ON VISCOPLASTIC REGULARIZATION OF STRAIN SOFTENING SOLIDS

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Dedicated to our mentors in mechanics Martti Mikkola and Eero-Matti Salonen

ABSTRACT

Some consequences of using viscoplastic material models in the failure analysis of strain softening solids are examined. Especially factors affecting the localisation band width under quasi-static loading conditions are studied.

1 INTRODUCTION

Strain localisation can be observed in the mechanical behaviour and failure of various materials. The origin of localisation lies at the micro-level of the material action. At a macroscopic level a softening response is observed. Classical, i.e. local continuum models embody an implicit assumption that the deformation of a body varies in a sufficiently smooth manner. This assumption is not valid when strain localisation occurs.

Application of descending relation between stress and strain in a rate independent solid within the classical continuum description results in the loss of well-posedness of the governing initial-boundary value problem. The governing hyperbolic equations of motion ceases to be hyperbolic and became elliptic in the softening regime. Therefore, the domain is split into an elliptic part, in which the waves have imaginary wave speeds and are not able to propagate (standing waves) and into a hyperbolic part with propagating waves. As a consequence, a spurious sensitivity to discretizations is observed in numerical simulations of localisation problems. Several approaches to resolve this problem have been introduced in the literature. Perhaps the simplest one which preserves well-posedness of the governing equations is incorporation of the strain rate dependence to the constitutive model. The viscosity contribution introduces a material length scale even though the constitutive equations do not explicitly contain a parameter with the dimension of length. Other strategies like using micropolar continuum (or more generally microcontinuum) or higher-order gradient continuum models result in much more complicated equations and their numerical treatment is also more involved.

2 VISCOPLASTIC MATERIAL MODELS

Two major categories in formulating rate-dependent or viscoplastic material models exists: the overstress format and the consistency approach. Today, two mostly used overstress models are the Perzyna and the Duvaut-Lions models. In the Perzyna model [1], the direction of viscoplastic flow is determined by the gradient of a plastic potential function calculated at the current stress point. In the Duvault-Lions [2] model the viscoplastic flow is determined by the difference between the current stress point and the closest point projection onto a static yield surface, also called as the back-bone model. For Perzyna viscoplastic model, use of the postulate of maximum dissipation is rather involved. In the consistency models, a dynamic rate-dependent yield surface is defined which allows the use of the postulate of maximum dissipation in a straightforward manner, as shown by Ristinmaa and Ottosen [3].

Sluys has analysed the properties of several regularizing techniques for strain softening solids in his dissertation [4]. He has concluded that the viscoplastic models regularize the governing equations of motion at deformation states in fracture zones (mode-I localisation) and in shear bands (mode-II localisation). Wave propagation in viscoplastic solid is dispersive, which is necessary to capture localisation phenomena.

In the Perzyna model the viscoplastic strain rate is defined by

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \frac{1}{\eta}\phi(f)\frac{\partial g}{\partial \sigma_{ij}}, \tag{1}$$

where η is the viscosity parameter and ϕ is some function of the yield function f and g is the plastic potential. Common choises for the overstress function ϕ are the power laws

$$\phi(f) = \left\langle \frac{f}{\sigma_0} \right\rangle^p \qquad \text{or} \qquad \phi(f) = \left\langle \frac{f}{\bar{\sigma}} \right\rangle^p, \tag{2}$$

in which p is a material parameter and $\bar{\sigma}, \sigma_0$ are the current yield stress and the initial value of it, respectively. The notation $\langle y \rangle$ refers to $yH(y)$ where H is the Heaviside unit step function.

3 ANALYSIS OF LOCALISATION – ONE-DIMENSIONAL PROBLEM

Needleman [5] studied the material rate dependency in localisation problems. He concluded that the imperfection or inhomogeneity completely determines the localisation length scale in quasi-static problems and in dynamic problems it is a characteristic length of propagation of elastic waves. As it will be seen in the following sections, this conclusion holds on only partially for quasi-static cases.

Wang investigated the width of the localisation zone in viscoplastic von Mises solid under dynamic loading [6]. In the absense of imperfections the key parameters which determine the width of the localisation band are the viscosity η and the softening modulus h. However, in imperfect cases the imperfection size l_{imp} dominates the width of the localisation band when the value l_{imp} is smaller than the material length scale l_{mat}. On the other hand the influence of the imperfection disappears when the imperfection size is larger than the material length scale [6], thus

$$l_{\mathrm{loc}} = \min(l_{\mathrm{mat}}, l_{\mathrm{imp}}). \tag{3}$$

Wang investigated the localisation bandwidth in the case of pure shearing under dynamic loading. Approximation to the width of the localisation zone is (linearized from eq. (6.42) in Ref. [6])

$$l_{\mathrm{mat}} \approx \frac{c\eta\sigma_0}{\frac{3}{4}G - h}, \tag{4}$$

where G is the shear modulus and $c = \sqrt{G/\rho}$ is the elastic shear wave velocity.

