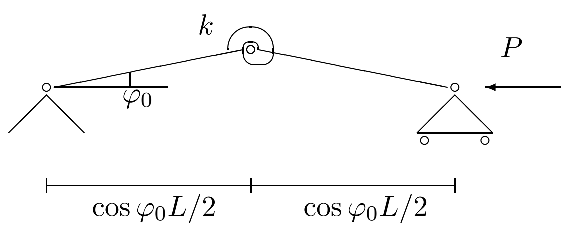
Stability of structures

2. exercise – equilibrium paths of discrete structural models

Problem 1. Determine all equilibrium paths of the structure consisting of two rigid bars and a linear elastic rotational spring. The structure has a geometrical imperfection ϕ_0 in its unloaded state. Investigate also the stability of all paths. Are there critical points on the paths?



Solution: The total potential energy expression is now

$$\begin{aligned}\Pi &= \frac{1}{2}k[2(\varphi-\varphi_0)]^2 - PL(\cos\varphi_0 - \cos\varphi)\\ \frac{\partial\Pi}{\partial\varphi} &= 4k(\varphi-\varphi_0) - PL\sin\varphi\\ \frac{\partial^2\Pi}{\partial\varphi^2} &= 4k - PL\cos\varphi\end{aligned}$$

The structure will be in equilibrium when the total potential energy attains its minimum, thus the first variation of the TPE will vanish.

$$\begin{aligned}\delta\Pi &= \frac{\partial\Pi}{\partial\varphi}\delta\varphi = 0\ \forall\ \delta\varphi\\ \Rightarrow \frac{\partial\Pi}{\partial\varphi} &= 0\\ \Rightarrow P &= \frac{4k(\varphi-\varphi_0)}{L\sin\varphi}\end{aligned}$$

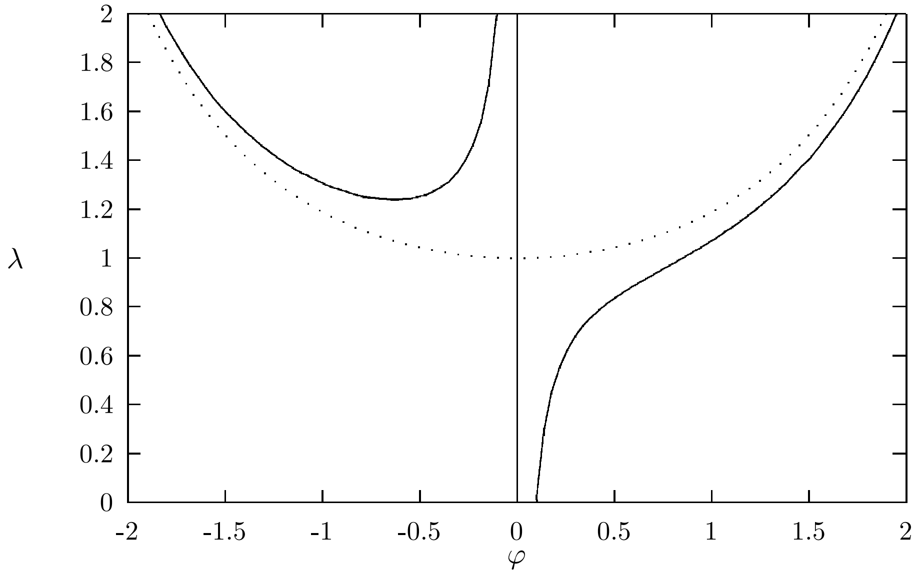
An equilibrium state is stable if the second variation of the TPE is positive

$$\begin{aligned}\delta^2\Pi &= \frac{\partial^2\Pi}{\partial\varphi^2}(\delta\varphi)^2 > 0\\ \Rightarrow \frac{\partial^2\Pi}{\partial\varphi^2} &> 0\\ \Rightarrow P &< \frac{4k}{L\cos\varphi}.\end{aligned}$$

Inserting the equilibrium equation $P = 4k(\varphi-\varphi_0)/L\sin\varphi$ in the expression above, gives the condition for stability

$$\begin{aligned}4k\left(1-\frac{\varphi-\varphi_0}{\tan\varphi}\right) &> 0\\ \Rightarrow \frac{\varphi-\varphi_0}{\tan\varphi} &< 1,\end{aligned}$$

which is valid for all non-negative values of φ. Thus this equilibrium path does not have critical points.

