SOME PROBLEMS IN NUMERICAL POST-BIFURCATION ANALYSIS

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ABSTRACT

A critical review of the existing methodology for the numerical treatment of post-bifurcation branches at a multiple bifurcation point is presented. In the critique the main emphasis is given to robustness, but also implementational issues and computational requirements as well unsolved problems are addressed.

1 INTRODUCTION

Path following is the most common procedure to analyze the stability behaviour of complex structures. In these methods a one dimensional equilibrium curve is traced. Difficulties are expected to exist with the basic continuation algorithms near critical points where the tangent stiffness matrix is ill-conditioned. If the multiplicity of the critical point is one, the numerical treatment of post-bifurcation paths is rather straightforward, but the situation changes dramatically when the multiplicity of the critical point is greater than one. Occurrencies of multiple bifurcation points in a numerically traced equilibrium path are quite rare. However, circumstances where the buckling loads are clustered in a small interval just above the smallest critical load are frequent in the stability analysis of various shell structures. Such an occurrence might lead to an exceedingly complicated situation, where the buckling modes interact with each other. Therefore it seems natural to require a branching procedure to be able to handle multiple bifurcation points (or near ones) in a continuation algorithm.

In the scientific literature there are only few papers concerning the numerical treatment of multiple bifurcation. Among them belong the works of Kearfott [1], Allgower and Chien [2] and Huitfeldt [3]. In this article those methods are briefly reviewed and the existing difficulties with these approaches are pointed out.

Also an approach which uses the Koiter type reduction technique is presented and its properties are compared to the above mentioned techniques.

Only the question of the computation of post-bifurcation branches is discussed in the present paper. Relevant related problems like location and computation of the critical point itself have been left out.

2 BASIC DEFINITIONS

Discretized form of static equilibrium equations under single load control can be written in the form

$$\boldsymbol{f}(\boldsymbol{q},\lambda) = \lambda \boldsymbol{p}_r - \boldsymbol{r}(\boldsymbol{q}) = \boldsymbol{0} \tag{1}$$

where $\boldsymbol{p}_r$ and $\boldsymbol{r}$ are the vectors of external and internal forces.

It is assumed that the magnitude of the loading is controlled by a single parameter λ, called the load parameter. The N-dimensional state variable vector is denoted as $\mathbf{q}$ and in most applications N is large. In the solution of the non-linear equilibrium equation (1) the tangent stiffness matrix $\boldsymbol{K}$ is usually needed (at step k)

$$\boldsymbol{K}_k = -\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\bigg|_{\boldsymbol{q}_k}.$$

In order to parametrize the path a length measure, i.e the path parameter s is defined as $\Delta s = \sum_{i=1}^{k} \sqrt{\boldsymbol{t}^T \boldsymbol{C}\boldsymbol{t}}$, where $\boldsymbol{t}^T = \left[\Delta \boldsymbol{q}^T \Delta\lambda\right]$ and $\boldsymbol{C}$ is a weighting matrix, see ref, [4].

It is now assumed that the critical point $(\boldsymbol{q}_{cr}, \lambda_{cr})$ is reached and located for prescribed accuracy between steps $k-1$ and k. Notification of new critical modes during the continuation can be obtained by monitoring the inertia of the tangent stiffness matrix. If the number of unstable modes associated with the appearance of the noticed critical point(s) is M, then

$$|\mathrm{p}(\boldsymbol{K}_k) - \mathrm{p}(\boldsymbol{K}_{k-1})| = M,$$

where $\mathrm{p}(\boldsymbol{K})$ stands for the number of positive eigenvalues. This does not necessarily mean that the lowest critical point itself is a M-fold critical point, i.e.

$$\dim(\ker \boldsymbol{K}_{cr}) = K \leq M.$$

However, it is assumed in the sequel that the rank deficiency of the tangent stiffness matrix at the critical point equals to M and no other critical points lie on the primary path in the last increment.

The number of positive eigenvalues can be determined using the Sylvester law of inertia by counting the number of positive diagonal elements in the $\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$-factorized stiffness matrix. If the linear system is solved by iterative methods, like preconditioned conjugate gradient (PCG) method, the inertia of the stiffness matrix is not easily obtainable. If the preconditioning matrix is denoted by $\boldsymbol{M}$, the outer eigenvalues of the preconditioned operator $\boldsymbol{M}^{-1}\boldsymbol{K}$ can be easily obtained from the tridiagonal matrix related to the underlying Lanczos iteration. However, it is not clear if these eigenvalue estimates can be used in the continuation algorithm.

