SOME OBSERVATIONS ON THE INTEGRATION OF INELASTIC CONSTITUTIVE MODELS WITH DAMAGE

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INTRODUCTION – Scalar model problems

Maxwell creep model $\dot{\epsilon}^{\mathrm{i}} = \tau_{\mathrm{vp}}^{-1}(\sigma/\sigma_{\mathrm{r}}) \quad \dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^{\mathrm{i}})$

$$\dot{\sigma} + \frac{E}{\tau_{\mathrm{vp}}}\frac{\sigma}{\sigma_{\mathrm{r}}} = E\dot{\epsilon}, \qquad \Longrightarrow \qquad \dot{y} + \frac{1}{\tau_{\mathrm{vp}}\epsilon_{\mathrm{r}}} y = \frac{\dot{\epsilon}}{\epsilon_{\mathrm{r}}}, \quad y(t_0) = y_0$$

where $y = \sigma/\sigma_{\mathrm{r}}$ and $\epsilon_{\mathrm{r}} = \sigma_{\mathrm{r}}/E$.

Kachanov/Rabotnov type damage model

$$\sigma = (1-D)E\epsilon, \quad \dot{D} = \frac{1+D}{\tau_{\mathrm{d}}}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^r \quad Y_{\mathrm{r}} = \frac{\sigma_{\mathrm{r}}^2}{2E}, \quad D(t_0) = D_0.$$

Solutions

For constant strain-rate loading $\dot{\epsilon} = \dot{\epsilon}_{\rm c}$.

The creep model

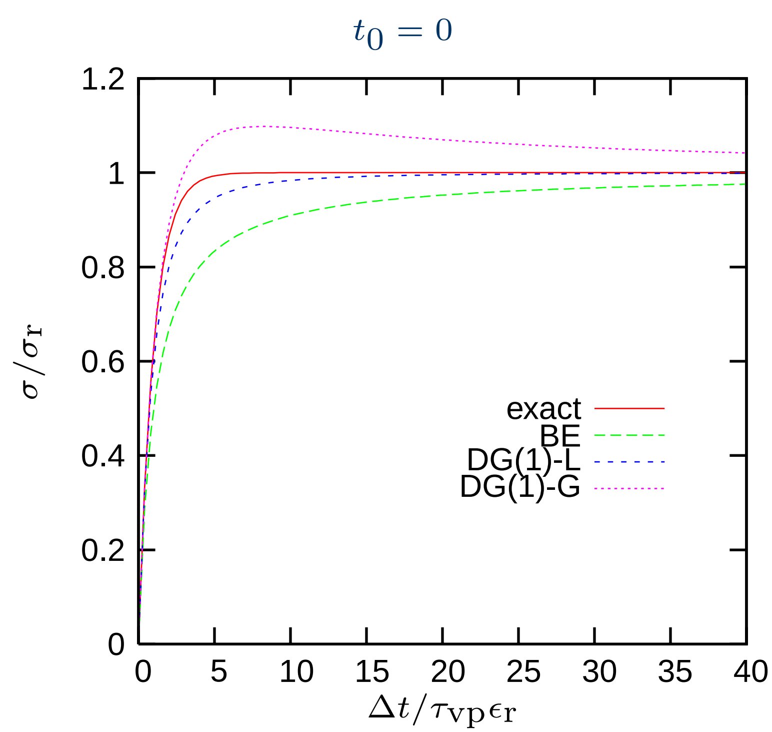
$$y(t) = \tau_{\rm vp}\dot{\epsilon}_{\rm c}\left[1 + \left(\frac{y_0}{\tau_{\rm vp}\dot{\epsilon}_{\rm c}} - 1\right)\exp\left(-\frac{t-t_0}{\tau_{\rm d}\epsilon_{\rm r}}\right)\right].$$

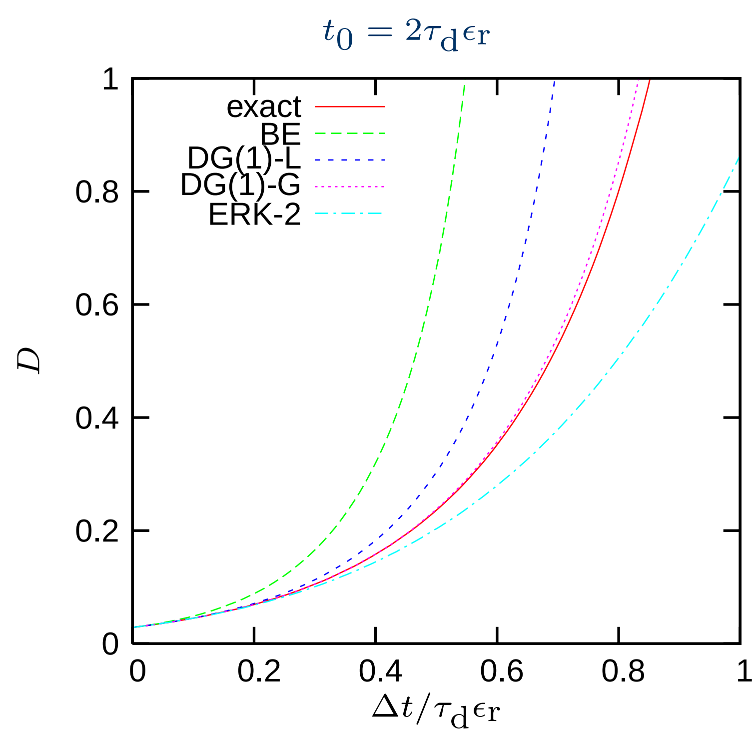
$y(t) \longrightarrow \tau_{\rm d}\dot{\epsilon}_{\rm c}$ as $t \longrightarrow \infty$.

