Continuum based fatigue modelling

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ZERO-DEFECT MANUFACTURING FOR GREEN TRANSITION IN EUROPE

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Outline

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- Continuum based fatigue model
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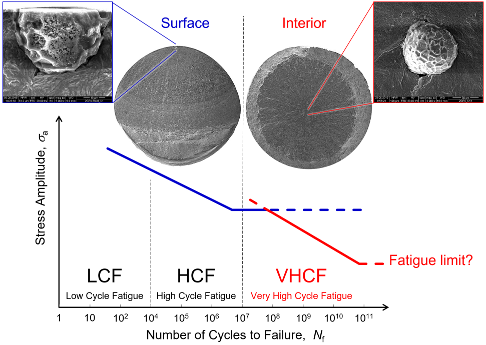
  - Anisotropic formats for the continuum based HCF model
  - Entension to better capture the form of the Haigh-diagram
  - Unified LCF-HCF model
  - Stress history as a stochastic process
- Concluding remarks and future directions

Intro to fatigue phenomenon



Courtesy B. Schönbauer.

- **Low-Cycle-Fatigue** (LCF) where substantial plastic deformations occur in the bulk material during the loading history. In terms of cycles, the fatigue failure occurs below the range of 10^3 to 10^4 cycles.
- **High-Cycle-Fatigue** (HCF) where deformation is mainly elastic and inelastic deformations take place only in macroscopically vanishing volumes near inclusions, voids or other material defects.
- **Very-High-Cycle-Fatigue** (VHCF) is currently under active research. More than $10^7 - 10^8$ cycles.

Continuum approach to HCF

A continuum approach for high-cycle fatigue (HCF) was proposed by Ottosen et al. in 2008 . It offers various advantages over traditional methods as:

- it is inherently multiaxial,
- no need to define a cycle, thus avoids heuristic rainflow type approaches,
- stress history need not to be stored,
- it can be easily extended to
  - anisotropic symmetries, like orthotropy and transverse isotropy (Kouhia et al. *EJMA* 2025),
  - to account gradient effects (Ottosen et al. *IJF* 2018),
  - stochastic loadings (Frondelius et al. *EJMA* 2022),
  - combine with cyclic plasticity models for low-cycle fatigue (LCF) analysis.

The key ingredients of the continuum based time continuous fatigue analysis method are: definition of an **endurance function** $\beta(\boldsymbol{\sigma}, \{\boldsymbol{\alpha}\}) = 0$, and **evolution equations** for the fatigue damage D and the internal variables $\{\boldsymbol{\alpha}\}$.

Endurance function, evolution eq. and conditions for evolution

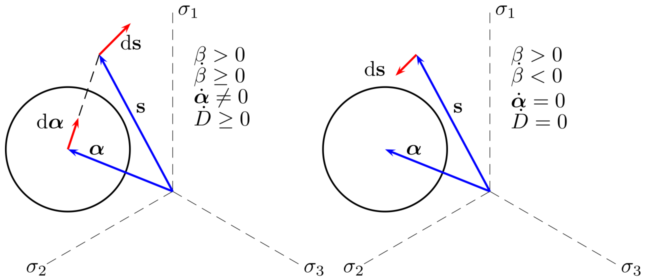
Original (Ottosen et al. IJF 2008) form for the endurance function

$$\beta(\boldsymbol{\sigma},\boldsymbol{\alpha}) = \frac{1}{\sigma_{\rm oe}}(\bar{\sigma}_{\rm eff} + AI_1 - \sigma_{\rm oe}), \quad I_1 = \operatorname{tr}\boldsymbol{\sigma}, \quad \bar{\sigma}_{\rm eff} = \sqrt{\tfrac{3}{2}\operatorname{tr}(\bar{\boldsymbol{s}})^2}, \quad \bar{\boldsymbol{s}} = \boldsymbol{s} - \boldsymbol{\alpha}, \quad \boldsymbol{s} = \boldsymbol{\sigma} - \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})\boldsymbol{I}.$$

Evolution equations

$$\dot{\boldsymbol{\alpha}} = C(\boldsymbol{s} - \boldsymbol{\alpha})\dot{\beta}, \quad \dot{D} = K\exp(L\beta)\dot{\beta}.$$

The set of internal variables contains now only one deviatoric second order tensor $\boldsymbol{\alpha}$.