Multiplicity of the critical point is here defined to be the dimension of the nullspace of the tangent stiffness matrix at the critical point. Other definitions exist. Bauer et al. [5] defined the multiplicity of the bifurcation point to be M if $2M+2$ half rays meet there. Since the number of emanating branches is not known a priori, the previously mentioned definition seems to be more appropriate. It is probably the most common definition adopted in the literature.

3 BRANCH SWITCHING ALGORITHMS

In this section a short review of the existing branch switching techniques for multiple bifurcations is given. The task of these algorithms is to seek solutions for the rate of load parameter $\dot{\lambda}$ and the rates of the projections of the tangent vectors $\dot{a}_i$ of the branches onto the critical eigenmodes $\boldsymbol{\phi}_i, i = 1, ..., M$.

Rheinboldt [6] developed an elegant and computationally favourable branch switching algorithm for simple bifurcation. He also described generalization of his method to multiple bifurcation. However, the question of initial values for the projections $\dot{a}$ remained unanswered.

Keller [7] presented four algorithms, which are denoted methods I-IV. The method I uses a perturbation approach and the solutions for the branch directions are obtained from the algebraic bifurcation equation (ABE). In the evaluation of the coefficients in the ABE, the second derivatives of the residual vector $\boldsymbol{f}$ are needed, or they need to be approximated by finite differences. This method will fail when the ABE is degenerate, e.g. at symmetric bifurcations. In order to avoid the determination of coefficents of the ABE, Keller proposes the method II where the idea is to seek solutions on some subset parallel to the tangent but displaced from the bifurcation point in some direction normal to the tangent. Obviously this method will work well in simple bifurcations, but the problem with multiple bifurcation is how to parametrize in a reasonable way the subset where the solution is to be found. The remaining two methods III and IV seems to be the most robust and also computationally the most demanding. Since they are described in ref. [7] only in the case of simple bifurcation and they have some resemblance with the Koiter's perturbation approach, only the connections to the proposed method are pointed out in the following discussion.

Kearfott [1] developed a technique, where in principle, all solution arcs can be found by locating the minima of $\|\boldsymbol{f}\|$ in the region near the critical point spanned by the critical eigenvectors, i.e finding the solutions branches on a sphere centered to the estimate of the critical point. Drawback of this method is that it needs numerous evaluations of the residual $\boldsymbol{f}$. Determination of the the necessary resolution needed to find all solutions is an open question. If the resolution to scan over the sphere is too low, the probability of missing some brances increases, however tightening the resolution increases the computational cost. Huitfeldt [3] included also the tangent vector of the primary path in the definition of the sphere where the minimization takes place. Pajunen [8] has used the residual minimization technique to solve double bifurcation problem of a truss structure.

Allgower and Chien [2] used the local perturbation method introduced by Georg [9] to multiple bifurcation problems. The idea is to introduce a perturbation near the bifurcation point and solve the perturbed problem

$$\boldsymbol{f}(\boldsymbol{q}, \lambda) + \tau \boldsymbol{b} = \boldsymbol{0} \tag{2}$$

from a point on the primary path and traverse a perturbed path until it is near a point on a branch. The theoretical foundation of this method is based on a version of a generalized Sard's theorem. For successful branching the choice of the perturbation

vectors plays a key role. In their numerical examples the components in the perturbation vectors are chosen in such a way that they oscillated correspondingly to those of the bifurcating solutions. This means that one should have a priori knowlegde of the solution of the problem which has to be solved. No specific theory or rules for the selection of the perturbation vectors was given in ref. [2], and the method seems to be used best as computing the solution curves interactively by trial and error fashion.