The damage model

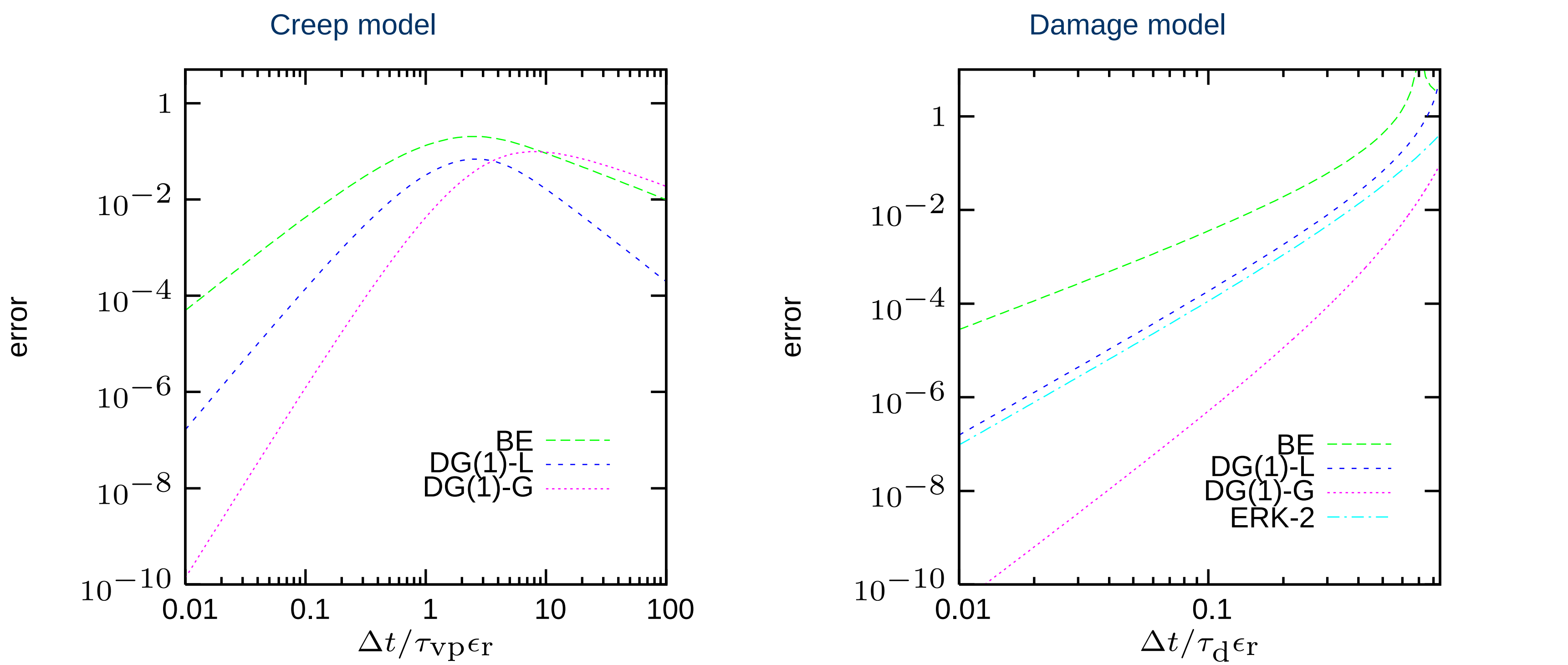
$$D = (1 + D_0)\exp\left\{\frac{\epsilon_{\rm r}}{(2r+1)\tau_{\rm d}\dot{\epsilon}_{\rm c}}\left[\left(\frac{\dot{\epsilon}_{\rm c}t}{\epsilon_{\rm r}}\right)^{2r+1} - \left(\frac{\dot{\epsilon}_{\rm c}t_0}{\epsilon_{\rm r}}\right)^{2r+1}\right]\right\} - 1.$$

Amplification factors = one-step solution





Amplification factors = one-step solution



An ideal integrator

for inelastic constitutive models should be:

1. L-stable
2. For $\dot{y} + \lambda y = 0$ (λ constant) the amplification factor should be
   (a) strictly positive
   (b) monotonous with respect to time step

Padé $(0, q)$-approximations of $\exp(-\lambda t)$ are positive and monotonous.
DG(1)-Lobatto = IRKL3C-2 = Padé-(0,2) for $\dot{y} + \lambda y = 0$

An ideal integrator

for damage and inelastic models with damage should be

- based on discontinuous Galerkin methods,
- an explicit-implicit split type methods
- or ????

CONSTITUTIVE MODEL

Clausius-Duhem Inequality $\gamma = -\rho\dot{\psi} + \sigma : \dot{\epsilon} \geq 0$

Helmholtz free energy $\psi(\epsilon_{\mathrm{e}}, \omega)$ where $\epsilon_{\mathrm{e}} = \epsilon - \epsilon_{\mathrm{i}}.$

Dissipation potential $\varphi(\sigma, Y)$ such that $\gamma = \dfrac{\partial\varphi}{\partial\sigma} : \sigma + \dfrac{\partial\varphi}{\partial Y} Y \geq 0$

$$\Longrightarrow \quad \left(\sigma - \rho\frac{\partial\psi}{\partial\epsilon_{\mathrm{e}}}\right) : \dot{\epsilon}_{\mathrm{e}} + \left(\dot{\epsilon}_{\mathrm{i}} - \frac{\partial\varphi}{\partial\sigma}\right) : \sigma + \left(-\dot{\omega} - \frac{\partial\varphi}{\partial Y}\right) Y = 0$$

$$\Longrightarrow \quad \sigma = \rho\frac{\partial\psi}{\partial\epsilon_{\mathrm{e}}} \qquad \dot{\epsilon}_{\mathrm{i}} = \frac{\partial\varphi}{\partial\sigma} \qquad \dot{\omega} = -\frac{\partial\varphi}{\partial Y}$$

