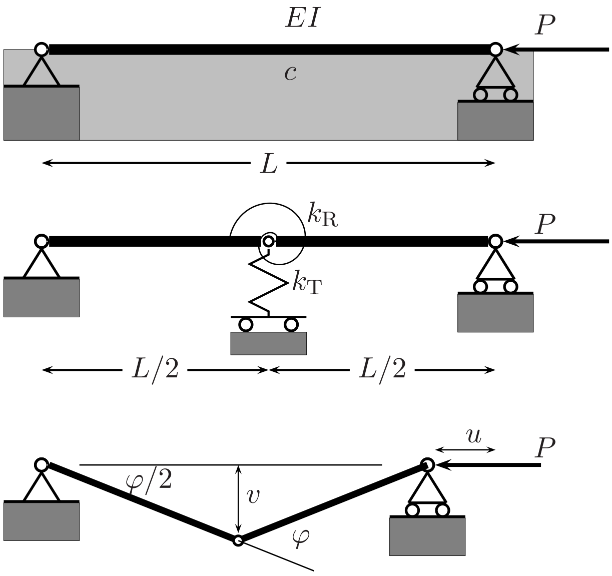
Stability of structures

1. exercise – equilibrium paths of simple structural models

Problem 1. Consider a beam on an elastic foundation. Idealize the beam as a discrete system of two equal length rigid bars connected by a linear rotational spring characterizing the bending rigidity of the beam. The foundation can be idealized with a linear translational spring. Determine all equilibrium paths and the critical load P_{cr}. The foundation coefficient is $c = \beta\pi^2 EI/L^4$, where β is a dimensionless constant. The spring constants are thus $k_{\mathrm{T}} = \frac{1}{2}cL$ and $k_{\mathrm{R}} = \frac{1}{4}\pi^2 EI/L$. Are the equilibrium paths near the critical point stable or unstable?



Solution. First, form the expressions for the horizontal displacement of the load point u and the vertical displacement of the joint:

$$u = 2 \cdot \frac{L}{2}(1 - \cos(\varphi/2)), \qquad v = \frac{L}{2}\sin(\varphi/2).$$

The total potential energy is thus

$$\begin{aligned}
\Pi &= \frac{1}{2}k_{\mathrm{R}}\varphi^2 + \frac{1}{2}k_{\mathrm{T}}v^2 - Pu \\
&= \frac{1}{2}k_{\mathrm{R}}\varphi^2 + \frac{1}{2}k_{\mathrm{T}}\frac{L^2}{4}\sin^2(\varphi/2) - PL(1 - \cos(\varphi/2)) \\
&= \frac{1}{8}\pi^2\frac{EI}{L}\varphi^2 + \frac{1}{16}\beta\pi^2\frac{EI}{L}\sin^2(\varphi/2) - PL(1 - \cos(\varphi/2)). \qquad (1)
\end{aligned}$$

It is often advisable to make the expressions dimensionless, therefore let us denote

$$P = \lambda\frac{\pi^2 EI}{L^2}, \qquad \tilde{\Pi} = \frac{L}{\pi^2 EI}\Pi,$$

which results in

$$\tilde{\Pi} = \frac{1}{8}\varphi^2 + \frac{1}{16}\beta\sin^2(\varphi/2) - \lambda(1 - \cos(\varphi/2)). \qquad (2)$$

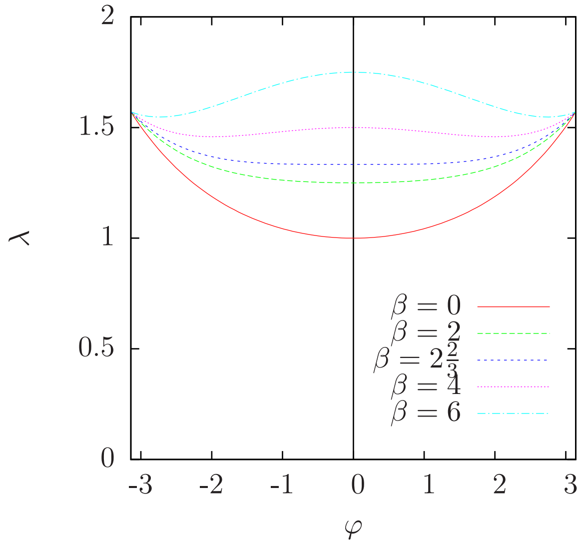


Figure 1: Equilibrium paths.

