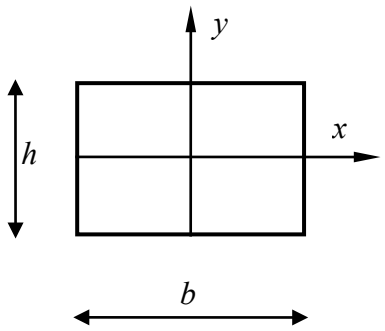


1. Laske oheisen suorakaidepoikkileikkauksen vääntöneliömomentti eli vääntöjäyhyys I_v käyttäen

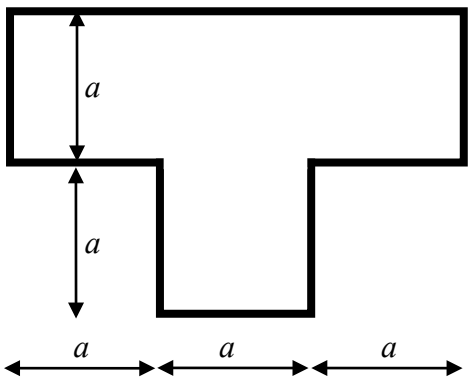
   a) potentiaalienergian minimin periaatetta sekä käyristymälle yritettä:

   $$\psi^* = a_1 xy$$

   b) komplementaaripotentiaalienergian minimin periaatetta sekä jännitysfunktiolle yritettä:

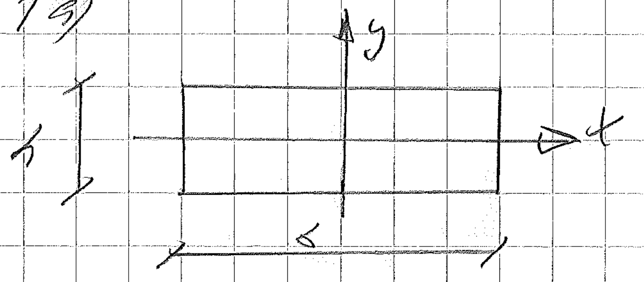
   $$\phi^{**} = b_1\left(x^2 - \frac{b^2}{4}\right)\left(y^2 - \frac{h^2}{4}\right)$$

   Laske alalikiarvot, kun $b = 4h$



2. Laske oheisen T-poikkileikkauksen vääntöneliömomentti I_v kappaleen 2.9 mukaisella hiekkakakkumenetelmällä.

1 a)



$\psi_1 = xy$

$\psi^* = a_1 \psi_1 = a_1 xy$ $D^2\psi^* = \frac{\partial^2 \psi^*}{\partial x^2} + \frac{\partial^2 \psi^*}{\partial y^2} = 0$

(2.233) $B_{11} = \int \left(\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{\partial \psi_1}{\partial y} \right) dA$ $\frac{\partial \psi_1}{\partial x} = y$ $\frac{\partial \psi_1}{\partial y} = x$

$= \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} (y^2 + x^2)\, dx\, dy$ (polaarinen momentti)

$= \int_{-h/2}^{h/2} \Big/_{-b/2}^{b/2} \left(y^2 x + \frac{1}{3}x^3 \right) dy = \int_{-h/2}^{h/2} \left(2y^2\frac{b}{2} + \frac{2}{3}\left(\frac{b}{2}\right)^3 \right)$

$= \int_{-h/2}^{h/2} \left(y^2 b + \frac{1}{12}b^3 \right) dy = \Big/_{-h/2}^{h/2} \left(\frac{1}{3}y^3 b + \frac{1}{12}b^3 y \right)$

$= \frac{2}{3}\left(\frac{h}{2}\right)^3 b + \frac{2}{12}b^3\frac{h}{2} = \frac{1}{12}h^3 b + \frac{1}{12}b^3 h \quad (= I_p)$

$C_1 = \int_A \left(-y\frac{\partial \psi_1}{\partial x} + x\frac{\partial \psi_1}{\partial y} \right) dA$

$= \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} (-y^2 + x^2)\, dx\, dy = \int_{-h/2}^{h/2} \Big/_{-b/2}^{b/2} \left(-y^2 x + \frac{1}{3}x^3 \right) dy$

$= \int_{-h/2}^{h/2} \left(-2y^2\left(\frac{b}{2}\right) + \frac{2}{3}\left(\frac{b}{2}\right)^3 \right) dy = \int_{-h/2}^{h/2} \left(-y^2 b + \frac{1}{12}b^3 \right) dy$

$= \Big/_{-h/2}^{h/2} \left(-\frac{1}{3}y^3 b + \frac{1}{12}b^3 y \right) = -\frac{1}{12}h^3 b + \frac{1}{12}b^3 h$

