A constitutive model for strain-rate dependent ductile-to-brittle transition

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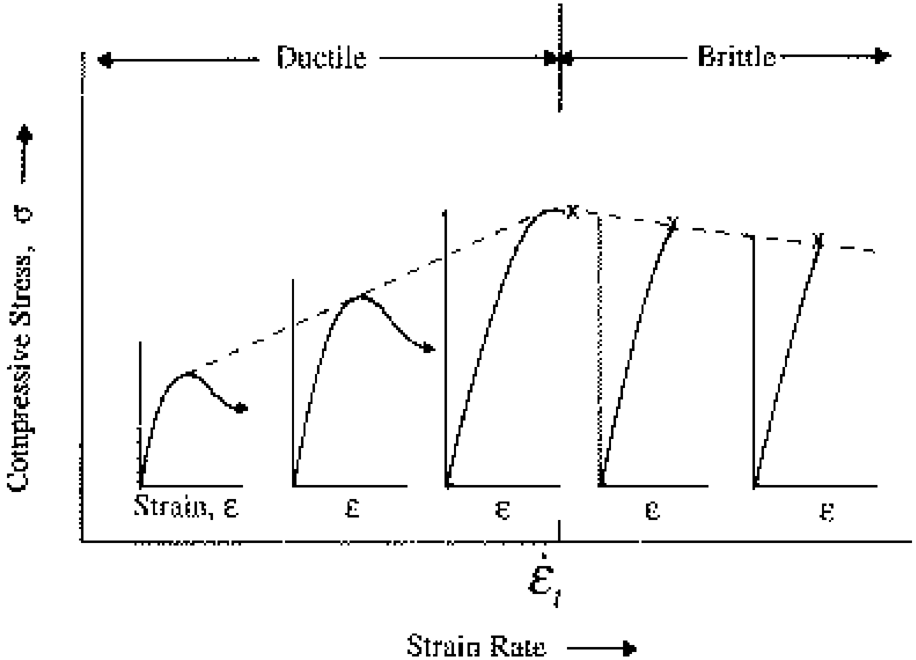
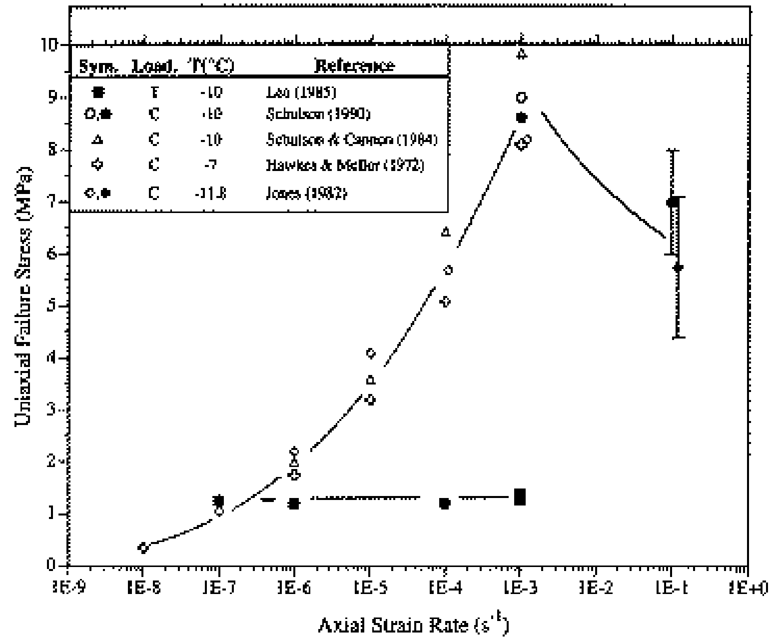
OUTLINE

- Motivation
- The model
  - Thermodynamic formulation
  - Helmholtz free energy
  - Dissipation potential
- Concluding remarks