$\Longrightarrow \quad \gamma \geq 0$ satisfies CDI

Particular model

$$\rho\psi = \tfrac{1}{2}(1-D)\epsilon_{\mathrm{e}} : C_{\mathrm{e}} : \epsilon_{\mathrm{e}} = \tfrac{1}{2}\omega\epsilon_{\mathrm{e}} : C_{\mathrm{e}} : \epsilon_{\mathrm{e}}$$

$$\varphi(\sigma, Y) = \varphi_{\mathrm{d}}(Y)\varphi_{\mathrm{tr}}(\sigma) + \varphi_{\mathrm{vp}}(\sigma)$$

$\varphi_{\mathrm{tr}} \geq 0 \quad \varphi_{\mathrm{tr}} \approx 0$ when $\|\dot{\epsilon}_{\mathrm{i}}\| < \eta$ and $\varphi_{\mathrm{tr}} > 1$ when $\|\dot{\epsilon}_{\mathrm{i}}\| > \eta$

$$\varphi_{\mathrm{d}} = \frac{1}{r+1}\frac{Y_{\mathrm{r}}}{\tau_{\mathrm{d}}\omega}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r+1}$$

$$\varphi_{\mathrm{tr}} = \frac{1}{pn}\left[\frac{1}{\tau_{\mathrm{vp}}\eta}\left(\frac{\bar{\sigma}}{\omega\sigma_{\mathrm{r}}}\right)^{p}\right]^{n} \qquad \sim \frac{\|\dot{\epsilon}_{\mathrm{i}}\|}{\eta}$$

$$\varphi_{\mathrm{vp}} = \frac{1}{p+1}\frac{\sigma_{\mathrm{r}}}{\tau_{\mathrm{vp}}}\left(\frac{\bar{\sigma}}{\omega\sigma_{\mathrm{r}}}\right)^{p+1}$$

Model characteristics

- Elastic stiffness is reduced monotonously due to damage
- The model does not include any specific yield stress
- In the absence of damage, the inelastic model behaves in a constant uniaxial strain-rate loading as

$$\sigma \longrightarrow (\tau_{\mathrm{vp}}\dot{\epsilon}_{\mathrm{c}})^{1/p}\sigma_{\mathrm{r}} \quad \text{when} \quad t \longrightarrow \infty$$

- The constraint for the integrity $\omega \in [0, 1]$ is satisfied automatically
- CDI is satisfied *a priori* for any admissible isothermal process
- The dissipation potential is a non-convex function with respect to the thermodynamic forces σ and Y

EVOLUTION EQUATIONS

$$\begin{cases} \dot{\boldsymbol{\sigma}} & = \boldsymbol{f}_\sigma(\boldsymbol{\sigma},\omega) = \omega \boldsymbol{C}_\mathrm{e}(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_\mathrm{i}) + \dfrac{f_\omega}{\omega}\boldsymbol{\sigma} = \omega \boldsymbol{C}_\mathrm{e}\left(\dot{\boldsymbol{\epsilon}} - g(\bar{\sigma},\omega)\dfrac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}\right) + \dfrac{f_\omega}{\omega}\boldsymbol{\sigma}, \\ \dot{\omega} & = f_\omega(\boldsymbol{\sigma},\omega) = -\dfrac{\varphi_\mathrm{tr}}{\tau_\mathrm{d}\omega}\left(\dfrac{Y}{Y_\mathrm{r}}\right)^r. \end{cases}$$

Non-dimensional form

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$$

where $\boldsymbol{y} = \left[(\boldsymbol{\sigma}/\sigma_\mathrm{r})^T, \omega\right]^T$

JACOBIAN MATRIX

In a uniaxial case the Jacobian of $\boldsymbol{f}$ is a 2×2 matrix
$\boldsymbol{y} = [\sigma/\sigma_\mathrm{r}, \omega]^T = [z, \omega]^T$ and $\boldsymbol{f} = [f_z, f_\omega]^T$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{bmatrix} \dfrac{\partial f_z}{\partial z} & \dfrac{\partial f_z}{\partial \omega} \\ \dfrac{\partial f_\omega}{\partial z} & \dfrac{\partial f_\omega}{\partial \omega} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \widetilde{J}_{11} - J_{22} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad \text{if} \quad \varphi_\mathrm{tr} \equiv 1$$

Properties

$$\widetilde{J}_{11} \le 0, \quad J_{12} > 0 \quad \text{when} \quad \dot{\epsilon}_\mathrm{c} > 0, \quad J_{21} < 0, \quad J_{22} \ge 0$$