A major improvement to the local perturbation algorithm is given by Huitfeldt [3]. He introduced an auxiliary equation which defines with the perturbed equilibrium equations (2) a closed one dimensional curve in a $N+2$-dimensional space. This curve passes exactly one point on each branch (or half branch) of the unperturbed equation (1). When passing such a point the perturbation parameter τ changes sign. The problem is then to locate the zero points of the perturbation parameter τ while traversing the branch connecting curve (BCC). Thus the branch switching problem is reduced to a path following task of the augmented system

$$\boldsymbol{h}(\boldsymbol{q},\lambda,\tau) = \left\{ \begin{array}{rcl} \boldsymbol{f}(\boldsymbol{q},\lambda) + \tau \boldsymbol{b} & = & \boldsymbol{0} \\ c_b(\boldsymbol{q},\lambda,\tau) & = & 0 \end{array} \right. , \tag{3}$$

which can be solved with standard continuation algorithms. A constraint that defines a closed surface around the critical point is of spherical (elliptical) form:

$$c_b(\boldsymbol{q},\lambda,\tau) = \tfrac{1}{2}\left(\|\boldsymbol{q}-\boldsymbol{q}_{cr}\|_W^2 + \alpha^2(\lambda-\lambda_{cr})^2 + \beta^2\tau^2 - \rho^2 \right), \tag{4}$$

where α, β are scaling factors and ρ is the radius of the sphere. In principle this method does not need expensive evaluation of the basis of the nullspace of the tangent stiffness matrix. Huitfeldt [3] used a random vector as perturbation $\boldsymbol{b}$.

There are some shortcomings with this conseptually simple and elegant method. It is not known if the branch connecting equation always defines a closed curve. It is believed, as also argued by Huitfeldt, that using a constraint which defines a closed surface, guarantees a closed path defined by the branch connecting equation (3,4). No mathematical proof of this is known to the authors. Secondly, there is no guarantee that all bifurcating branches have been found. This obviously depends on the choice of the perturbation. In addition, the computational expense can be very high for large problems, fortunately it grows only linearily with respect to the emanating branches from the bifurcation point[1]. However, the number of branches in multimode buckling with higher multiplicity can be very large as will be explained in the following.

An essential feature for the construction of a reliable bifurcation procedure is the determination of the number of possible solutions branches emanating from the critical point. This problem has been explored in the late 60's by Sewell [10], [11], Johns and Chilver [12],[13]. Depending on the symmetry properties of the system, the maximum number of different post-buckling branches is

$$2^M - 1 \quad \text{or} \quad \tfrac{1}{2}(3^M - 1)$$

[1]It is assumed that for reliable detection of the zeros of the perturbation parameter on the BCC, a minimum number of steps, say 4-5, has to separate two consecutive roots.

for a system without symmetry or perfectly symmetric, respectively. The minimum number of post-buckling paths is 1 for the former case and M for the latter. The complexity of a multi-mode buckling problem grows enormously with the multiplicity of the critical point.

In order to develope a robust branch switching algorithm it seems natural to reduce the problem into a smaller one and to try to get as much information as possible from the reduced system, see e.g. [14]. Koiter's initial postbuckling theory is based on perturbation formulation resulting in a strongly reduced potential energy function, the variables being the amplitudes of the relevant buckling modes. The number of "post-buckling equilibrium equations" derived from the reduced potential energy expression equals the multiplicity of the buckling load or the number of pertinent interacting modes. A series expansion for the displacement field is used in the form [2]

$$\boldsymbol{q} = \lambda \boldsymbol{q}_r + \sum_{i=1}^{M} a_i(\lambda)\boldsymbol{q}_i + \sum_{i,j=1}^{M} a_i(\lambda)a_j(\lambda)\boldsymbol{q}_{ij},$$

where $\boldsymbol{q}_r$ and $\boldsymbol{q}_i$'s denote the reference displacement vector and buckling modes, $\boldsymbol{q}_{ij}$'s are the second order post-buckling fields and a_i's are the unknown amplitudes. The Koiter's approach consists of the following steps:

1. solution of the eigenvalue problem in order to get the relevant eigenmodes,
2. solving the second-order displacement fields,[3]
3. evaluation of the coefficients of the reduced system,
4. solution of the reduced set of equilibrium equations.

Since the dimension of the reduced problem is very small, any robust solution scheme can be applied. Notice that these equations are polynomial, hence, it is possible to find all the solutions with algorithms described in ref. [15].

Solving the amplitude equation in the vicinity of the critical point gives the local form of the equilibrium surface of the structure. The most severe limitation is that the range of validity of the results obtained is difficult to judge. Therefore the perturbation method has primarily been considered as an "analytical tool" to get qualitative picture of the behaviour of the initial post-buckling regime.