Photograph: Kari Kolari, Helsinki, Feb 2007

MOTIVATION





E. M. Schulson: Brittle failure of ice, *Engineering Fracture Mechanics* **68** (2001) 1839–1887.

Brittle failure in compression

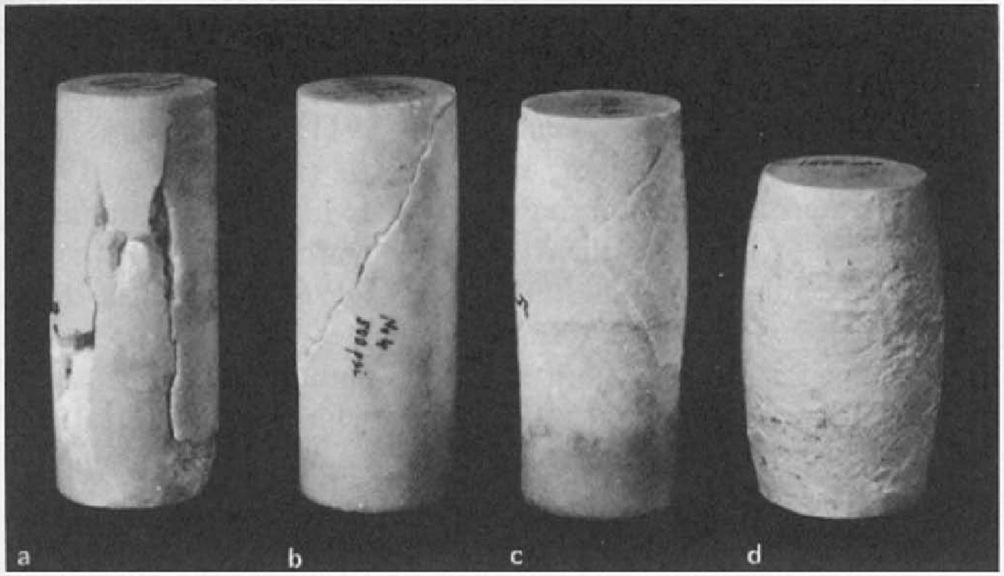
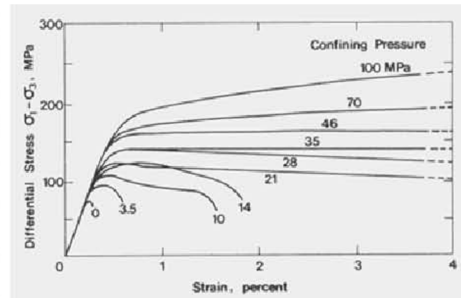


Fig. 48a–d. Types of fractures or flow in Wombeyan marble at various confining pressures: **a** axial splitting failure at atmospheric pressure; **b** single shear failure at 3.5 MPa (35 bars); **c** conjugate shears at 35 MPa (350 bars); **d** ductile behaviour at 100 MPa (1 kbar). (From experiments of the author; cf. Paterson, 1958)



M.S. Paterson, Experimental deformation and faulting in Wombeyan marble, *Bull. Geol. Soc. Am.*, **69** (1958) 465–476.

Occurrence to splitting mode is sensitive to strain rate: J.F. Dorris (1985)

THERMODYNAMIC FORMULATION

Helmholtz free energy $\psi(\boldsymbol{\epsilon}_{\mathrm{e}}, \boldsymbol{d})$ where $\boldsymbol{\epsilon}_{\mathrm{e}} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}_{\mathrm{i}}$.