Initially, when $z = \sigma/\sigma_\mathrm{r} \ll 1$ and $\omega \approx 1$ then

$$|J_{12}| \gg |J_{11}| \gg |J_{22}| \sim |J_{21}|$$

EIGENVALUES OF THE JACOBIAN

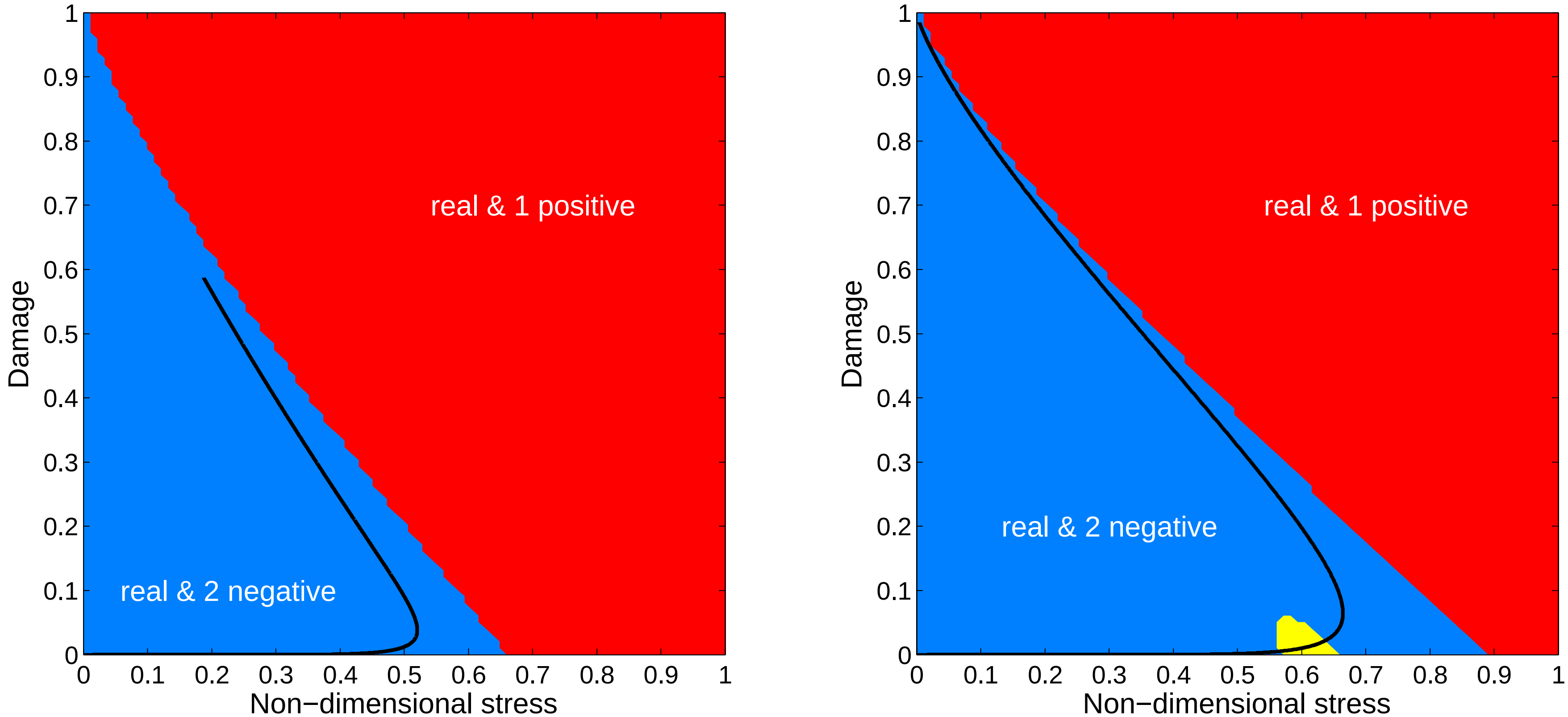
$$\lambda_{1,2} = \frac{1}{2}\widetilde{J}_{11} \pm \sqrt{(\frac{1}{2}\widetilde{J}_{11} - J_{22})^2 + J_{12}J_{21}}$$

Three possible cases:

1. both eigenvalues are real and negative
2. eigenvalues are complex conjugates with negative real part ($\widetilde{J}_{11} < 0$)
3. eigenvalues are real, but of opposite sign