Equilibrium equation is

$$\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}\varphi} = \frac{1}{4}\varphi + \frac{1}{16}\beta\sin(\varphi/2)\cos(\varphi/2) - \frac{1}{2}\lambda\sin(\varphi/2) = 0. \tag{3}$$

It is immediately noticed that $\varphi = 0$ is a solution for all values of the load parameter λ, the second solution is

$$\lambda = \frac{\varphi/2}{\sin(\varphi/2)} + \frac{1}{8}\beta\cos(\varphi/2). \tag{4}$$

These two equilibrium paths intersect at $\varphi = 0$, $\lambda = 1 + \frac{1}{8}\beta$, which is the bifurcation point. The critical load is then

$$P_{\mathrm{cr}} = (1 + \tfrac{1}{8}\beta)\frac{\pi^2 EI}{L^2}.$$

Equilibrium paths with different β-values are shown in Fig. 1. For the stability investigation of these paths, we need the second derivative of the potential

$$\begin{aligned} \frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2} &= \frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{4}\varphi + \frac{1}{16}\beta\sin(\varphi/2)\cos(\varphi/2) - \frac{1}{2}\lambda\sin(\varphi/2)\right) \\ &= \frac{1}{4} + \frac{1}{32}\beta\cos\varphi - \frac{1}{4}\lambda\cos(\varphi/2) \end{aligned} \tag{5}$$

Path I. On the primary path $\mathcal{P}_I$ when $\varphi = 0$, thus

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_I} = \frac{1}{4} + \frac{1}{32}\beta - \frac{1}{4}\lambda. \tag{6}$$

Thus

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_I} > 0, \quad \text{when} \quad \lambda < 1 + \tfrac{1}{8}\beta, \tag{7}$$

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_I} = 0, \quad \text{when} \quad \lambda = 1 + \tfrac{1}{8}\beta, \tag{8}$$

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_I} < 0, \quad \text{when} \quad \lambda > 1 + \tfrac{1}{8}\beta, \tag{9}$$

and we can conclude that

$$\mathcal{P}_I \quad \text{is} \quad \begin{cases} \text{stable when} & \lambda < 1 + \tfrac{1}{8}\beta, \\ \text{unstable when} & \lambda > 1 + \tfrac{1}{8}\beta. \end{cases} \tag{10}$$

Stability of the bifurcation point $\lambda = 1 + \frac{1}{8}\beta$ cannot be determined from the second derivative.

Path II. Definition to the secondary equilibrium path $\mathcal{P}_{II}$ is given in (4) and substituting it into (5) gives

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_{II}} = \frac{1}{4} + \frac{1}{32}\beta\cos\varphi - \frac{1}{4}\left(\frac{\varphi/2}{\sin(\varphi/2)} + \frac{1}{8}\beta\cos(\varphi/2)\right)\cos(\varphi/2) \tag{11}$$

$$= \frac{1}{4} - \frac{\varphi/8}{\tan(\varphi/2)} + \frac{1}{32}\beta\left(\cos\varphi - \cos^2(\varphi/2)\right) \tag{12}$$

$$= \frac{1}{4}\left(1 - \frac{\varphi/2}{\tan(\varphi/2)} - \frac{1}{8}\beta\sin^2(\varphi/2)\right). \tag{13}$$

It can be seen that depending on the value of β, the secondary path $\mathcal{P}_{II}$ can be either stable or unstable.

Let us investigate stability of the secondary path near the critical point $\varphi = 0$, $\lambda = 1 + \frac{1}{8}\beta$. Remember that

$$\begin{aligned} \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots, \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots, \\ \tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots, \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots. \end{aligned}$$

Series expansion of (5) is

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_{II}} = \frac{1}{4}\left(1 - \frac{\varphi/2}{\varphi/2 + \frac{1}{3}(\varphi/2)^3}\right) - \frac{1}{32}\beta(\varphi/2)^2\left(1 - \frac{1}{6}(\varphi/2)^2\right) + \text{h.o.t.}$$

$$= \frac{1}{4}\left(\frac{1}{3} - \frac{1}{8}\beta\right)(\varphi/2)^2 + \text{h.o.t.} \tag{14}$$

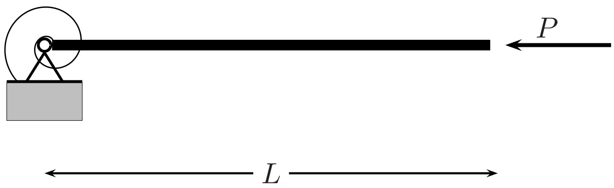
Thus, it can be concluded that when $|\varphi| \ll 1$ then $\mathcal{P}_{II}$ is initially stable if $\beta < 8/3 = 2\frac{2}{3}$ and unstable if $\beta > 2\frac{2}{3}$.