Another problem in the initial post-buckling method is to decide how many eigen-modes are relevant in the expansion. If one interacting mode is left out from the expansion, it will appear in the second order field [16]. However, the range of validity can be extremely small in those cases. An example of that is given in ref. [16] where a T-beam is analysed. The interacting buckling modes comprise two local and one overall mode, the critical load of which is higher than the loads corresponding the local modes. If the overall mode is left out from the series expansion, the resulting two mode analysis deviates rapidly from the three mode analysis after the secondary bifurcation point, which lies in the immediate vicinity of the primary bifurcation point.

[2] Here the behaviour on the primary path is assumed to be almost linear.

[3] In Keller's approach III [7] the second-order fields have to be solved from a non-linear equation system. It is also unclear how the amplitudes in his method are determined in multimode buckling problems.

4 EXAMPLE

A well known example of multiple bifurcation is the double bifurcation of a compressed flat simply supported plate, fig. 1. A plate with aspect ratio $\sqrt{2}$ is chosen as a test example. This problem has also been analysed by Huitfeldt [3] and Lidström [17], however, no post-bifurcation paths have been presented. The plate is discretized by uniform 20×10 quadrilateral mesh using bilinear stabilized MITC type elements with drilling rotations (1304 dof). The stabilization parameter (shear reduction) for the MITC element has been 0.4 [18], [19], [20]. Full 2×2 Gaussian integration is used in evaluation of the element stiffness matrices and internal force vectors. The loaded edges are constrained to remain straight, but the in-plane deflections are allowed for the longitudinal sides (Hemp type boundary conditions). The analytical buckling load has the value $P_{cr} = 4.5\pi^2 D/L$, where L is the length of the loaded side and D is the bending rigidity of the plate $D = Et^3/12(1-\nu^2)$. The buckling modes corresponding to this double bifurcation load have one or two half waves in the x-axis direction. The length to thickness ratio is $L/t = 100$ and the Poisson's ratio has the value $\nu = 0.3$. In the numerical computation the value obtained is $4.39\pi^2 D/L$ interpolated from the zero point of the lowest eigenvalue of the tangent stiffness matrix, which is easily computed at the beginning of each increment by applying few inverse iterations. If an eigenvalue buckling analysis is performed, the double eigenvalue will split in two separate eigenvalues with values 4.32 and $4.40\pi^2 D/L$.

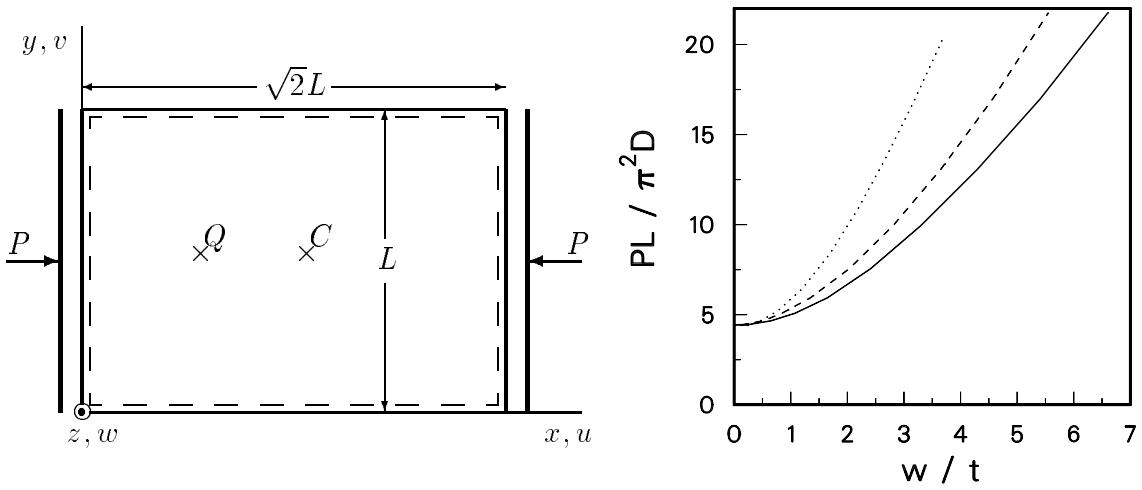

Figure 1: (a) Simply supported plate, (b) load deflection curves; solid line = w_C mode 1 branch, dashed line = w_Q mode 1 branch, dotted line = w_Q mode 2 branch.

This example is not particularly difficult, since there are only two post-buckling branches, deformation patterns of which are just like the buckling modes. The load deflection paths are shown in fig. 1. C and Q refer to the center and quarter points of the plate. Deformed shapes at the end of the continuation are shown in fig. 2.