From the numerical experiments it can be concluded, that the average strain rate has an influence on the localisation length scale in quasi-static cases. To investigate this phenomenon a simple tensile bar of length L under quasi static loading is considered. The other end of the bar is clamped and the axial displacement is prescribed at the other end with a constant rate. Since there are no body forces and the modulus of elasticity is assumed to be constant, the equilibrium equation is simply

$$\sigma_{,x} = E(\epsilon_{,x} - \epsilon^{\mathrm{vp}}_{,x}) = 0. \tag{5}$$

Taking the kinematical equation into account, the equilibrium equation can be written as

$$u_{,xx} = \epsilon^{\mathrm{vp}}_{,x}, \tag{6}$$

where u is the axial displacement. The viscoplastic strain can be solved from the rate equation

$$\dot{\epsilon}^{\mathrm{vp}} = \frac{1}{\eta}\frac{\sigma - \bar{\sigma}}{\sigma_0} \qquad \text{where} \qquad \bar{\sigma} = \sigma_0 + h\epsilon^{\mathrm{vp}}, \tag{7}$$

and h is the hardening/softening modulus. It is assumed for simplicity, that the exponent p has the value $p = 1$. The initial yield stress σ_0 is a function of the position x

$$\sigma_0(x) = \hat{\sigma}_0(1 - \theta(x)), \tag{8}$$

where the perturbation $\theta(x)$ is small, i.e. $\theta(x) \ll 1$. Substituting the constitutive equation $\sigma = E(\epsilon - \epsilon^{\mathrm{vp}})$ into (7) gives a first order ordinary diffrential equation for the viscoplastic strain

$$\dot{\epsilon}^{\mathrm{vp}} + \frac{E + h}{\eta\sigma_0}\epsilon^{\mathrm{vp}} = \frac{1}{\eta}\left(\frac{E}{\sigma_0}\epsilon - 1\right). \tag{9}$$

Solution of this equation can be written as

$$\epsilon^{\mathrm{vp}}(x,t) = \mathrm{e}^{-a(x)t}\epsilon^{\mathrm{vp}}(x,t_0) + \int_{t_0}^{t} \mathrm{e}^{-(a(x)t - a(x)s)}\frac{1}{\eta}\left(\frac{E}{\sigma_0(x)}\epsilon(x,s) - 1\right) ds, \tag{10}$$

where

$$a(x) = \frac{E + h}{\eta\sigma_0(x)}. \tag{11}$$

Substituting this expression into the equilibrium equation (6) gives an integro-differential equation

$$u_{,xx}(x,t) = \frac{\partial}{\partial x}\left[\mathrm{e}^{-a(x)t}\epsilon^{\mathrm{vp}}(x,t_0) + \int_{t_0}^{t} \mathrm{e}^{-(a(x)t - a(x)s)}\frac{1}{\eta}\left(\frac{E}{\sigma_0(x)}u_{,x}(x,s) - 1\right) ds\right] \tag{12}$$