$B_{11} a_1 = -C_1 \Rightarrow \frac{1}{12}(h^3 b + b^3 h) a_1 = \frac{1}{12}(h^3 b - b^3 h)$

$a_1 = \frac{h^3 b - b^3 h}{h^3 b + b^3 h} = \frac{h^2 - b^2}{h^2 + b^2} =$

$$I_v^* = I_p + \sum_{i=1}^{n} a_i c_i = I_p + a_1 c_1$$

$$a_1 c_1 = -\frac{c_1^2}{B_{11}} = -\left(\frac{1}{12} b^3 h - \frac{1}{12} h^3 b\right)^2 / I_p$$

$$I_v^* = I_p - \frac{c_1^2}{I_p} < I_p$$

Todetaan, että $a_1 = 0$ jos $b = h$

Lasku $b = 4h$

$$I_p = \frac{1}{12}(h^3 b + b^3 h) = \frac{1}{12}(4h^4 + 64h^4) = \frac{17}{3} h^4 = 5{,}6667 h^4$$

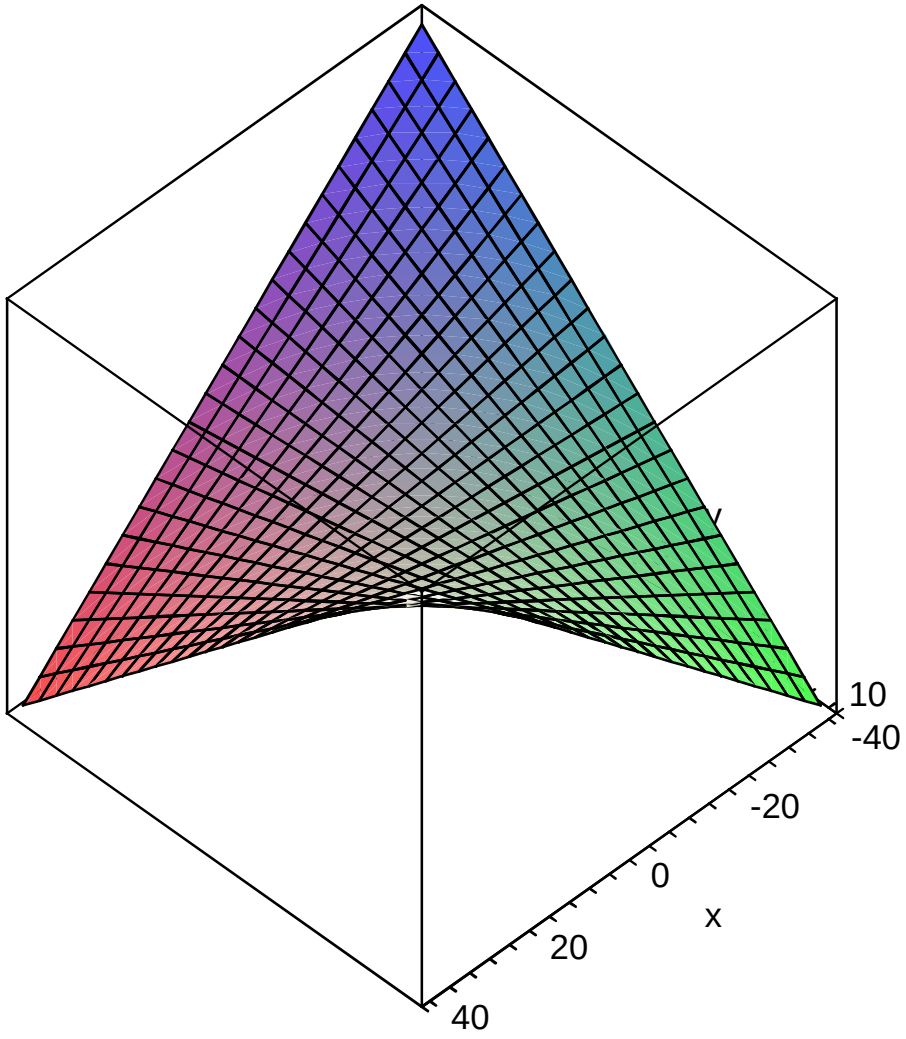
$$c_1 = \frac{1}{12}(b^3 h - h^3 b) = \frac{1}{12}(64h^4 - 4h^4) = 5h^4$$