Anisotropic format for the continuum based HCF model

Orthotropic symmetry can be specified by a symmetry group

$$\mathcal{G}_{\text{orth}} = \{\boldsymbol{Q} \in \mathcal{O}(V) \,|\, \boldsymbol{Q}(\boldsymbol{m}^{(1)} \otimes \boldsymbol{m}^{(1)})\boldsymbol{Q}^T = \boldsymbol{m}^{(1)} \otimes \boldsymbol{m}^{(1)}, \\ \boldsymbol{Q}(\boldsymbol{m}^{(2)} \otimes \boldsymbol{m}^{(2)})\boldsymbol{Q}^T = \boldsymbol{m}^{(2)} \otimes \boldsymbol{m}^{(2)}, \boldsymbol{Q}(\boldsymbol{m}^{(3)} \otimes \boldsymbol{m}^{(3)})\boldsymbol{Q}^T = \boldsymbol{m}^{(3)} \otimes \boldsymbol{m}^{(3)}\}.$$

The unit vectors $\boldsymbol{m}^{(i)}(1 = 1, 2, 3)$ defines the normals of the three orthogonal symmetry planes. The corresponding structural tensors $\boldsymbol{M}^{(i)}$ are defined by $\boldsymbol{M}^{(i)} = \boldsymbol{m}^{(i)} \otimes \boldsymbol{m}^{(i)}$ (no summation over$i$).

The functional bases for scalar Υ_{s} and symmetric tensor Υ_{t} invariants belonging to the orthotropy symmetry group are

$$\Upsilon_{\text{s}} = \{\underline{\operatorname{tr} \boldsymbol{A}}, \underline{\operatorname{tr} \boldsymbol{A}^2}, \operatorname{tr} \boldsymbol{A}^3, \operatorname{tr}(\boldsymbol{A}\boldsymbol{M}^{(1)}), \operatorname{tr}(\boldsymbol{A}\boldsymbol{M}^{(2)}), \operatorname{tr}(\boldsymbol{A}\boldsymbol{M}^{(3)}), \\ \operatorname{tr}(\boldsymbol{A}^2\boldsymbol{M}^{(1)}), \operatorname{tr}(\boldsymbol{A}^2\boldsymbol{M}^{(2)}), \operatorname{tr}(\boldsymbol{A}^2\boldsymbol{M}^{(3)})\},$$

$$\Upsilon_{\text{t}} = \{\underline{\boldsymbol{I}}, \underline{\boldsymbol{A}}, \underline{\boldsymbol{A}^2}, \boldsymbol{M}^{(1)}, \boldsymbol{M}^{(2)}, \boldsymbol{M}^{(3)}, \\ \boldsymbol{A}\boldsymbol{M}^{(1)} + \boldsymbol{M}^{(1)}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{M}^{(2)} + \boldsymbol{M}^{(2)}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{M}^{(3)} + \boldsymbol{M}^{(3)}\boldsymbol{A}, \\ \boldsymbol{A}\boldsymbol{M}^{(1)}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{M}^{(2)}\boldsymbol{A},\ \boldsymbol{A}\boldsymbol{M}^{(3)}\boldsymbol{A}\}.$$

The endurance function β and the evolution equations for the internal variables $\boldsymbol{\alpha}$ and D can then be written in the most general form. Notice: $\boldsymbol{M}^{(1)} + \boldsymbol{M}^{(2)} + \boldsymbol{M}^{(3)} = \boldsymbol{I}$.