Notice that increasing the foundation stiffness, i.e. increasing β, increases the critical bifurcation load, but it makes the secondary paths (also known as post-buckling paths) more unstable!

Problem 2. Determine all equilibrium paths of the simple rigid bar-spring system. Investigate stability of these paths. Determine also the possible critical points and the corresponding load values P_{cr}. The constitutive relation of the spring is

$$M = k_1\varphi + k_2\varphi^3, \qquad k_1 > 0.$$

Investigate the effect of the nonlinear term k_2, i.e. use different values of the ratio $\alpha = k_2/k_1$.



Solution. The total potential energy is

$$\begin{aligned}\Pi &= \int_0^\varphi M \mathrm{d}\varphi - PL(1-\cos\varphi) \\ &= \frac{1}{2}k_1\varphi^2 + \frac{1}{4}k_2\varphi^4 - PL(1-\cos\varphi). \end{aligned} \tag{15}$$

Equilibrium equation is

$$\frac{\mathrm{d}\Pi}{\mathrm{d}\varphi} = k_1\varphi(1+\alpha\varphi^2) - PL\sin\varphi = 0. \tag{16}$$

Immediately we notice that $\varphi = 0$ is a solution. Defining dimensionless load parameter as $P = \lambda k_1/L$, i.e. $\lambda = PL/k_1$, we have for the secondary path

$$\lambda = \frac{\varphi(1+\alpha\varphi^2)}{\sin\varphi}. \tag{17}$$

The primary- and secondary paths intersect at $\varphi = 0$, $\lambda = 1$, which is the critical bifurcation point.

Let us investigate stability of the paths.

$$\tilde{\Pi} = \Pi/k_1 = \frac{1}{2}\varphi^2 + \frac{1}{4}\alpha\varphi^4 - \lambda(1-\cos\varphi), \tag{18}$$

$$\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}\varphi} = \varphi + \alpha\varphi^3 - \lambda\sin\varphi = 0, \tag{19}$$

$$\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2} = 1 + 3\alpha\varphi^2 - \lambda\cos\varphi. \tag{20}$$

Path I. Primary path $\mathcal{P}_I$: $\varphi = 0$, then

$$\left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_I} \begin{cases} > 0 & \text{when } \lambda < 1 \quad \text{stable} \\ = 0 & \text{when } \lambda = 1 \quad ? \\ < 0 & \text{when } \lambda > 1 \quad \text{unstable} \end{cases} \tag{21}$$

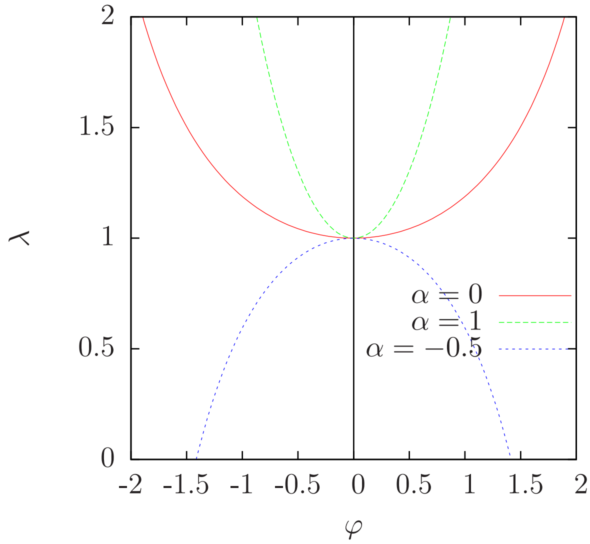


Figure 2: Equilibrium paths with different values of α.

Path II. Secondary path $\mathcal{P}_{II}$ defined in (17), then on $\mathcal{P}_{II}$

$$\begin{aligned} \left.\frac{\mathrm{d}^2\tilde{\Pi}}{\mathrm{d}\varphi^2}\right|_{\mathcal{P}_{II}} &= 1 + 3\alpha\varphi^2 - \frac{\varphi}{\tan\varphi} - \alpha\frac{\varphi^3}{\tan\varphi} \\ &= 1 + 3\alpha\varphi^2 - (1 - \tfrac{1}{3}\varphi^2) - \alpha\varphi^2(1 - \tfrac{1}{3}\varphi^2) + \text{h.o.t.} \\ &= (\tfrac{1}{3} + 2\alpha)\varphi^2 + \text{h.o.t.} \end{aligned} \tag{22}$$

Secondary path $\mathcal{P}_{II}$ is thus stable if $\frac{1}{3} + 2\alpha > 0$.