$$I_v = I_p - \frac{c_1^2}{I_p} = \frac{64}{51} h^4 = \underline{1{,}255 h^4}$$

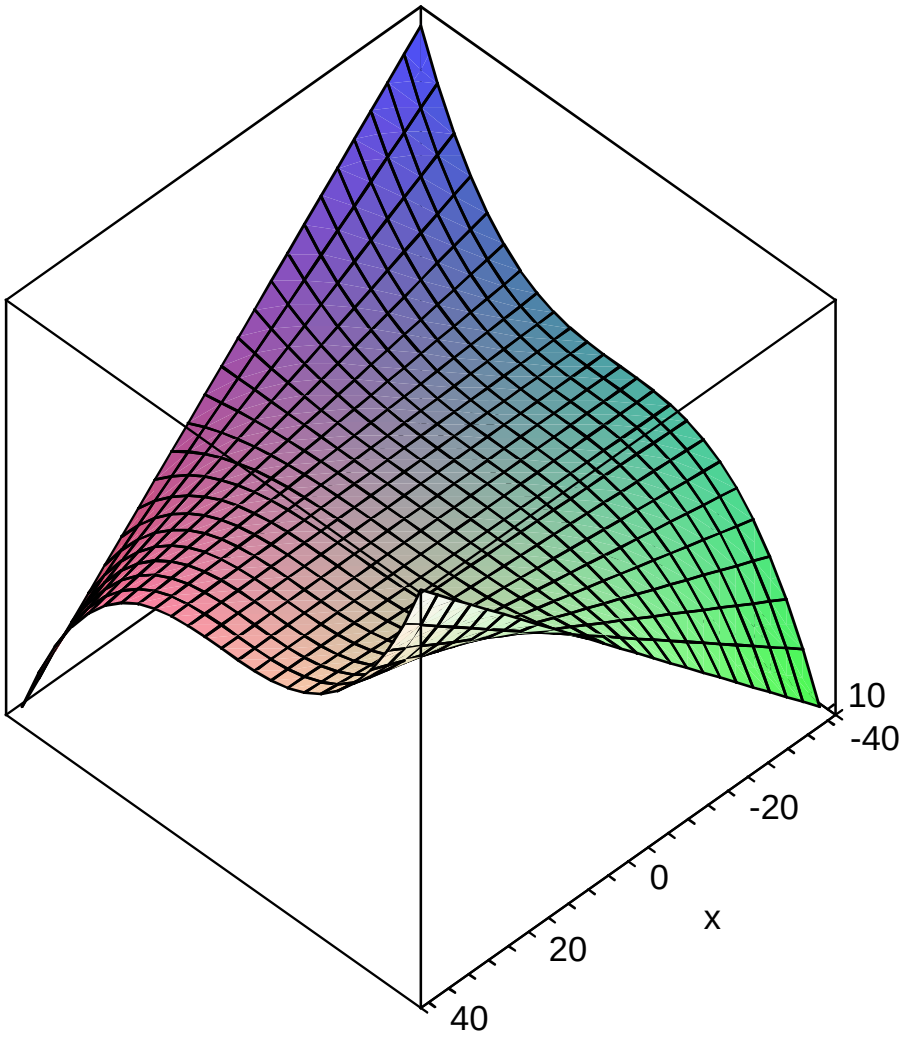
Lujam. $h/b = 4$ $\overset{h=4b}{\Rightarrow}$ $\alpha = 0{,}281$

$$I_v = 0{,}281 b^3 h = 4 \cdot 0{,}281 b^4 = 1{,}124 h^4$$

```
> with(linalg):
> # Deplanaatiofunktio
> plot3d(x*y, x=-40..40, y=-10..10, axes=boxed);
```




```
> # Deplanaatiofunktio 2
> plot3d(y^3*x, x=-40..40, y=-10..10, axes=boxed);
```




```
> psi1:=x*y;psi2:=y^3*x;
```

$$\psi 1 := x\,y$$

$$\psi 2 := y^3\,x$$

```
> # Haetaan kertoimet B11, B12 ja B22
```

```
> B11:=(int(int(diff(psi1,x)*diff(psi1,x)+diff(psi1,y)*diff(psi1,y),y=-h/2..
  h/2),x=-b/2..b/2));
```

$$B11 := \frac{1}{12}\,h^3\,b + \frac{1}{12}\,h\,b^3$$

```
> B12:=simplify(int(int(diff(ps1,x)*diff(psi2,x)+diff(psi1,y)*diff(psi2,y),
  =-h/2..h/2),x=-b/2..b/2));
```

$$B12 := \frac{1}{48}\,h^3\,b^3$$

```
> B22:=simplify(int(int(diff(psi2,x)*diff(psi2,x)+diff(psi2,y)*diff(psi2,y)
  y=-h/2..h/2),x=-b/2..b/2));
```