Clausius-Duhem Inequality $\gamma = -\rho\dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \geq 0$

Dissipation potential $\varphi(\boldsymbol{\sigma}, \boldsymbol{y})$ such that $\gamma = \dfrac{\partial \varphi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma} + \dfrac{\partial \varphi}{\partial \boldsymbol{y}} \cdot \boldsymbol{y} \geq 0$

$$\Longrightarrow \quad \left(\boldsymbol{\sigma} - \rho\frac{\partial \psi}{\partial \boldsymbol{\epsilon}_{\mathrm{e}}}\right) : \dot{\boldsymbol{\epsilon}}_{\mathrm{e}} + \left(\dot{\boldsymbol{\epsilon}}_{\mathrm{i}} - \frac{\partial \varphi}{\partial \boldsymbol{\sigma}}\right) : \boldsymbol{\sigma} + \left(\dot{\boldsymbol{d}} - \frac{\partial \varphi}{\partial \boldsymbol{y}}\right) \cdot \boldsymbol{y} = 0$$

$$\Longrightarrow \quad \boldsymbol{\sigma} = \rho\frac{\partial \psi}{\partial \boldsymbol{\epsilon}_{\mathrm{e}}} \qquad \dot{\boldsymbol{\epsilon}}_{\mathrm{i}} = \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} \qquad \dot{\boldsymbol{d}} = \frac{\partial \varphi}{\partial \boldsymbol{y}}$$

$\Longrightarrow \quad \gamma \geq 0$ satisfies CDI

Helmholtz free energy

Integrity basis

$$I_1 = \operatorname{tr}\boldsymbol{\epsilon}_{\mathrm{e}}, \quad I_2 = \tfrac{1}{2}\operatorname{tr}\boldsymbol{\epsilon}_{\mathrm{e}}^2, \quad I_3 = \tfrac{1}{3}\operatorname{tr}\boldsymbol{\epsilon}_{\mathrm{e}}^3, \quad I_4 = \|\boldsymbol{d}\|, \quad I_5 = \boldsymbol{d}\cdot\boldsymbol{\epsilon}_{\mathrm{e}}\cdot\boldsymbol{d}, \quad I_6 = \boldsymbol{d}\cdot\boldsymbol{\epsilon}_{\mathrm{e}}^2\cdot\boldsymbol{d}$$

$$\begin{aligned}\rho\psi &= (1 - I_4)\left(\tfrac{1}{2}\lambda I_1^2 + 2\mu I_2\right) \\ &+ H(\sigma^\perp)\frac{\lambda\mu}{\lambda + 2\mu}(I_4 I_1^2 - 2I_1 I_5 I_4^{-1} + I_5^2 I_4^{-3}) + (1 - H(\sigma^\perp))(\tfrac{1}{2}\lambda I_4 I_1^2 + \mu I_5^2 I_4^{-3}) \\ &+ \mu\left(2I_4 I_2 + I_5^2 I_4^{-3} - 2I_6 I_4^{-1}\right)\end{aligned}$$

Heaviside function $H(\sigma^\perp)$ takes into account the “crack” opening/closure

Helmholtz free energy - alternative expression

Integrity basis

$$I_1 = \mathrm{tr}\,\boldsymbol{\epsilon}_\mathrm{e}, \quad I_2 = \tfrac{1}{2}\mathrm{tr}\,\boldsymbol{\epsilon}_\mathrm{e}^2, \quad I_3 = \tfrac{1}{3}\mathrm{tr}\,\boldsymbol{\epsilon}_\mathrm{e}^3, \quad I_4 = \|\boldsymbol{d}\|, \quad \hat{I}_5 = \hat{\boldsymbol{d}}\cdot\boldsymbol{\epsilon}_\mathrm{e}\hat{\cdot}\boldsymbol{d}, \quad \hat{I}_6 = \hat{\boldsymbol{d}}\cdot\boldsymbol{\epsilon}_\mathrm{e}^2\hat{\cdot}\boldsymbol{d}$$

where

$$\hat{\boldsymbol{d}} = \boldsymbol{d}/\|\boldsymbol{d}\| = \boldsymbol{d}/I_4$$

$$\begin{aligned}\rho\psi = {} & (1 - I_4)\left(\tfrac{1}{2}\lambda I_1^2 + 2\mu I_2\right) \\ & + H(\sigma^\perp)\frac{\lambda\mu}{\lambda + 2\mu} I_4(I_1^2 - 2I_1\hat{I}_5 + \hat{I}_5^2) + (1 - H(\sigma^\perp))I_4(\tfrac{1}{2}\lambda I_1^2 + \mu\hat{I}_5^2) \\ & + \mu I_4\left(2I_2 + \hat{I}_5^2 - 2I_6\right)\end{aligned}$$