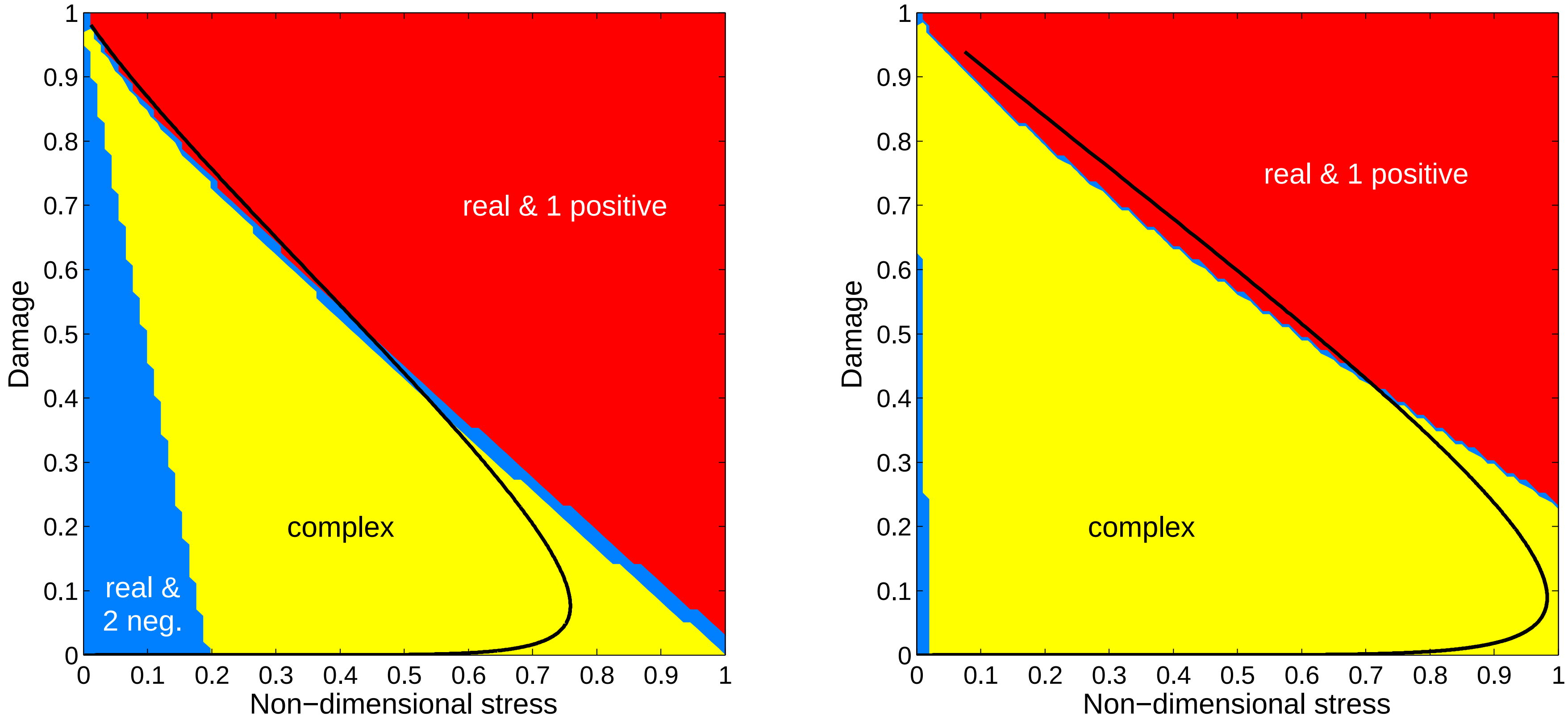
$$\varphi_{\text{tr}} \equiv 1$$

$$\dot{\epsilon}_c = 10^{-4} \qquad \dot{\epsilon}_c = 4.05 \cdot 10^{-4}$$



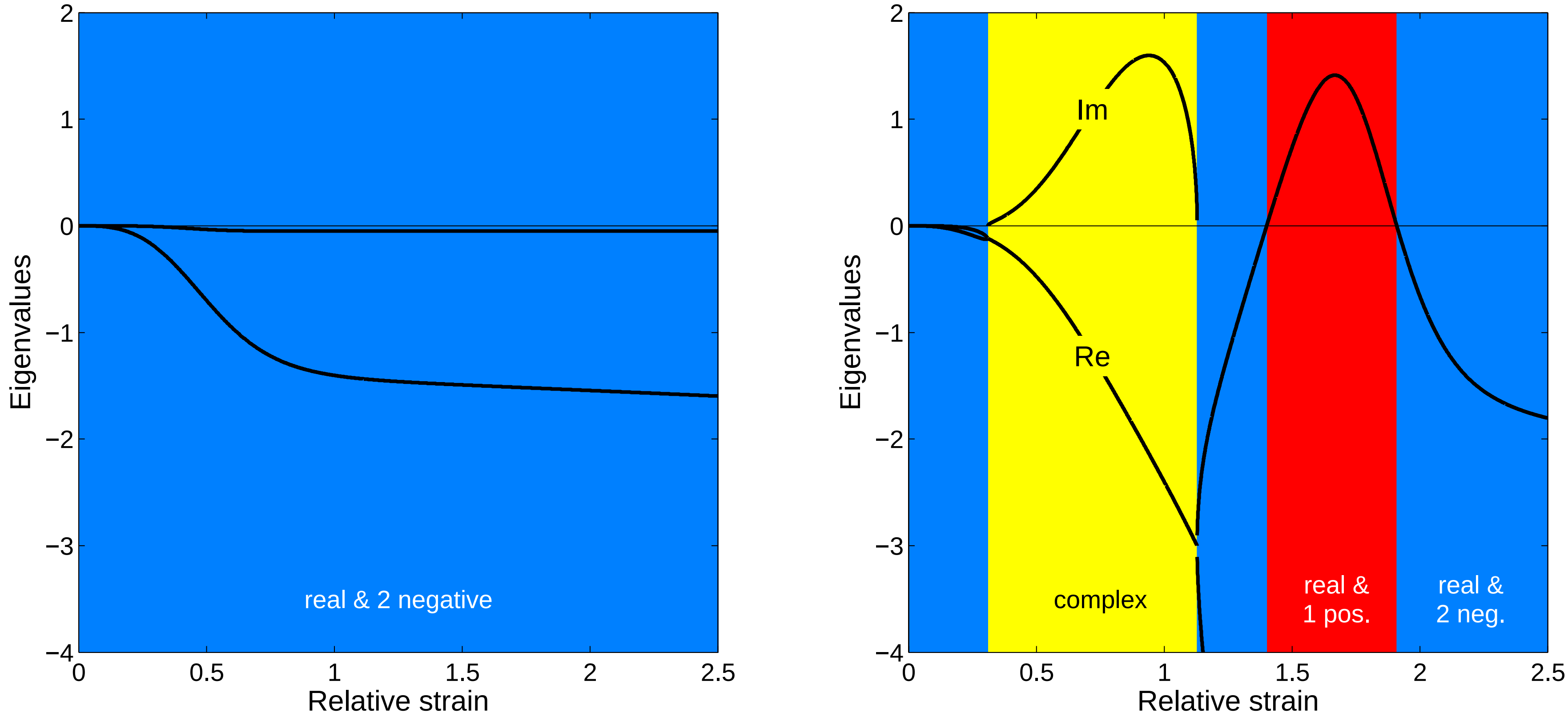
$$\varphi_{\rm tr} \equiv 1$$

$$\dot{\epsilon}_{\rm c} = 10^{-3} \qquad \dot{\epsilon}_{\rm c} = 7.5 \cdot 10^{-3}$$



$$\varphi_{\mathrm{tr}} \equiv 1$$

$$\dot{\epsilon}_{\mathrm{c}} = 10^{-4} \qquad \dot{\epsilon}_{\mathrm{c}} = 6.5 \cdot 10^{-4}$$