Orthotropic form of the endurance function

Endurance function splitted in deviatoric and hydrostatic parts:

$$\beta = \frac{1}{\sigma_{-1}^{(1)}} \left(\beta_{\rm d} + \beta_{\rm h} - \sigma_{-1}^{(1)} \right),$$

where $\sigma_{-1}^{(1)}$ is the fatigue strength for alternating uniaxial stress in direction $\boldsymbol{m}^{(1)}$. Deviatoric part:

$$\begin{aligned}\beta_{\rm d}^2 =& F\,(\bar{J}_1^{(1)} - \bar{J}_1^{(2)})^2 + G\,(\bar{J}_1^{(1)} - \bar{J}_1^{(3)})^2 + H\,(\bar{J}_1^{(2)} - \bar{J}_1^{(3)})^2 \\ &+ (L+M-N)\left(\bar{J}_2^{(1)} - (\bar{J}_1^{(1)})^2\right) + (N+L-M)\left(\bar{J}_2^{(2)} - (\bar{J}_1^{(2)})^2\right) + (M+N-L)\left(\bar{J}_2^{(3)} - (\bar{J}_1^{(3)})^2\right),\end{aligned}$$

where

$$\bar{J}_1^{(i)} = \ {\rm tr}(\bar{\boldsymbol{s}}\boldsymbol{M}^{(i)}), \quad \bar{J}_2^{(i)} = \ {\rm tr}(\bar{\boldsymbol{s}}^2\boldsymbol{M}^{(i)}), \quad i = 1,2,3.$$

The hydrostatic part is then chosen as

$$\beta_{\rm h} = A^{(1)} I_1^{(1)} + \frac{\sigma_{-1}^{(1)}}{\sigma_{-1}^{(2)}} A^{(2)} I_1^{(2)} + \frac{\sigma_{-1}^{(1)}}{\sigma_{-1}^{(3)}} A^{(3)} I_1^{(3)},$$

where

$$I_1^{(i)} = {\rm tr}(\boldsymbol{\sigma}\boldsymbol{M}^{(i)}), \quad i = 1,2,3.$$

Orthotropic form of the evolution equations

The most general form of the evolution equation for $\boldsymbol{\alpha}$ is

$$\dot{\boldsymbol{\alpha}} = \dot{\beta}\big[a_1\boldsymbol{M}^{(1)} + a_2\boldsymbol{M}^{(2)} + a_3\boldsymbol{M}^{(3)} + a_4(\boldsymbol{m}^{(1)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(1)} + \overline{\boldsymbol{s}}\boldsymbol{m}^{(1)} \otimes \boldsymbol{m}^{(1)}) + \\ + a_5(\boldsymbol{m}^{(2)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(2)} + \overline{\boldsymbol{s}}\boldsymbol{m}^{(2)} \otimes \boldsymbol{m}^{(2)}) + a_6(\boldsymbol{m}^{(3)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(3)} + \overline{\boldsymbol{s}}\boldsymbol{m}^{(3)} \otimes \boldsymbol{m}^{(3)}) + \\ + a_7\overline{\boldsymbol{s}}\boldsymbol{m}^{(1)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(1)} + a_8\overline{\boldsymbol{s}}\boldsymbol{m}^{(2)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(2)} + a_9\overline{\boldsymbol{s}}\boldsymbol{m}^{(3)} \otimes \overline{\boldsymbol{s}}\boldsymbol{m}^{(3)}\big],$$

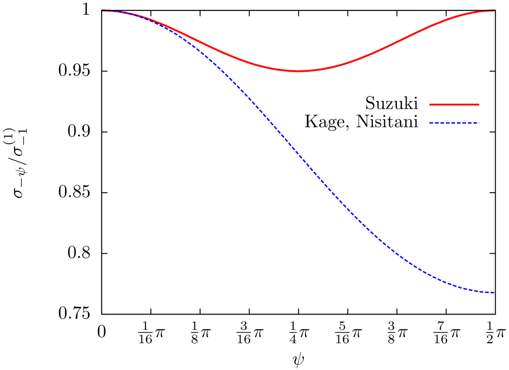
where the coefficients $a_1, \dots, a_9$ can be functions of the scalar invariants in the basis set Υ_{s} given before and $\overline{\boldsymbol{s}} = \boldsymbol{s} - \boldsymbol{\alpha}$. Difficult to get enough data and thus the following simplification is used