$$B22 := \frac{1}{448} h^7 b + \frac{3}{320} h^5 b^3$$

```
> # Muodostetaan kerroinmatriisi B
```

```
> B:=matrix(2,2, [B11,B12,B12,B22]);
```

$$B := \begin{bmatrix} \frac{1}{12} h^3 b + \frac{1}{12} h b^3 & \frac{1}{48} h^3 b^3 \\ \frac{1}{48} h^3 b^3 & \frac{1}{448} h^7 b + \frac{3}{320} h^5 b^3 \end{bmatrix}$$

```
> # lasketaan kerroinvektorin kompomentit C1 ja C2
```

```
> C1:=simplify(int(int(-diff(psi1,x)*y+diff(psi1,y)*x,y=-h/2..h/2),x=-b/2..
  /2));
```

$$C1 := -\frac{1}{12} h^3 b + \frac{1}{12} h b^3$$

```
> C2:=simplify(int(int(-diff(psi2,x)*y+diff(psi2,y)*x,y=-h/2..h/2),x=-b/2..
  /2));
```

$$C2 := -\frac{1}{80} h^5 b + \frac{1}{48} h^3 b^3$$

```
> # Kerroinvektori C
```

```
> C:=matrix(2,1, [C1,C2]);
```

$$C := \begin{bmatrix} -\frac{1}{12} h^3 b + \frac{1}{12} h b^3 \\ -\frac{1}{80} h^5 b + \frac{1}{48} h^3 b^3 \end{bmatrix}$$

```
> # Lasketaan apusuure C*a
```

```
> vast:=det(simplify(evalm(transpose(C)&*inverse(B)&*C)));
```

$$vast := \frac{h b (69 h^6 - 96 h^4 b^2 - 15 h^2 b^4 + 70 b^6)}{30 (15 h^4 + 78 h^2 b^2 + 28 b^4)}$$

```
> # polaarinen neliömomentti
```

```
> Ip:=1/12*(h^3*b+b^3*h);
```

$$Ip := \frac{1}{12} h^3 b + \frac{1}{12} h b^3$$

```
> # vääntöneliömomentti Iv
```

```
> Iv:=simplify(Ip-vast);
```

$$Iv := \frac{h^3 b (-63 h^4 + 657 h^2 b^2 + 560 b^4)}{60 (15 h^4 + 78 h^2 b^2 + 28 b^4)}$$

```
> # Lasketaan Iv kun h=10 ja b=40
> evalf(subs({h=10, b=40},Iv));
                                    12162.17926
>
>
>
>
> # Lasketaan vielä Iv, kun käytetään vain yhtä esitimaattia \psi1
> evalf(subs({h=10, b=40},Ip-C1*C1/B11));
                                    12549.01961
> # Lasketaan vielä Iv, kun käytetään vain yhtä esitimaattia \psi2
> evalf(subs({h=10, b=40},Ip-C2*C2/B22));
                                    29620.07168
>
>
```

1б)

$$\varphi = \beta\varphi_1 \qquad \varphi_1 = (x^2 - \frac{b^2}{4})(y^2 - \frac{h^2}{4})$$

$$\frac{\partial\varphi}{\partial x} = 2x(y^2 - \frac{h^2}{4})$$

$$\frac{\partial\varphi}{\partial y} = 2y(x^2 - \frac{b^2}{4})$$

$$D_{11} = \int_A \left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \left(\frac{\partial\varphi_1}{\partial y}\right)^2\right) dA$$

$$D_{11} = \frac{1}{90}(b^3h^5 + h^3b^5)$$

$$E_1 = \int_A \varphi_1 dA = \frac{1}{36} b^3h^3$$

$$\beta_1 = 2G\frac{E_1}{D_{11}} = \frac{5G}{h^2+b^2}$$

$$I_v^{**} = \frac{2}{G}\beta_1 E_1 = \frac{5b^3h^3}{18(h^2+b^2)}$$

$$b = 4h \Rightarrow I_v^{**} = \frac{160}{153}h^4 \approx 1{,}0457h^4$$

$$I_v^{**} < I_v < I_v^{*}$$

$$1{,}0457 \cdot h^4 < 1{,}124h^4 < 1{,}255h^4$$

```
> ? int
> \phi:=(x^2-(b/2)^2)*(y^2-(h/2)^2);
> fx:=diff(\phi,x);
> fy:=diff(\phi,y);
```