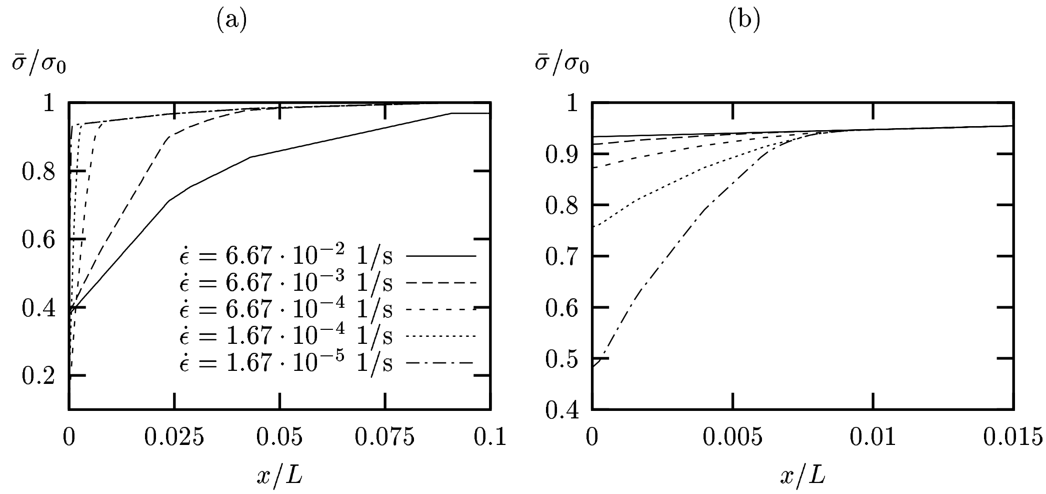

Figure 1: One-dimensional problem; current yield stress as a function of position: (a) band development with different strain rates, (b) evaluation of a localisation band, average strain rate $\dot{\epsilon} = 6.67 \cdot 10^{-4}$ 1/s.

which has to be solved in the viscoplastic domain. In the elastic unloading part the right hand side of equation (12) vanishes. The boundary conditions are

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = \dot{u}_L\, t, \tag{13}$$

where $\dot{u}_L$ is the prescribed constant velocity of the right end of the bar.

From equation (12) it seems more tractable to assume the initial yield stress to be constant over the bar and to consider the imperfection in the form of initial viscoplastic strain. Wang [6] used constant reduction of the yield strength in the imperfect part of the domain in analysing the width of the localisation zone in dynamic case. Numerical quasi-static test cases reveal that in the case of a constant reduction of the yield stress the size of the imperfect domain totally determines the localisation zone in the one-dimensional case. However, this can be considered to be in accordance to the results of Wang [6], see equation (3), since the material lenght scale in the quasi-static case is infinite.

However, if the reduction of the yield stress is not constant, the width of the localisation band depends on the loading rate, see Fig. 1a, where the current yield stress $\bar{\sigma}$ is plotted along the imperfect part of the bar. The imperfection has been an initial viscoplastic strain near the clamped end of the bar reaching to the point $x = 9L/100$. The initial viscoplastic strain distribution is obtained from an *a priori* analysis in which the bar is loaded with a linearly varying distributed force. Uniform mesh with 4000 linear elements is used in the computations.

In Fig. 2a the localisation band width is shown as a function of average strain rate as observed in the numerical computations. Dependency on the strain rate is almost linear, except near origin at very low strain rates. It seems that the loading rate in quasi-static problems functions as a stress wave speed in the dynamic case, see equation 3.

Also the effect of the stress exponent p on the localisation band width has been studied and the result is shown in Fig. 2b for values $p \in (1, 2.25)$. For larger p-values the viscoplastic strain spread out the imperfect zone. However, in these cases the rate of the viscoplastic

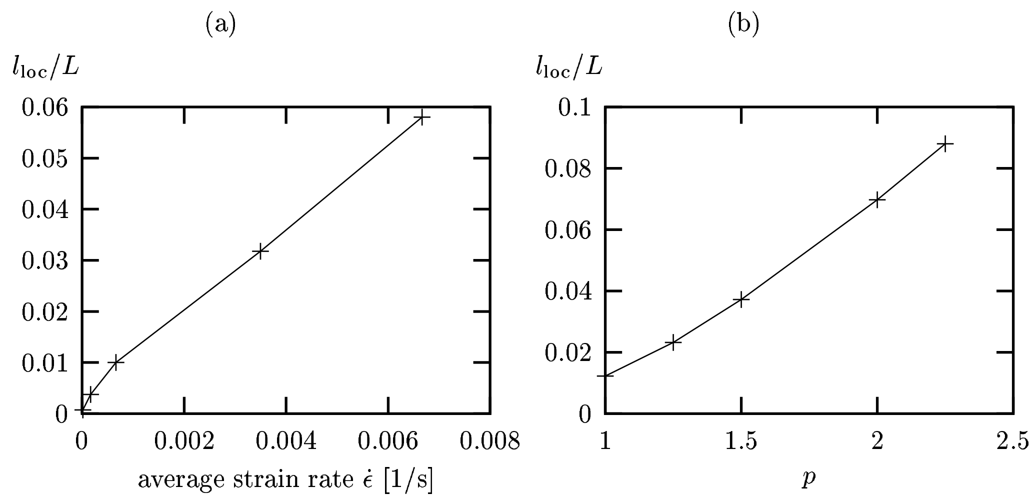

Figure 2: (a) Width of the localisation zone as a function of average strain rate and (b) as a function of the exponent p (loading rate $\dot{\epsilon} = 6.67 \cdot 10^{-4}$).

strain is larger in the imperfect area of the bar. The imperfection has been a linear reduction (10 % reduction at the clamped end) of the initial yield stress in the part $x \in (0, 0.09L)$.