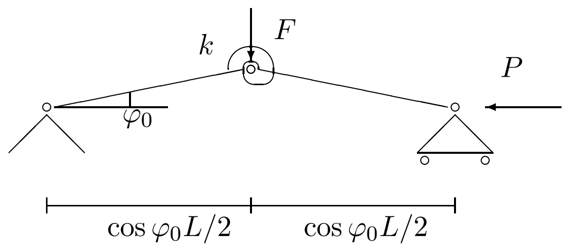


In the figure above, dotted line shows the equilibrium path of the perfect structure $\varphi_0 = 0$, and solid line indicates the stable path when $\varphi_0 > 0$. The path in the negative part of φ axis shown by a solid line is a complementary path. The load parameter λ is defined as

$$\lambda = P/P_{\mathrm{kr}} = PL/(4k). \tag{1}$$

Notice, that the complementary path is not stable everywhere. Determine the unstable and stable parths of the complementary path! Note too, that this means an existence of a critical point on the complementary path.

Problem 2. Determine all equilibrium paths starting from the unloaded state of the structure consisting of two rigid bars (length $L/2$) and a linear elastic rotational spring. The structure has a geometrical imperfection ϕ_0 in its unloaded state ($P = F = 0$). Investigate also the stability of all paths. The perturbation load $F = \epsilon 4k/L$, where ϵ is a dimensionless (second) perturbation parameter.



Solution. The total potential energy of the structure Π is

$$\Pi(\varphi;\varphi_0,\epsilon) = \frac{1}{2}k[2(\varphi-\varphi_0)]^2 - PL(\cos\varphi_0-\cos\varphi) - \frac{1}{2}FL(\sin\varphi_0-\sin\varphi) \quad (2)$$

A necessary condition of an equilibrium state is the stationarity of the TPE, thus the first variation of the total potential energy must vanish

$$\delta\Pi = \frac{d\Pi}{d\varphi}\delta\varphi = \left[4k(\varphi-\varphi_0) - PL\sin\varphi + \frac{1}{2}FL\cos\varphi\right]\delta\varphi = 0 \qquad \forall\ \delta\varphi \neq 0 \quad (3)$$

An equilibrium path is thus defined by

$$P = 4\left(\frac{k}{L}\right)\frac{\varphi-\varphi_0+\frac{1}{2}\epsilon\cos\varphi}{\sin\varphi} \quad (4)$$

This equation determines a unique path with respect to φ if the perturbation parameters does not satisfy the condition $\epsilon = 2\varphi_0$. In such a case the structure is a straight bar of length L at the "unloaded" state $P = 0$. Let us examine this special case first.

Case $\epsilon = 2\varphi_0$ The equilibrium equation is now

$$\frac{d\Pi}{d\varphi} = (\varphi-\varphi_0) - PL\sin\varphi + 4k\varphi_0\cos\varphi \quad (5)$$

$$= 4k\varphi - PL\sin\varphi + 4k\varphi_0(\cos\varphi - 1) = 0 \quad (6)$$

and the two solutions are

$$\varphi = 0 \qquad \text{primary path} \quad \mathcal{P}_{\mathcal{I}}, \quad (7)$$

$$P = 4\left(\frac{k}{L}\right)\frac{\varphi+\varphi_0(\cos\varphi-1)}{\sin\varphi} \qquad \text{secondary path} \quad \mathcal{P}_{\mathcal{II}} \quad (8)$$