$$\dot{\boldsymbol{\alpha}} = \dot{\beta}\big\{C_\sigma\big[\mathrm{tr}(\overline{\boldsymbol{s}}\boldsymbol{M}^{(1)})\boldsymbol{M}^{(1)} + \mathrm{tr}(\overline{\boldsymbol{s}}\boldsymbol{M}^{(2)})\boldsymbol{M}^{(2)} + \mathrm{tr}(\overline{\boldsymbol{s}}\boldsymbol{M}^{(3)})\boldsymbol{M}^{(3)}\big] \\ + C_\tau\big[\boldsymbol{M}^{(1)}\overline{\boldsymbol{s}} + \overline{\boldsymbol{s}}\boldsymbol{M}^{(1)} + \boldsymbol{M}^{(2)}\overline{\boldsymbol{s}} + \overline{\boldsymbol{s}}\boldsymbol{M}^{(2)} + \boldsymbol{M}^{(3)}\overline{\boldsymbol{s}} + \overline{\boldsymbol{s}}\boldsymbol{M}^{(3)}\big]\big\}.$$

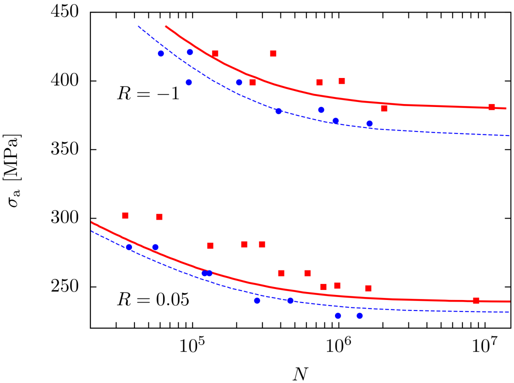
Damage evolution:

$$\dot{D} = \dot{\beta}\left(K_0 + K^{(1)}\frac{\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}^{(1)})}{\mathrm{tr}(\boldsymbol{\sigma}^2)} + K^{(2)}\frac{\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}^{(2)})}{\mathrm{tr}(\boldsymbol{\sigma}^2)} + K^{(3)}\frac{\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}^{(3)})}{\mathrm{tr}(\boldsymbol{\sigma}^2)}\right)\exp(L_0\beta).$$

Orthotropy - some results



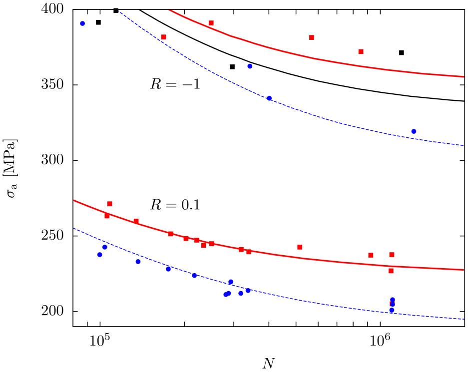
Fat. strengths in different orientations.



Suzuki data, red lines correspond to the 0° direction and the blue dashed line to the 45° direction.

Anisotropic format - transverse isotropy (cont'd)

SN-curves for L-PBF manufactured stainless steel 316L (data from Akfhami et al. 2021 and Blinn et al. 2019)



Kouhia et al. *EJMA* 2025

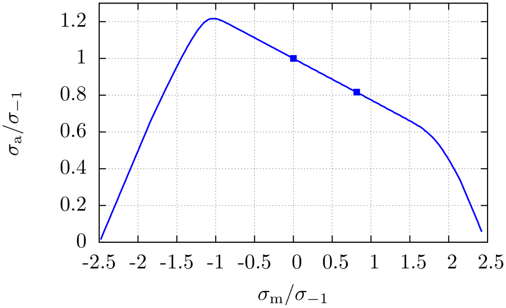
Extensions to better capture the form of the Haigh-diagram