Stresses are continuous when the “crack” closes

Dissipation potential

Decomposed as

$$\varphi(\boldsymbol{\sigma}, \boldsymbol{y}) = \varphi_{\mathrm{d}}(\boldsymbol{y})\varphi_{\mathrm{tr}}(\boldsymbol{\sigma}) + \varphi_{\mathrm{vp}}(\boldsymbol{\sigma})$$

where

$$\varphi_{\mathrm{d}} = \frac{1}{2(r+1)}\frac{Y_{\mathrm{r}}}{\tau_{\mathrm{d}}(1-I_4)}H(\epsilon_1 - \epsilon_{\mathrm{tresh}})\left(\frac{(\boldsymbol{y}+\boldsymbol{y}_0)\cdot\boldsymbol{M}\cdot(\boldsymbol{y}+\boldsymbol{y}_0)}{Y_{\mathrm{r}}^2}\right)^{r+1}$$

$$\varphi_{\mathrm{tr}} = \frac{1}{pn}\left[\frac{1}{\tau_{\mathrm{vp}}\eta}\left(\frac{\bar{\sigma}}{(1-I_4)\sigma_{\mathrm{r}}}\right)^{p}\right]^{n}$$

$$\varphi_{\mathrm{vp}} = \frac{1}{p+1}\frac{\sigma_{\mathrm{r}}}{\tau_{\mathrm{vp}}}\left(\frac{\bar{\sigma}}{(1-I_4)\sigma_{\mathrm{r}}}\right)^{p+1}$$

and

$$\boldsymbol{M} = \boldsymbol{n}\otimes\boldsymbol{n}, \quad \boldsymbol{y}_0 = \beta Y_{\mathrm{r}}\boldsymbol{n}$$

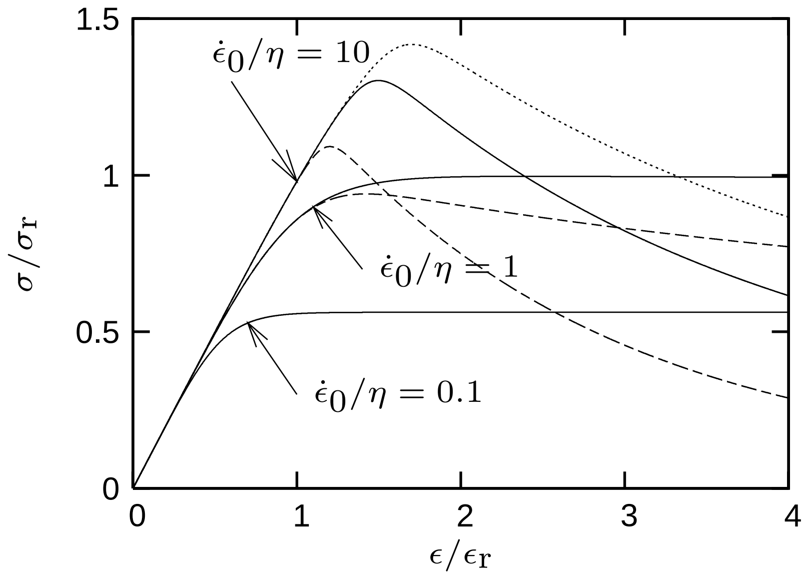
MODEL CHARACTERISTICS

- Elastic stiffness is reduced monotonously due to damage
- Qualitatively predicts correct brittle failure mode in compression/tension
  Animation
- The constraint for the damage $\|\boldsymbol{d}\| \in [0, 1]$ is satisfied automatically
- The transition function $\varphi_{\rm tr}$ deals with the change in the mode of deformation through the damage evolution such that