An equilibrium state is stable if the second variation of Π:

$$\delta^2\Pi = \frac{d^2\Pi}{d\varphi^2}(\delta\varphi)^2 = (4k - PL\cos\varphi - 4k\varphi_0\sin\varphi)(\delta\varphi)^2 \qquad \forall\ \delta\varphi \neq 0 \quad (9)$$

is positive. Let us first examine stability of the primary path, i.e. when $\varphi = 0$, thus

$$\delta^2\Pi|_{\mathcal{P}} = \frac{d^2\Pi}{d\varphi^2}\big|_{\mathcal{P}}(\delta\varphi)^2 = (4k - PL)(\delta\varphi)^2 \tag{10}$$

The primary path is thus stable when $P < 4k/L$ and unstable when $P > 4k/L$, and the critical load is thus $P_{\mathrm{cr}} = 4k/L$. Let us examine wheather the critical point is a symmetric or asymmetric bifurcation point. The expression of the third variation of the TPE is

$$\delta^3\Pi|_{\mathcal{P}} = \frac{d^3\Pi}{d\varphi^3}\big|_{\mathcal{P}}(\delta\varphi)^3 \tag{11}$$

where

$$\frac{d^3\Pi}{d\varphi^3} = -PL\sin\varphi - 4k\varphi_0\cos\varphi \tag{12}$$

At the critical point the value of the third derivatve of the TPE is on

$$\frac{d^3\Pi}{d\varphi^3}\big|_{\mathrm{kr}} = -4k\varphi_0 \neq 0 \tag{13}$$

thus the critical point is an asymmetric bifurcation point. The equilibrium path is drawn in figure .

Case $\epsilon \neq 2\varphi_0$ Let us examine stability of the equilibrium path, defined in (4). The second variation of the TPE

$$\delta^2\Pi = \frac{d^2\Pi}{d\varphi^2}(\delta\varphi)^2 \tag{14}$$

is obtained from the expression of the first variation (3). An equilibrium state is stable if the second variation of the TPE is positive for all kinematically admissible variations $\delta\varphi$, thus in this single degree of freedom example it is sufficient to investigate the sign of the second derivative of the TPE

$$\frac{d^2\Pi}{d\varphi^2} = 4k - PL\cos\varphi - 2k\epsilon\sin\varphi \tag{15}$$

Let's insert the expression of the equilibrium path (4) in the expression above, gives

$$\frac{d^2\Pi}{d\varphi^2} = 4k\frac{\sin\varphi - (\varphi - \varphi_0)\cos\varphi - \frac{1}{2}\epsilon}{\sin\varphi} \tag{16}$$

Let us examine the cases $\epsilon > 2\varphi_0$ and $\epsilon < 2\varphi_0$ separately.

In the case $\epsilon > 2\varphi_0$, the structure is below the horizonal line defined by the supports before applying the compressive load, thus the structure will continue to displace below the support line, thus $\varphi < 0$. Let us define $\epsilon = 2\varphi_0 + \bar{\epsilon}$, and the expression (16) gives

$$\frac{d^2\Pi}{d\varphi^2} = 4k\frac{\sin\varphi - \varphi\cos\varphi - \varphi_0(1 - \cos\varphi) - \bar{\epsilon}}{\sin\varphi} \tag{17}$$

Since now $\varphi < 0$ and both the nominator and denominator are negative, thus δ^{Π} is always positive, i.e. the path is stable when $\epsilon > 2\varphi_0$.

The case $\epsilon < 2\varphi_0$ is more interesting. Now $\varphi > 0$ and the denominator of the expression (16) is always positive but the nominator can have zero points. These roots can be solved from the transcendental equation

$$\sin\varphi - (\varphi - \varphi_0)\cos\varphi - \tfrac{1}{2}\epsilon = 0. \tag{18}$$

Since analytical solution is impossible, let's try the asymptotic analysis assuming that the angles φ and φ_0 are small, thus

$$\sin\varphi \approx \varphi - \tfrac{1}{6}\varphi^3, \qquad \cos\varphi \approx 1 - \tfrac{1}{2}\varphi^2,$$

and the expression (18) will has a form

$$\tfrac{1}{3}\varphi^3 - \tfrac{1}{2}\varphi_0\varphi^2 + (\varphi_0 - \tfrac{1}{2}\epsilon) = 0 \tag{19}$$