The hydrostatic part of the endurance function $\beta = \frac{1}{\sigma_{-1}}(\bar{\sigma}_{\text{eff}} + \beta_{\text{h}}(I_1))$ as a solution of the diffusion-reaction equation ($k = \sigma_{\text{u}}^2/\delta^2, s = 1$)

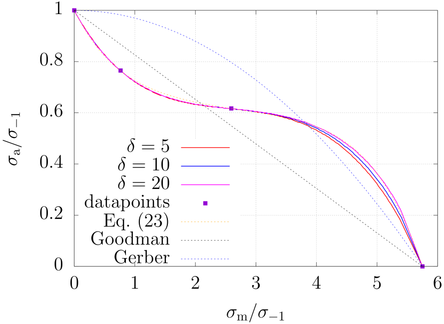
$$-k\frac{\mathrm{d}^2\beta_{\text{h}}}{\mathrm{d}I_1^2} + s\beta_{\text{h}} = \sum_{k=0}^{n} r_k (I_1/\sigma_{\text{u}})^k$$

$$-k\frac{\mathrm{d}^2\beta_{\text{h}}}{\mathrm{d}I_1^2} + s\beta_{\text{h}} = \tilde{r}_0 + \tilde{r}_1(I_1/\sigma_{\text{u}}) + \tilde{r}_2\left[(I_1 - I_1^*)/\sigma_{\text{u}}\right]^3$$

AISI4340 steel (n=1)



aluminium alloy LY12CZ



Eq. (23): $\sigma_{\text{af}} = \frac{1}{2}\left(\xi - \sigma_{\text{m}} + \sqrt{(\xi - \sigma_{\text{m}})^2 + 4\eta\xi\sigma_{\text{m}}}\right)$, Hou, Cai, Xu, *IJMS*, 2015

Unified LCF-HCF model

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Couples with the plasticity model, Chaboche type model adopted:

$$f(\boldsymbol{\sigma}, \boldsymbol{X}, R) = \sqrt{\tfrac{3}{2}(\boldsymbol{s} - \boldsymbol{X}) : (\boldsymbol{s} - \boldsymbol{X})} - (\sigma_{\mathrm{y}} + R) = 0$$

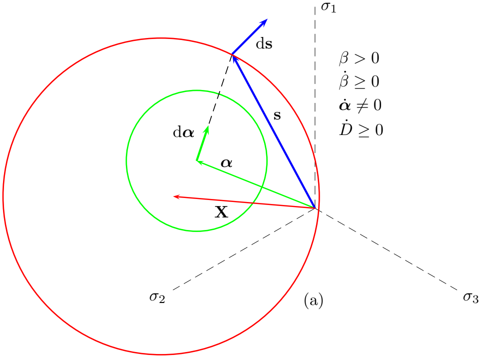
$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \qquad \dot{\varepsilon}^{\mathrm{p}}_{\mathrm{eff}} = \sqrt{\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}}$$

$$R = \sum R_i, \qquad \dot{R}_i = \gamma R_{\infty,i}\left(1 - R_i/R_{\infty,i}\right)\dot{\varepsilon}^{\mathrm{p}}_{\mathrm{eff}}$$

$$\boldsymbol{X} = \sum \boldsymbol{X}_i, \qquad \dot{\boldsymbol{X}}_i = \tfrac{2}{3}X_{\infty,i}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \gamma_i \dot{\varepsilon}^{\mathrm{p}}_{\mathrm{eff}}\boldsymbol{X}_i$$

Fatigue damage evolution equation:

$$\frac{\mathrm{d}D}{\mathrm{d}t} = K\exp(L\beta)\frac{\mathrm{d}\beta}{\mathrm{d}t} + M\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(Q\beta)\varepsilon^{\mathrm{p}}_{\mathrm{eff}}\right),$$



Stress history as a stochastic process (1/2)

Assume that the periodic stress history can be decomposed to the deterministic and noise parts as

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_{\mathrm{p}}(t) + \boldsymbol{\sigma}_{\mathrm{n}}(t),$$

and we have assumed that the noise is an Ornstein-Uhlenbeck process, which can be written as

$$\mathrm{d}\boldsymbol{\sigma}_{\mathrm{n}}(t) = \boldsymbol{\lambda}\left[\boldsymbol{\mu} - \boldsymbol{\sigma}_{\mathrm{n}}(t)\right]\mathrm{d}t + \boldsymbol{\eta}\mathrm{d}\boldsymbol{W}(t),$$

where $\boldsymbol{W}$ denotes the Wiener process (white noise).