NUMERICAL SOLUTION

Backward Euler + Newton linearisation

$$\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{h}_{12} \\ \boldsymbol{h}_{21}^T & H_{22} \end{bmatrix} \left\{ \begin{array}{c} \delta\boldsymbol{\sigma} \\ \delta\omega \end{array} \right\} = \Delta t \left\{ \begin{array}{c} \boldsymbol{f}_\sigma \\ f_\omega \end{array} \right\} - \left\{ \begin{array}{c} \Delta\boldsymbol{\sigma} \\ \Delta\omega \end{array} \right\}$$

$$\boldsymbol{H}_{11} = \boldsymbol{I} - \Delta t \frac{\partial \boldsymbol{f}_\sigma}{\partial \boldsymbol{\sigma}} \quad \boldsymbol{h}_{12} = -\Delta t \frac{\partial \boldsymbol{f}_\sigma}{\partial \omega} \quad \boldsymbol{h}_{21}^T = -\Delta t \frac{\partial f_\omega}{\partial \boldsymbol{\sigma}} \quad H_{22} = 1 - \Delta t \frac{\partial f_\omega}{\partial \omega}$$

ATS $\quad \boldsymbol{C} = \omega(\boldsymbol{H}_{11} - \boldsymbol{h}_{12} H_{22}^{-1} \boldsymbol{h}_{21}^T)^{-1} \boldsymbol{C}_\mathrm{e}$

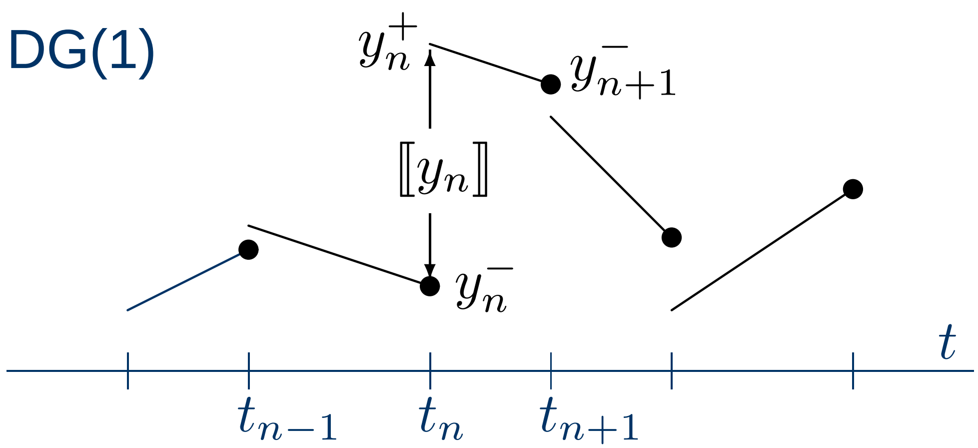
Discontinuous Galerkin for

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$$

DG(q): find $\boldsymbol{y}$ (polynomial of degree q, as test functions $\hat{\boldsymbol{y}}$) such that

$$\int_{t_n}^{t_{n+1}} (\dot{\boldsymbol{y}} - \boldsymbol{f}(\boldsymbol{y}))^T \hat{\boldsymbol{y}} \, \mathrm{d}t + [\![\boldsymbol{y}_n]\!]^T \hat{\boldsymbol{y}}_n^+ = 0$$

DG(1)



Explicit-implicit split

Jacobian $\boldsymbol{J} = \partial \boldsymbol{f} / \partial \boldsymbol{y}$.
Similarity transformation $\boldsymbol{J} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1}$,
where $\boldsymbol{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\boldsymbol{J}$.
Transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{y}$ results in

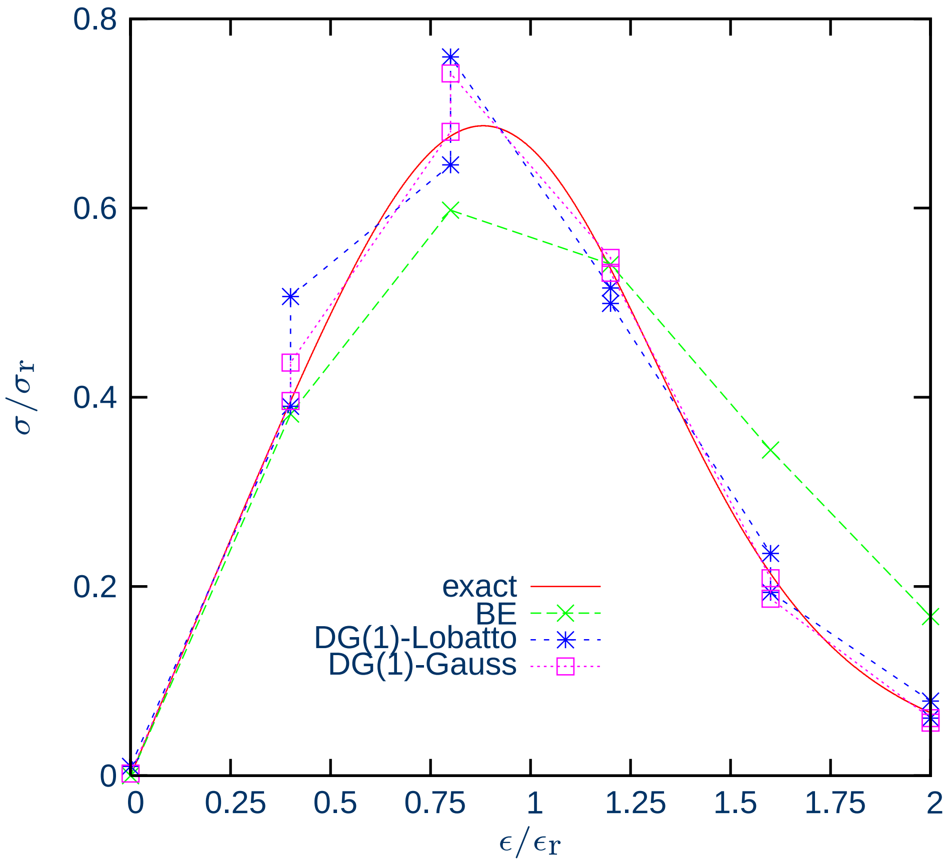
$$\dot{\boldsymbol{z}} = \boldsymbol{T}\boldsymbol{f}(\boldsymbol{T}^{-1}\boldsymbol{z}).$$