Only Huitfeldt's approach and the branch-switching scheme based on Koiter's initial post-buckling theory are used in the computations. Results based on Huitfeldt's approach are reported first. Plot of the perturbation parameter τ with respect to the

Figure 2: Deformed states at post-buckling branches 1 and 2.

arc-length along the branch connection curve is shown in fig. 3 as well as the distance from the starting point of the branch connecting curve. The branch connecting curve is traced with 56 increments and it has six roots for the perturbation parameter (two on primary path and four on post-buckling branches) and it constitutes the main computational effort, since the primary path and one branch is traversed within 14 steps.

Magnitude of the perturbation load is chosen in such a way that the maximum deflection caused by the perturbing load equals to the value $0.2t$. In computations, shown in figs. 1, 3 the perturbation load vector is chosen to have only one point load at the quarter point Q. The distance d from the begining of the BCC tracing is defined by

$$d = \left(\|\boldsymbol{q} - \boldsymbol{q}^*\|_W^2 + \alpha^2(\lambda - \lambda^*)^2 + \beta^2\tau^2 \right)^{1/2},$$

where $\boldsymbol{q}^*, \lambda^*$ is the starting point on the primary path.

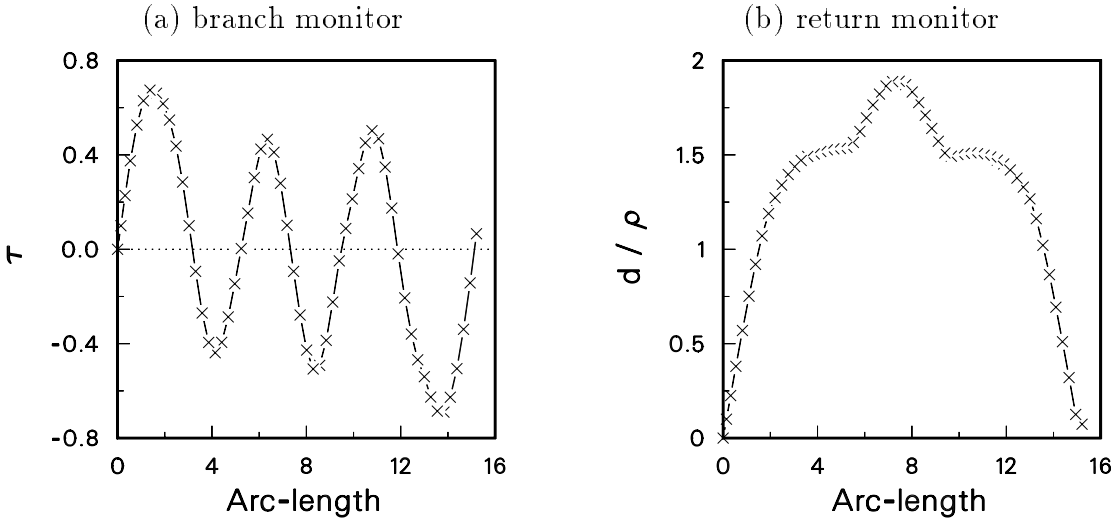

Figure 3: Double bifurcation of simply supported plate. Traversing branch connecting curve. The arc-length is normalized with respect to the radius ρ of the spherical constraint.

If a random vector is used for the pertubation, the BCC-tracing failed for several magnitudes tried. For values $\|\boldsymbol{q}_b\|_{W,max} = 0.2t, 0.3t$ the iterations did not converge even if the step size was halved five times[4]. When using the value $\|\boldsymbol{q}_b\|_{W,max} = 0.1t$ the BCC-tracing was stopped after 200 steps and after finding 28 zeros[5] for the perturbation parameter. Even if the closed surface constraint for the BCC is used, this particular perturbation vector does not seem to define a closed path. The perturbation parameter with respect to the normalized arc-length as well as the distance from departure are shown in fig. 4.