This stochastic differential equation is solved by Euler-Maruyama method

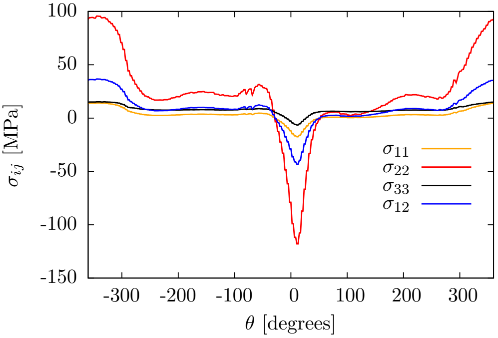
$$\boldsymbol{\sigma}_{\mathrm{n}}(t_{k+1}) = \boldsymbol{\sigma}_{\mathrm{n}}(t_k) + \boldsymbol{\lambda}\left[\boldsymbol{\mu} - \boldsymbol{\sigma}_{\mathrm{n}}(t_k)\right](t_{k+1} - t_k) + \boldsymbol{\eta}\left[\boldsymbol{W}(t_{k+1}) - \boldsymbol{W}(t_{k+1})\right].$$

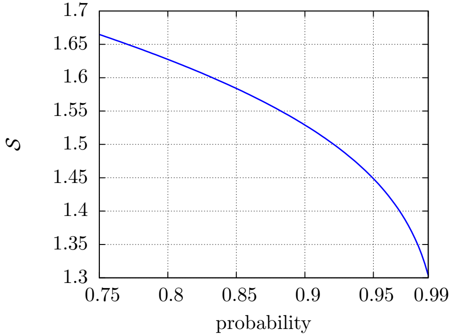
Stress history as a stochastic process (2/2)

The safety factor for infinite lifetime can be defined as $\mathcal{S} = 1/(1 + M^*)$, where $M^* = \max \beta(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}^*; \sigma_{-1})$; the stable state $\boldsymbol{\alpha}^*$ can be found by the min-max problem

$$\min_{\boldsymbol{\alpha}} \max_{t \geq 0} \beta(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}; \sigma_{-1}), \quad \max_{t \geq 0} \beta(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}; \sigma_{-1}) \leq 0.$$

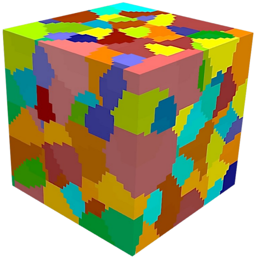
Connecting rod: computed stress history based on measured pressure curves - Frondelius et al. *EJMA* 2022



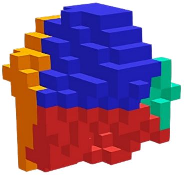


Ongoing/future work

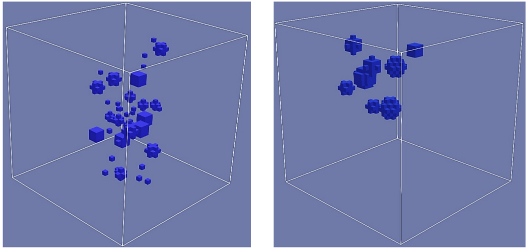
- *Macroscopic parameters* as *functions of microstrucural features*.



Single Prior austenite with 4 martensite packets



Different inclusion distributions



- Endurance function depending on defect size (damage dependency)
- Different stochastic processes
- Applications to other materials, e.g. composites

Concluding remarks

- continuum based macroscopic fatigue analysis method provide a flexible methodology
- new features can be easily added
- inherently multiaxial
- does not need cycle-counting methods

Human fatigue illustrated by Akseli Gallen-Kallela 1894

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