$$\phi := \left(x^2 - \frac{b^2}{4}\right)\left(y^2 - \frac{h^2}{4}\right)$$

$$fx := 2\,x\left(y^2 - \frac{h^2}{4}\right)$$

$$fy := 2\left(x^2 - \frac{b^2}{4}\right)y$$

```
>
> D11:=simplify(int(int(fx^2+fy^2,x=-b/2..b/2),y=-h/2..h/2));
```

$$D11 := \frac{1}{90}\,b^3\,h^5 + \frac{1}{90}\,h^3\,b^5$$

```
> E1:=simplify(int(int(\phi,x=-b/2..b/2),y=-h/2..h/2));
```

$$E1 := \frac{b^3\,h^3}{36}$$

```
> b1:=simplify(2*G*E1/D11);
```

$$b1 := \frac{5\,G}{h^2 + b^2}$$

```
> Ipp:=simplify(expand(simplify(2/G*b1*E1)));
```

$$Ipp := \frac{5\,b^3\,h^3}{18\,(h^2 + b^2)}$$

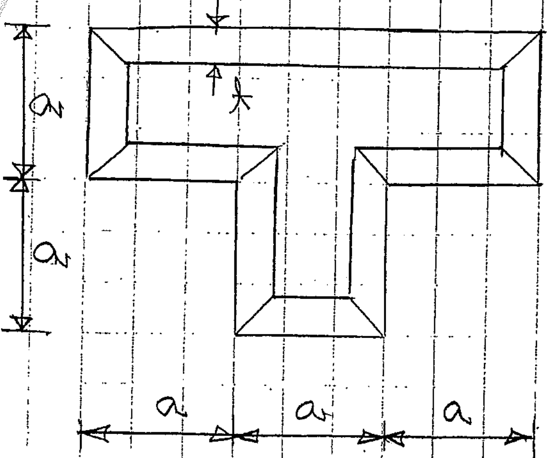
```
> vast:=simplify(subs(b=4*h,Ipp));
```

$$vast := \frac{160\,h^4}{153}$$

```
> evalf(vast);
```

$$1.045751634\,h^4$$

```
>
```

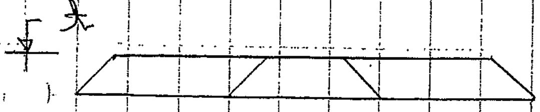


Arvioidaan kuvan profiilin vääntöneliömomenttia I_v:tä ortogonaalisten jännitystilakenttien avulla.

$$I_v = \sum \{ \sum I_{va} \}$$

$$I_{va} = \frac{[2 V_A]^2}{A_A^*} \quad (2.305)$$

jossa V_A on keon tilavuus
A_A^* on lappeiden vaaka-ala



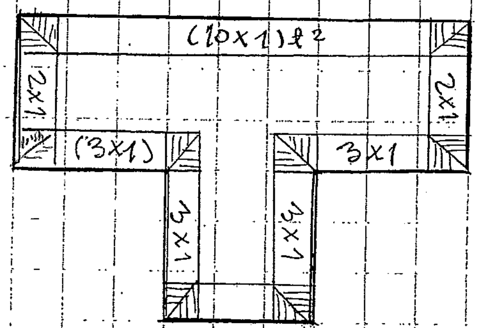
(1) ensimmäinen keko $t = a/4$

$$V_1 = \frac{1}{3}\{(4t)^2 * 2t - (2t)^2 * t\} + \frac{1}{2}\{4t*8t*2t - 2t*8t*t\} + \frac{1}{2}\{(4t)^2 * 2t - 4t*2t*t\}$$

$$V_1 = 45,333\, t^3$$

$$A_1 = [4t*16t - 2t*14t] = 36 t^2$$

$$I_{v1} = \frac{(2*45,333 t^3)^2}{36 t^2} = 228,35\, t^4$$



Lappeiden suorakaiteet

$(10t*t) + 4*(3t*t) + 3*(2t*t)$

$0,312*10t^4 + 4*0,263*3t^4 + 3*0,229*2t^4 = 7,650\, t^4$

$$I_{v,lappeet} = 7,650\, t^4$$

Loppuosa (katto) tulkitaan pitkäksi suorakaiteeksi

$$I_{v,katto} = \frac{1}{3} * 14t * (2t)^3 = 37,333\, t^4$$

$$I_{v,tot} = I_{v1} + I_{v,lappeet} + I_{v,katto} = 273,333\, t^4 = 1,068\, a^4$$

$V = 2t \cdot 4t \cdot t + 10t \cdot 2t \cdot t + 10t^2 t/2 + 4 \cdot 3t^2 t/2 + 3 \cdot 2t^2/2$
$+ 2\frac{1}{3}(2t)^2 t = 44,67$