The third order polynomial above can have both negative and positive values for positive values of φ. To show that, let us fisrt calculate the minumum point

$$\varphi^2 - \varphi_0\varphi = 0 \qquad \Longrightarrow \qquad \varphi = \varphi_0. \tag{20}$$

The minimum value of the function defined in (19) (kun $\varphi > 0$) and the condition for the nagativity we get an inequality (let's define $\epsilon = \eta\varphi_0$)

$$-\tfrac{1}{3}\varphi_0^2 + 1 - \tfrac{1}{2}\eta < 0 \qquad \Longrightarrow \qquad \eta > 2 - \tfrac{1}{3}\varphi_0^2$$

Taking the condition $\epsilon < 2\varphi_0$ into account we'll get a condition for the perturbation parameter $\epsilon = \eta\varphi_0$:

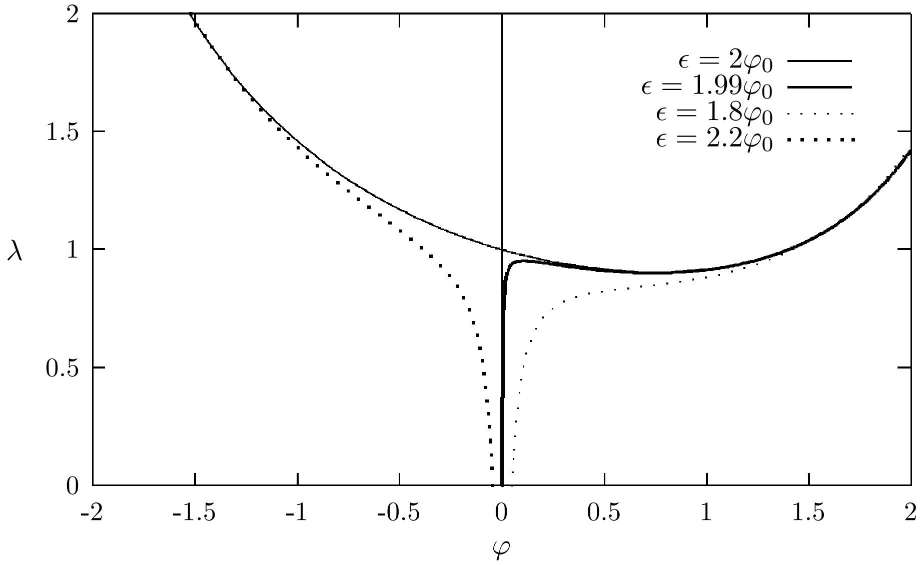
$$2 > \eta > 2 - \tfrac{1}{3}\varphi_0^2 \qquad \text{i.e.} \qquad 2\varphi_0 > \epsilon > (2 - \tfrac{1}{3}\varphi_0^2)\varphi_0$$

for the existence of a limit point on the equilibrium path. In the following figure, some equilibrium paths are shown for some values of the perturbation parameter ϵ

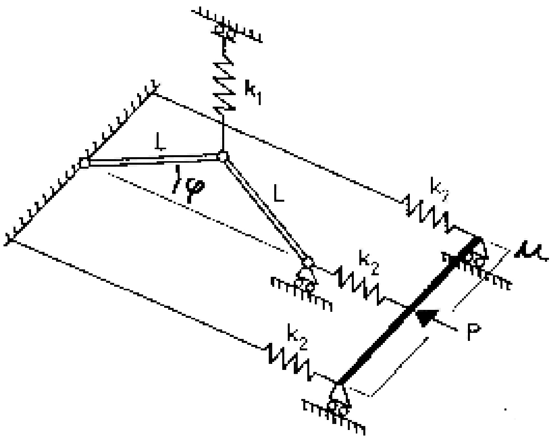
To sum up, the equilibrium paths of this structure can have

- a trivial equilibrium path and an asymmetric bifurcation point if $\epsilon = 2\varphi_0$. The secondary path is defined in equation (8).
- A stable equilibrium path without critical points if $\epsilon > 2\varphi_0$ or if $\epsilon \lessapprox (2 - \frac{1}{3}\varphi_0^2)\varphi_0$.
- An equilibrium path has a limit point if $(2 - \frac{1}{3}\varphi_0^2)\varphi_0 \lessapprox \epsilon \lessapprox 2\varphi_0$,

Equilibrium paths shown in the figure below: $\lambda = P/P_{\text{cr}} = PL/(4k)$.