$$\varphi_{\rm tr} \geq 0 \quad \text{and} \quad \varphi_{\rm tr} \approx 0 \text{ when } \|\dot{\boldsymbol{\epsilon}}_{\rm i}\| < \eta \quad \text{and} \quad \varphi_{\rm tr} > 1 \text{ when } \|\dot{\boldsymbol{\epsilon}}_{\rm i}\| > \eta;$$

- CDI is satisfied *a priori* for any admissible isothermal process
- The dissipation potential is a non-convex function with respect to the thermodynamic forces $\boldsymbol{\sigma}$ and $\boldsymbol{y}$

Uniaxial stress-strain behaviour



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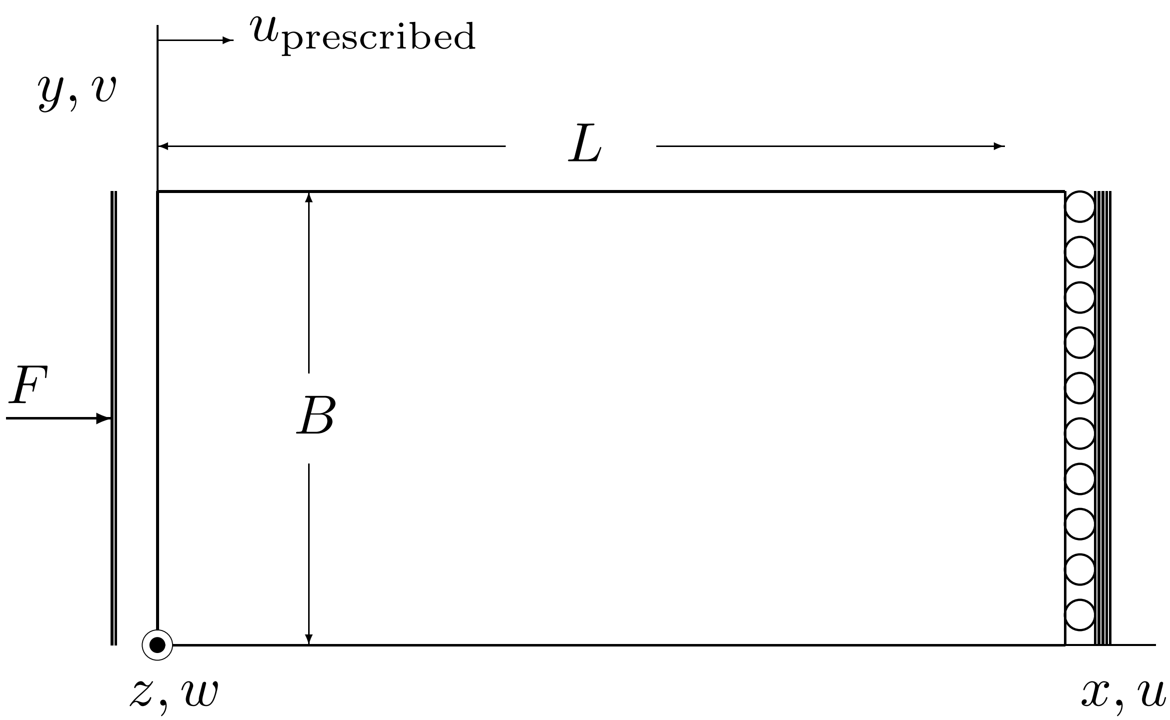
NUMERICAL EXAMPLE

von-Mises solid $\bar{\sigma} = \sigma_{\text{eff}}$, E = 40 GPa, $\nu = 0.3$, σ_{r} = 20 MPa, $\tau_{\text{vp}} = 1000$ s
transition strain rate $\eta = 10^{-3}$ s^{-1}, $p = r = n = 4$