$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_- \\ \boldsymbol{z}_+ \end{pmatrix}$$

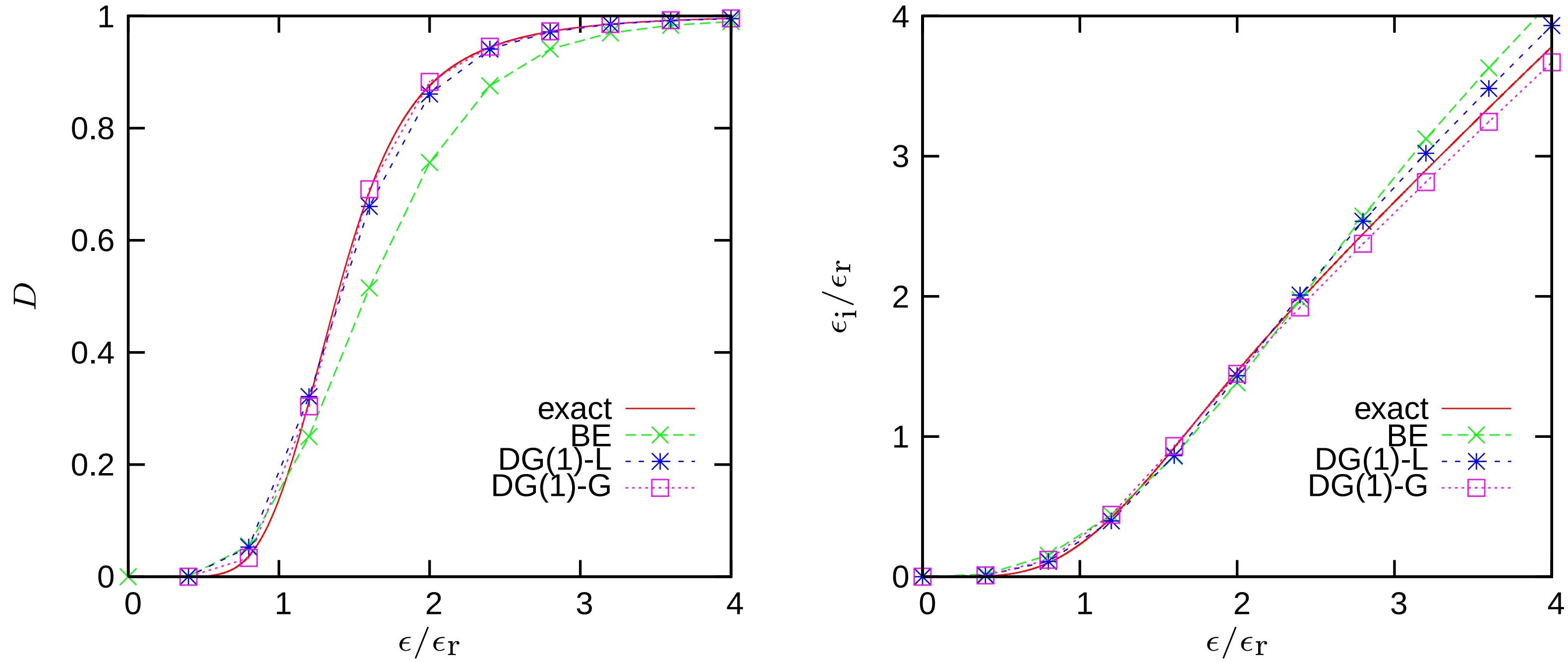
- Explicit scheme for $\boldsymbol{z}_+$ and
- implicit scheme for $\boldsymbol{z}_-$.

NUMERICAL EXAMPLE

Uniaxial constant strain-rate loading $\dot{\epsilon}_{\rm c} = 5 \cdot 10^{-4}$ 1/s, and $\varphi_{\rm tr} \equiv 1$