2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
λ
φ
$\epsilon = 2\varphi_0$
$\epsilon = 1.99\varphi_0$
$\epsilon = 1.8\varphi_0$
$\epsilon = 2.2\varphi_0$

Problem 3. Determine the equilibrium paths of the simple structure shown, consisting of rigid bars and elastic springs. Investigate also the stability of the equilibrium paths. Investigate especially cases $k_1 = k_2$ and $k_1 = 5k_2$. What kind of real structures these models imitate?



Solution. The total potential energy (TPE) expression is

$$\begin{aligned} \Pi &= U + V \\ U &= \frac{1}{2}k_1 L^2 \sin^2\varphi + k_2 u^2 + \frac{1}{2}k_2[u - 2L(1-\cos\varphi)]^2 \\ V &= -Pu \end{aligned} \tag{21}$$

The equilibrium paths can be obtained from the stationarity condition of the TPE:

$$\delta\Pi = \frac{\partial \Pi}{\partial \varphi}\delta\varphi + \frac{\partial \Pi}{\partial u}\delta u = 0 \tag{22}$$

Since the variations of the displacement u and rotation φ are arbitrary, the equilibrium paths are obtained from equations

$$\begin{aligned} \frac{\partial \Pi}{\partial \varphi} &= k_1 L^2 \sin\varphi\cos\varphi + k_2[u - 2L(1-\cos\varphi)](-2L\sin\varphi) = 0 \\ \frac{\partial \Pi}{\partial u} &= 2k_2 u + k_2[u - 2L(1-\cos\varphi)] - P = 0 \end{aligned} \tag{23}$$

After some manipulations we get

$$\sin\varphi[k_1 L^2\cos\varphi - 2Lk_2 u + 4k_2 L^2(1-\cos\varphi)] = 0 \tag{24}$$

$$u = \frac{P}{3k_2} + \frac{2}{3}L(1-\cos\varphi) \tag{25}$$

Equation (24) is satisfied, if

$$\sin\varphi = 0 \qquad \text{tai} \qquad (k_1 - 4k_2)L^2\cos\varphi + 4k_2L^2 - 2k_2Lu = 0 \tag{26}$$

If equation (25) is put into equation (26) and define $k_2 = k$ ja $k_1 = \alpha k$, we get

$$\Rightarrow \quad P = kL\left(4 + (\tfrac{3}{2}\alpha - 4)\cos\varphi\right) \tag{27}$$

which is the projection of the secondary path onto the (φ, P)-plane. Accordingly from equation (25) we get

$$\cos\varphi = 1 + \frac{P}{2kL} - \frac{3}{2}\frac{u}{L},$$

which is substituted into (27)

$$\Rightarrow \quad P = \frac{kL}{4-\alpha}\left[2\alpha + (8-3\alpha)\frac{u}{L}\right],$$

which describes the projection of the secondary path onto the (u, P)-plane.