4 ALGORITHMIC TREATMENT

For rate-dependent solids implicit time integrators are preferable. The critical time step of explicit methods for the Perzyna type viscoplastic model is of order $\Delta t_{\rm cr} \sim \eta\sigma_0/(pE)$, which results in a value of order 10^{-3} s for the material parameters used in the example in section 5. Especially for quasi-static cases it is untolerably small. In this study the backward Euler scheme is used to integrate the viscoplastic constitutive models.

Incremental relation for stress and strain can be written as

$$\Delta\boldsymbol{\sigma} = \mathbf{C}^{\rm el}(\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^{\rm vp}) \tag{14}$$

where the viscoplastic strain increment is

$$\Delta\boldsymbol{\epsilon}^{\rm vp} = \Delta\lambda\mathbf{m}, \qquad \text{where} \qquad \mathbf{m} = \frac{\partial g}{\partial\boldsymbol{\sigma}} \qquad \text{and} \qquad \Delta\lambda = \frac{\Delta t}{\eta}\phi(f). \tag{15}$$

The algorithmically consistent tangent matrix, which is needed in the global equilibrium equations, can be derived from the iterative counterpart of (15) and the iterative change of the viscoplastic strain is

$$\delta\boldsymbol{\epsilon}^{\rm vp} = \Delta\lambda\frac{\partial\mathbf{m}}{\partial\boldsymbol{\sigma}}\delta\boldsymbol{\sigma} + (\Delta\lambda\frac{\partial\mathbf{m}}{\partial\lambda} + \mathbf{m})\delta\lambda \tag{16}$$

Substituting expression (16) into the iterative counterpart of equation (14) results in

$$\delta\boldsymbol{\sigma} = \mathbf{H}\delta\boldsymbol{\epsilon} - \mathbf{H}\left(\mathbf{m} + \Delta\lambda\frac{\partial\mathbf{m}}{\partial\lambda}\right)\delta\lambda, \qquad \text{where} \qquad \mathbf{H} = \left((\mathbf{C}^{\rm el})^{-1} + \Delta\lambda\frac{\partial\mathbf{m}}{\partial\boldsymbol{\sigma}}\right)^{-1}. \tag{17}$$

The change in the viscoplastic multiplier can be solved from a scalar non-linear equation

$$r(\boldsymbol{\sigma}, \lambda) = \Delta\lambda - \frac{\Delta t}{\eta}\phi(\boldsymbol{\sigma}, \lambda) = 0. \tag{18}$$

By using the Newton's method the iterative change $\delta\lambda$ can be solved, giving

$$\delta\lambda = -\frac{r}{a} + \frac{\Delta t}{a\eta}\mathbf{n}^T\mathbf{H}\delta\boldsymbol{\epsilon}, \tag{19}$$

where

$$a = 1 + \frac{\Delta t}{\eta}\left[\mathbf{n}^T\mathbf{H}\left(\mathbf{m} + \Delta\lambda\frac{\partial \mathbf{m}}{\partial\lambda}\right) - \frac{\partial\phi}{\partial\lambda}\right] \qquad \text{and} \qquad \mathbf{n} = \frac{\partial f}{\partial\boldsymbol{\sigma}}. \tag{20}$$

Substituting the iterative change of the viscoplastic multiplier (19) back to the equation (17) gives the desired Jacobian matrix

$$\mathbf{C} = \mathbf{H} - \frac{\Delta t}{a\eta}\mathbf{H}\left(\mathbf{m} + \Delta\lambda\frac{\partial \mathbf{m}}{\partial\lambda}\right)\mathbf{n}^T\mathbf{H}. \tag{21}$$

This algorithmic tangent matrix is necessary for the Newton's method to obtain asymptotically quadratic convergence of the global equilibrium equations.

The local nonlinear problem (18) is also solved by the Newton's method. Increasing the power p in the constitutive model, makes the local problem more difficult and more iterations are needed in the Newton process to reach the asymptotic convergence domain.

