'(0) \\
2C_1 k^3 &= -6C_5 + 2k^2 C_7 \\
C_5 &= -\frac{1}{3} k^3 C_1 + \frac{1}{3} k^2 C_7
\end{aligned}
$$

$$
\begin{aligned}
M_1(0) &= M_2(0) \\
-2EI v_1''(0) &= -EI v_2''(0) \\
2C_2 k^2 &= -2C_6 \;\Rightarrow\; C_6 = -k^2 C_2
\end{aligned}
$$

$$
\begin{aligned}
v_1'(0) &= v_2'(0) \\
C_1 k + C_3 &= C_7 \Rightarrow C_5 = -\frac{1}{3} k^3 C_1 + \frac{1}{3} k^2 (C_1 k + C_3) = \frac{1}{3} k^2 C_3
\end{aligned}
$$

$$
\begin{aligned}
v_1(0) &= v_2(0) \\
C_2 + C_4 &= C_8 \\
v_1\left(-\frac{L}{2}\right) &= 0 \Rightarrow C_4 = C_1 \sin\frac{kL}{2} - C_2\cos\frac{kL}{2} + C_3\frac{L}{2} \\
v_1'\left(-\frac{L}{2}\right) &= 0 \Rightarrow C_3 = -k(C_1\cos\frac{kL}{2} + C_2\sin\frac{kL}{2}) \\
v_2\left(\frac{L}{2}\right) &= 0 \Rightarrow \frac{1}{3}k^2C_3\left(\frac{L}{2}\right)^3 - k^2C_2\left(\frac{L}{2}\right)^2 + (C_1k + C_3)\frac{L}{2} + C_2 + C_4 = 0 \\
&\Rightarrow \left[\frac{kL}{2} - kL\cos\frac{kL}{2}\left(1 + \frac{1}{24}(kL)^2\right) + \sin\frac{kL}{2}\right]C_1 \\
&+ \left[1 - \frac{1}{4}(kL)^2 - \cos\frac{kL}{2} - kL\sin\frac{kL}{2}\left(1 + \frac{1}{24}(kL)^2\right)\right]C_2 = 0 \\
v_2'\left(\frac{L}{2}\right) &= 0 \Rightarrow k^2C_3\left(\frac{L}{2}\right)^2 - 2k^2C_2\frac{L}{2} + C_1k + C_3 = 0 \\
&\Rightarrow \left[1 - \left(1 + \frac{1}{4}(kL)^2\right)\cos\frac{kL}{2}\right]kC_1 + \left[-kL - \left(1 + \frac{1}{4}(kL)^2\right)\sin\frac{kL}{2}\right]C_2 = 0 \\
&\Rightarrow \begin{bmatrix} a(kL) & b(kL) \\ c(kL) & d(kL) \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\
&\det = 0 \Rightarrow kL \approx 7.55 \Rightarrow P_{kr} = 114\frac{EI}{L^2}
\end{aligned}
$$