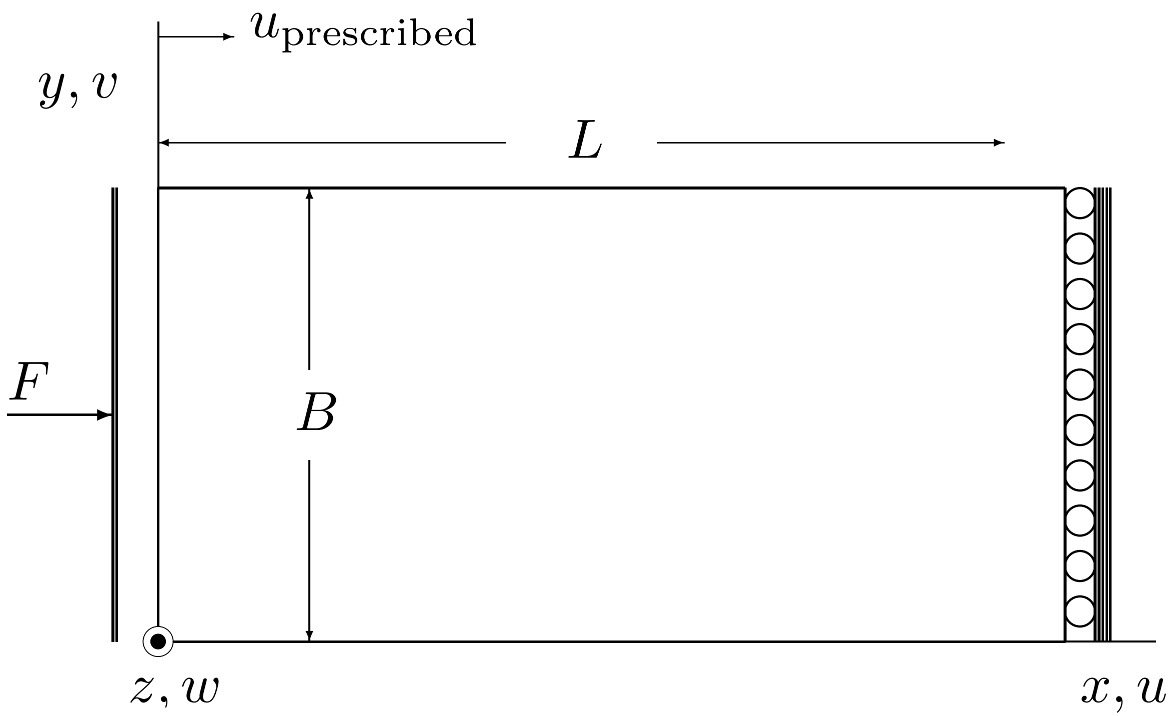


$\dot{\epsilon}_{\rm c} = 5 \cdot 10^{-4}$ 1/s, and $\varphi_{\rm tr} \equiv 1$

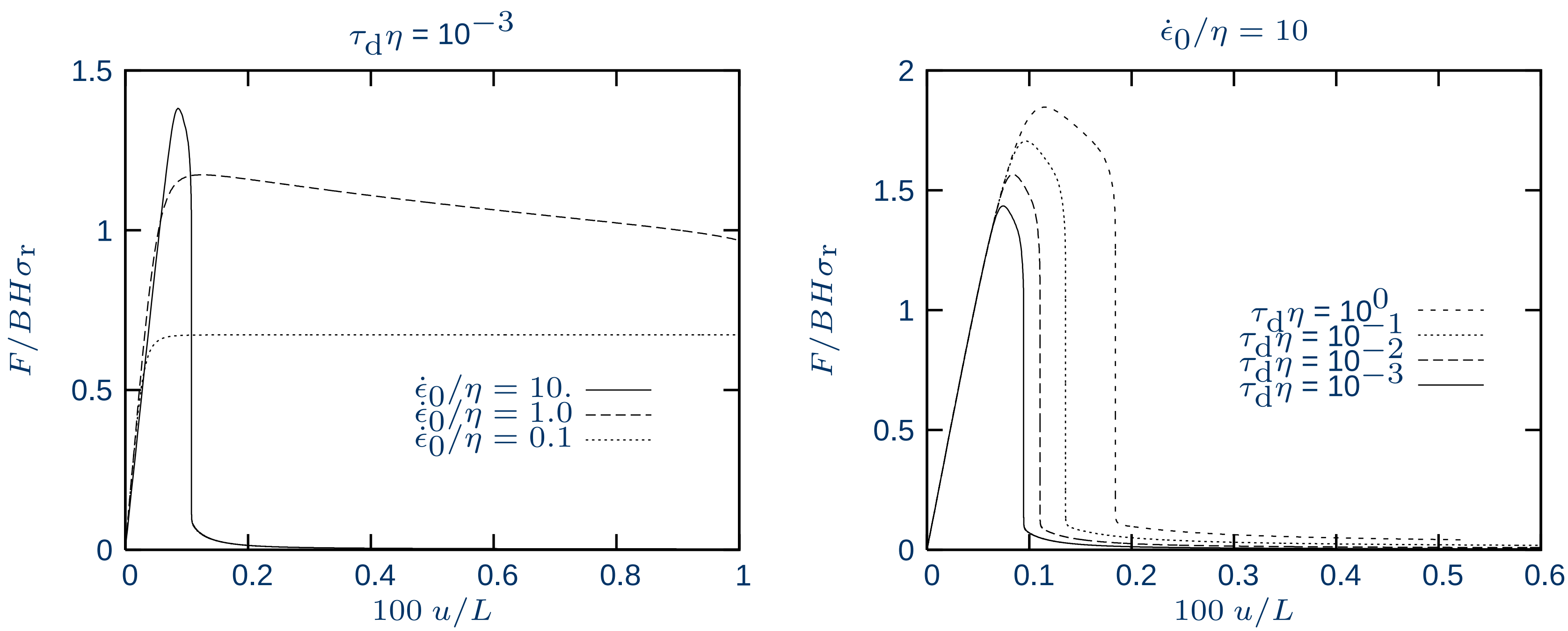


3D-EXAMPLE

von-Mises solid $\bar{\sigma} = \sigma_{\mathrm{eff}}$, E = 40 GPa, $\nu = 0.3$, σ_{r} = 20 MPa, $\tau_{\mathrm{vp}} = 1000$ s
transition strain rate $\eta = 10^{-3}$ s^{-1}, $p = r = n = 4$



Load-displacement curves, 12×6 mesh



CONCLUSIONS AND FURTHER DEVELOPMENTS

- The implicit Euler method is not good for damage
- The discontinuous Galerkin methods seems to work well for damage models
- Algorithm for switching strategy
- Evaluation of explicit-implicit split algorithms
- Alternative constitutive formulation using $\psi^*(\sigma, \alpha)$ and $\varphi(\dot{\epsilon}_{\mathrm{i}}, Z)$!