The primary paths are defined as

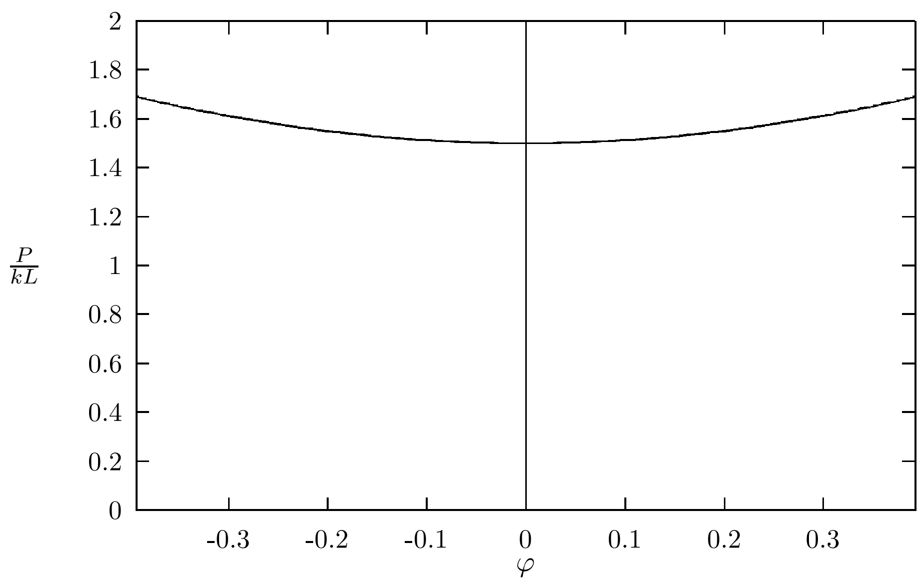
$$\begin{cases} \varphi &= 0 \\ u &= \dfrac{P}{3k} \end{cases}$$

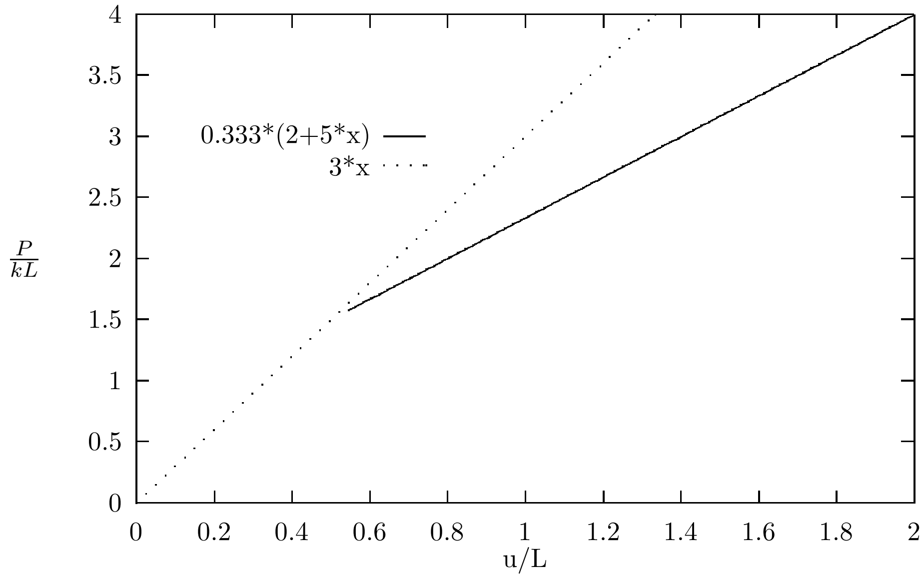
and the secondary paths

$$\begin{cases} P = [4 + (\frac{3}{2}\alpha - 4)\cos\varphi]kL \\ P = \dfrac{kL}{4-\alpha}\left[2\alpha + (8-3\alpha)\dfrac{u}{L}\right] \end{cases}$$

Let us investigate the cases $\alpha = 1$ ja $\alpha = 5$.