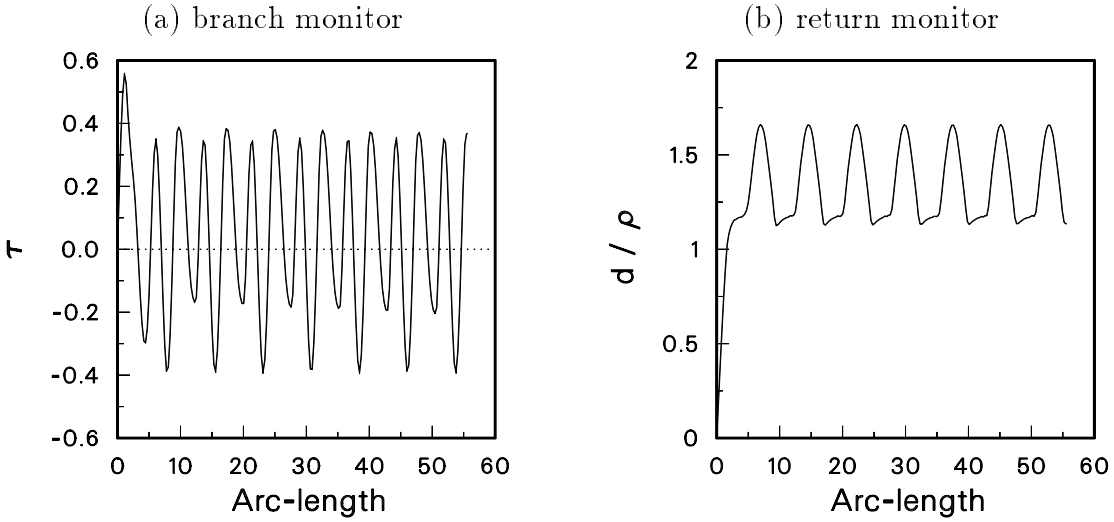

Figure 4: Failure in traversing BCC; the arc-length is normalized with respect to the radius ρ of the spherical constraint.

In the Koiter type perturbation method the reduced potential energy has the form

$$V(a_1, a_2) = \tfrac{1}{2}\left[(1 - (\lambda/\lambda_{cr}))\left(a_1^2 + a_2^2\right)\right] + A_{1111}a_1^4 + A_{1122}a_1^2a_2^2 + A_{2222}a_2^4,$$

where a_1 and a_2 are the dimensionless amplitudes of the buckling modes. In this simple example the two post-buckling branches emanating from the critical point can be solved analytically and they are simply:

- mode 1 branch:
$$\lambda/\lambda_{cr} = 1 + 4A_{1111}a_1^2, \quad \text{and} \quad a_2 \equiv 0,$$
- mode 2 branch:
$$\lambda/\lambda_{cr} = 1 + 4A_{2222}a_2^2, \quad \text{and} \quad a_1 \equiv 0.$$

The values a_1 and a_2 which are used in the prediction step onto the post-buckling branch can be defined by fixing the maximum displacement of the predictor. Solution time

[4]The norm $\|\boldsymbol{q}_b\|_{W,max}$ denotes the maximum norm taken from pure displacement components (rotations excluded). $\boldsymbol{q}_b$ is the displacement vector caused by the perturbing load $\boldsymbol{b}$.

[5]In this example the BCC tracing should be stopped after finding seven zeros without returning to the point of departure.

which is needed for this kind of branch-switching algorithm is a fraction compared to the Huitfeldt's method. Nevertheless, the Huitfeldt's approach can be used to solve the small reduced polynomial equation system instead of using the polynomial continuation methods.

5 DISCUSSION

So far all existing branch switching techniques which can be used in multiple bifurcation problems have some annoying features. In principle Huitfeldt's approach for traversing the branch connecting curve requires only a path following routine, no other specific algorithms are needed. This is in contrast to other branch switching methods which requires the basis of the nullspace of the tangent stiffness matrix, i.e. the eigenmodes. However, in practise also with Huitfeldt's approach, some knowledge on the critical eigenmodes seems to be necessary in order to construct a proper perturbation load.

To solve the eigenvalue problem at the critical point is the most time consuming part of the proposed branch switching algorithm which uses the Koiter-type reduction method. It is believed that this approach is also much more economical with respect to computing time than Kearfott's minimization procedure, in which a lot of residual computations are needed. The price which has to be paid is the formulation of the "second-order" load vectors, where the second order derivatives of the residual appear.

As mentioned before, the range of applicability of the Lyapunov-Schmidt-Koiter type reduction can be very narrow due in the case that some relevant interacting modes are left out from the series expansion. However, this usually manifests itself by the appearance of secondary bifurcations close to the primary one. It is extremely difficult to automate the selection of the relevant buckling modes. Thus, human expertise in performing stability computations involving interactive buckling phenomena is crucial for successful analysis.

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