5 NUMERICAL EXAMPLE

A biaxial compressed specimen is analysed where strain localisation into a shear band takes place at the onset of softening. The vertical displacement at the upper edge is prescribed at constant rate and constrained to remain horizontal. Associative viscoplastic flow is assumed with von Mises "yield function" f. The constitutive parameters have the following values: elastic shear modulus $G = 4000$ MPa, Poisson's ratio $\nu = 0.49$, initial yield stress $\sigma_0 = 100$ MPa, the linear softening modulus $h = -G/10$, the viscosity $\eta = 0.1$ s, the exponent $p = 1$. The width and height of the specimen are 60 mm and 240 mm, respectively. In Ref. [7] the same specimen is analysed as a rate independent solid by using both classical and gradient continuum formulations. To trigger the unstable localisation an imperfection is introduced with a 10 % reduction of the yield stress in one element at the bottom left-hand corner of the specimen. Four noded bilinear elements with mean dilatation formulation [8] are used in the computations.

In Fig. 3 deformed shapes of the specimen are shown from computations with average strain rate $\dot{\epsilon} = 10^{-2}$ s (Fig. 3a) and 10^{-3} s (Fig. 3b). It is clearly seen that the localisation band is wider for the higher strain rate case. Also the width of the localisation band near the imperfection is dominated by the size of the imperfect domain, but gradually widens when moving away from the imperfection. The behaviour is thus similar to the dynamic case [6].

Deformed states shown in Fig. 3 correspond to the computations where uniform meshes are used. However, there is no visible difference in the load-deflection curves if highly irregular meshes (internal nodal positions disturbed randomly) are used.

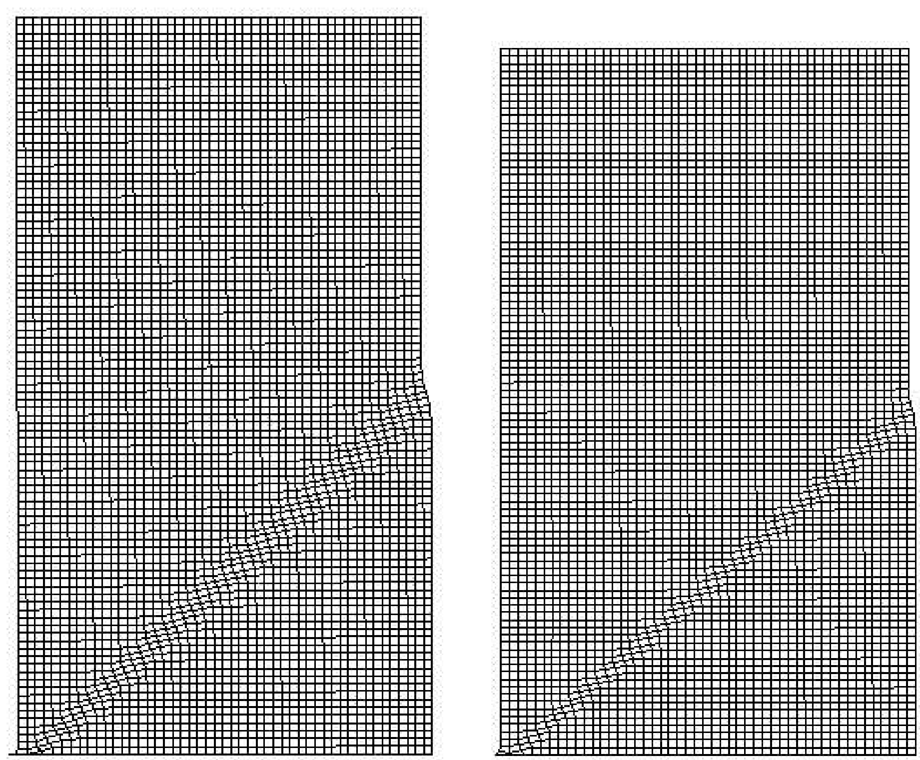

Figure 3: Deformed states: (a) everage strain rate $\dot{\epsilon} = 10^{-2}$ (displacements magnified 5 times) (b) $\dot{\epsilon} = 10^{-3}$ (magnification 7 times).

6 CONCLUDING REMARKS

Localisation of deformation of viscoplastic solid under quasi-static loading conditions has been studied. Size of the imperfection completely determines the localisation band width if the imperfection is distributed uniformly. However, if the imperfection is non-uniform, the width of the localisation zone is almost linearly dependent on the loading rate. Also the localisation band width grows linearly with the stress exponent p in the interval $p \in (1, 2.25)$. Naturally, at vanishing loading rate the localisation takes place at a point/line or plane depending on the dimension of the problem.

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