

An investigation into the behaviour of the Kachanov-Rabotnov type continuum damage model

H. Askes¹ , J. Hartikainen², K. Kolari³, **R. Kouhia**² T. Saksala², J. Vilppo²

¹ University of Twente , The Netherlands

²Tampere University, Finland

³VTT Technical Research Centre of Finland Ltd

reijo.kouhia@tuni.fi

XIV Finnish Mechanics Days, Lappeenranta, May 18-20, 2022

Introduction

In 1958 Kachanov introduced a model to describe continuous degradation of a material.

He introduced a single variable - damage index or integrity - which continuously reduces the elastic properties.

For the evolution of the integrity ϕ , he proposed the following kinetic law ([modified by Rabotnov 1959](#))

$$\dot{\phi} = -\frac{A}{\phi^p} \left(\frac{\sigma}{\phi} \right)^n, \quad \sigma = \phi E \varepsilon \quad (1)$$

For an undamaged material $\phi = 1$ and at fully damaged state $\phi = 0$.

Thermodynamic formulation

- The specific free energy ψ (per unit mass) depends on strain $\boldsymbol{\varepsilon}$ and damage $D = 1 - \phi$: Videlly used simple form of the free energy is

$$\rho\psi(\boldsymbol{\varepsilon}, D) = \frac{1}{2}(1 - D)\mathbf{C}\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = (1 - D)W_e. \quad (2)$$

- The dissipative behaviour is modelled using the dual form of the dissipation potential depending on the **dissipative force** $Y_d = -Y_e = W_e$ as

$$\varphi^*(Y_d; D) = \frac{1}{r+1} \frac{Y_r}{(1-D)^p t_d} \left(\frac{Y_d}{Y_r} \right)^{r+1}, \quad Y_r = \frac{1}{2} E \varepsilon_r^2 = \frac{\sigma_r^2}{2E} \quad (3)$$

- The Coleman-Noll procedure yields the constitutive equations:

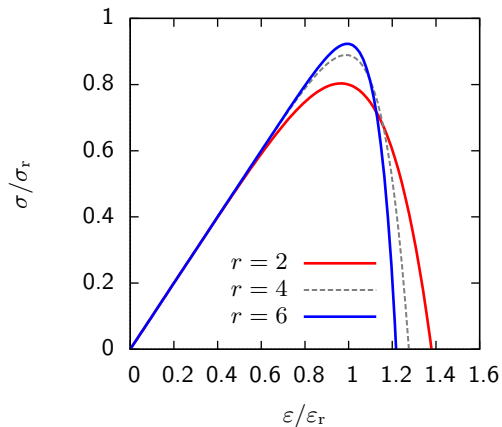
$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = (1 - D)\mathbf{C} : \boldsymbol{\varepsilon} \quad \text{and} \quad \dot{D} = \frac{\partial \varphi^*}{\partial Y_d} = \frac{1}{t_d(1-D)^p} \left(\frac{Y_d}{Y_r} \right)^r. \quad (4)$$

Motivation - understand the model behaviour

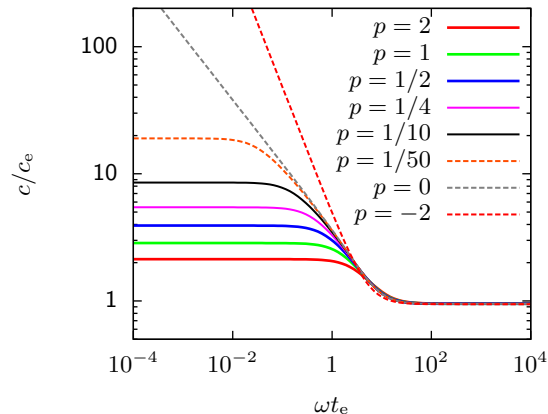
Uniaxial constant strain rate loading $\varepsilon(t) = \dot{\varepsilon}_0 t$. Stress-strain and dispersion behaviour of damaging bar.

H. Askes, J. Hartikainen, K. Kolari, R. Kouhia, T. Saksala, J. Vilppo. On the Kachanov-Rabotnov continuum damage model. *Rakenteiden Mekaniikka*, 20(2)2020, 125-144.

$$p = 0, \dot{\varepsilon}_0 t_d / \varepsilon_r = 1$$



$$\text{at peak stress, } r = 4, t_e = L/c_e$$



Localisation study

Definition:

$$\text{localisation width: } l_{\text{loc}} = \text{meas} \{x | D^* \leq D \leq 1\} \quad (5)$$

D^* damage value at fracture stress for quasi-static constant strain-rate loading

$$D^* = 1 - \left(\frac{2r - 1}{2r + p + 2} \right)^{1/(p+1)}, \quad (6)$$

independent of the applied strain-rate.