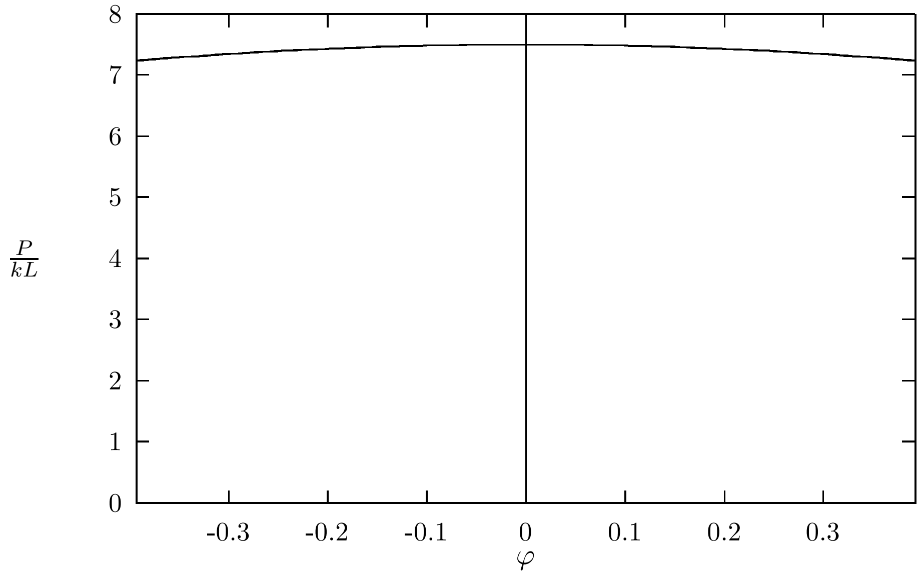
$$\alpha = 1 \Rightarrow \begin{cases} P = (4 - \frac{5}{2}\cos\varphi)kL \\ P = \dfrac{1}{3}kL\left(2 + 5\dfrac{u}{L}\right) \end{cases}$$

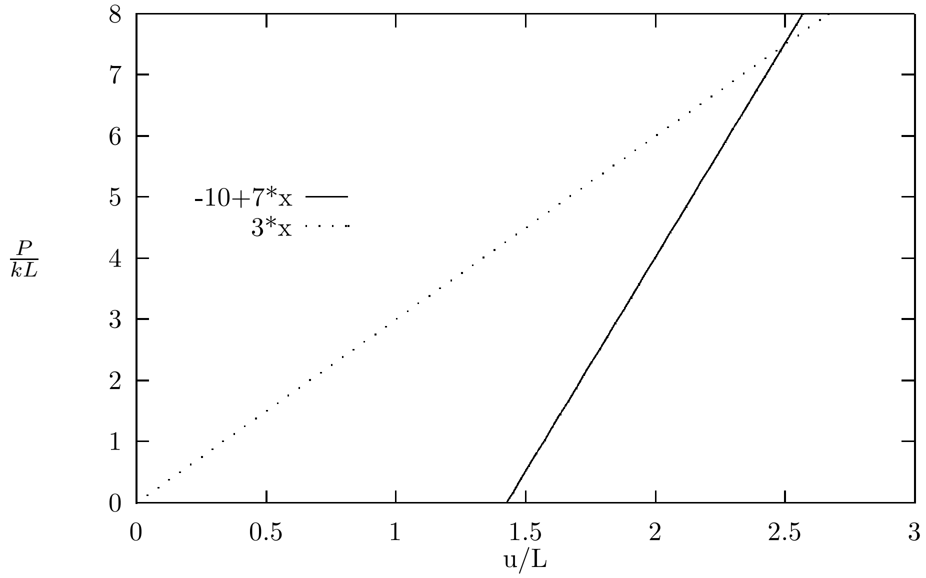




We notice, that displacements are increasing more rapidly on the secondary path than in the primary path. However, the load can still be increased over the critical value at the bifurcation point. ($P_{\mathrm{kr}} = \frac{3}{2}kL$, thus the secondary path is stable. In compressed thin plates such kind of behaviour can be obtained. The strong stability of the secondary paths can be utilized also in design for some cases.

$$\alpha = 5 \Rightarrow \begin{cases} P = (4 + \frac{7}{2}\cos\varphi)kL \\ P = -kL\left(10 - 7\dfrac{u}{L}\right) \end{cases}$$





In this case the bifurcation load is much higher than in the previous one. However, the secondary equilibrium path is now unstable. Shells, especially exhibit such kind of unstable behaviour after bifurcation. If the post-buckling regime is unstable, such structures are imperfection sensitive, which means that the critical load of an imperfect structure is much lower than the theoretical bifurcation load. Imperfections are due to eccentricities, geometrical deviations etc.