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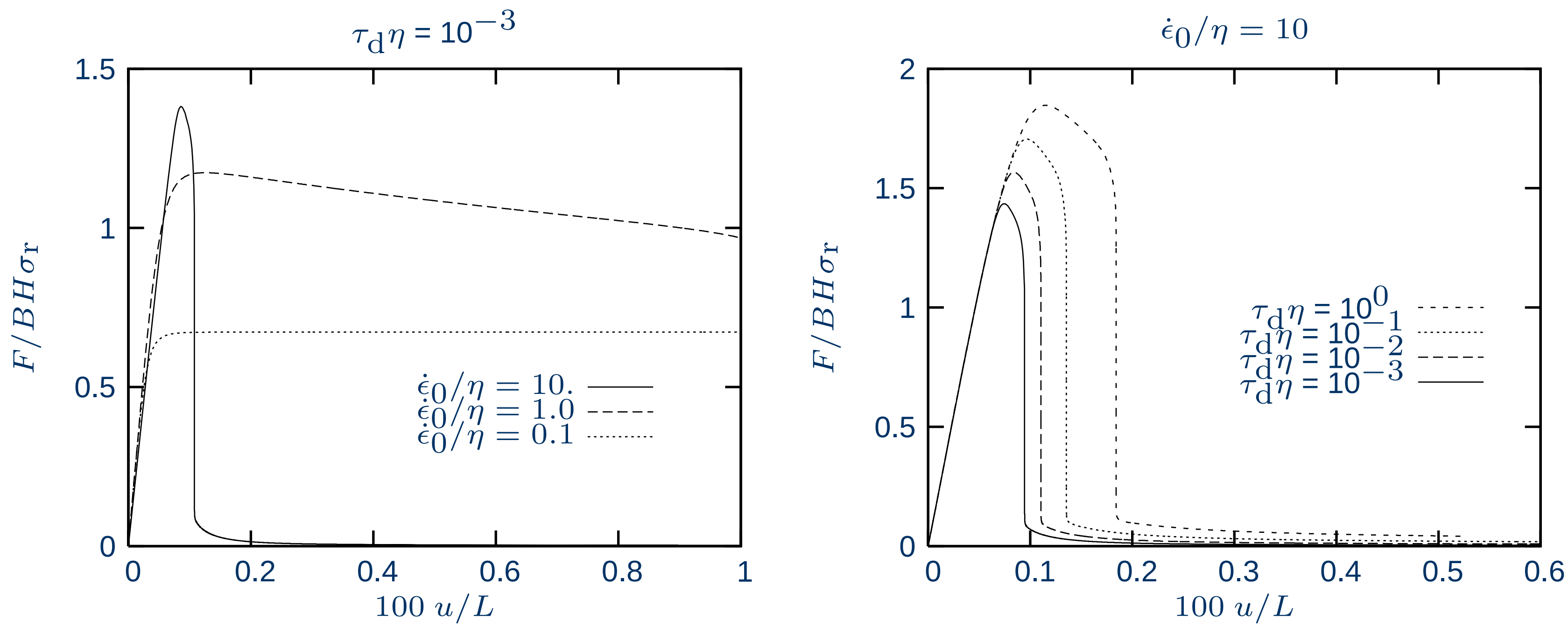
Load-displacement curves, 12×6 mesh



CONCLUSIONS AND FURTHER DEVELOPMENTS

- Thermodynamically consistent formulation
- Predicts correct failure modes in tension/compression
- Easily extensible to more realistic creep and plasticity models
- Length scale ??
- Alternative formulation using $\psi^*(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ and $\varphi(\dot{\boldsymbol{\epsilon}}_\mathrm{i}, \boldsymbol{Z})$!!