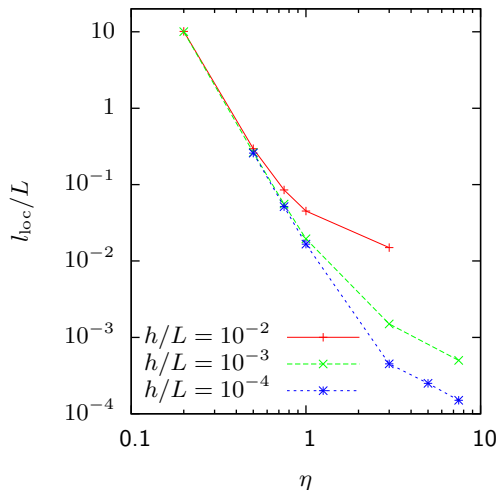
FE-study of a semi-infinite bar with prescribed displacement

$$u(0, t) = \eta \varepsilon_r L t / t_d \quad \text{and} \quad L = c_e t_d, \quad c_e = \sqrt{E/\rho} \quad (7)$$

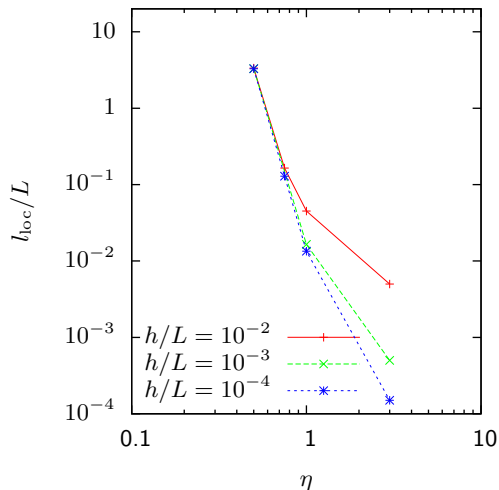
Linear finite elements, central-difference time integration.

Damage localisation width - prescribed displacement

$$r = 2, \quad p = 1$$

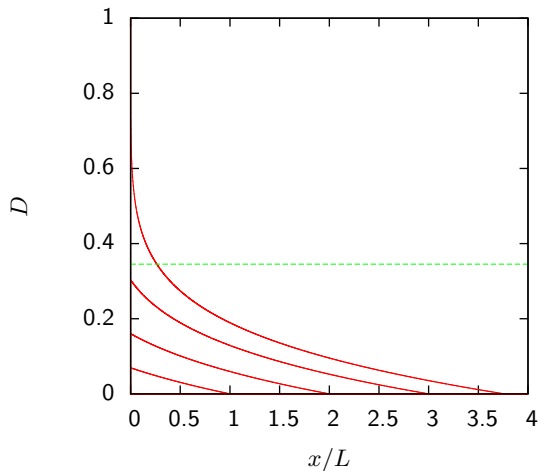


$$r = 4, \quad p = 1$$



Damage profiles

At times $t_d, 2t_d, 3t_d$ and $3.76t_d$, $p = 1, r = 2, \eta = 0.5$ and $h = L/1000$.



Gradient enhancement - balance equations

The generalized principle of virtual power is stated as

$$\mathcal{P}_{\text{int}} + \mathcal{P}_{\text{ine}} + \mathcal{P}_{\text{mech}} = 0 \quad (8)$$

where the power of internal and inertial forces are

$$\mathcal{P}_{\text{int}} = - \int_V \left(\boldsymbol{\sigma} : \text{sym}(\text{grad } \mathbf{v}) + Y \dot{D} + \mathbf{Z} \cdot \text{grad } \dot{D} + X \Delta \dot{D} \right) dV, \quad \mathcal{P}_{\text{ine}} = - \int_V \rho \dot{\mathbf{v}} \cdot \mathbf{v} dV, \quad (9)$$

and the power of external mechanical forces is

$$\mathcal{P}_{\text{mech}} = \int_V \rho \mathbf{b} \cdot \mathbf{v} dV + \int_S \mathbf{t} \cdot \mathbf{v} dS + \int_V \rho b_D \dot{D} dV + \int_S t_D \dot{D} dS + \int_S x_D \text{grad } \dot{D} \cdot \mathbf{n} dS \quad (10)$$

gives the balance equations

$$\text{div } \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dot{\mathbf{v}}, \quad \text{in } V, \quad \text{and} \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \quad \text{in } S, \quad (11)$$

and

$$- \Delta X + \text{div } \mathbf{Z} - Y + \rho b_D = 0, \quad \text{in } V, \quad \text{and} \quad (\mathbf{Z} - \nabla X) \cdot \mathbf{n} = t_D, \quad X = x_D \quad \text{in } S. \quad (12)$$

Gradiend enhancement - potential functions and power of dissipation

Assume now that the specific Helmholtz free energy is expressed as

$$\psi(\boldsymbol{\varepsilon}, D, \text{grad } D, \Delta D) \quad (13)$$

and the thermodynamic forces are additively decomposed into energetic and dissipative components as

$$Y = Y_e + Y_d, \quad \mathbf{Z} = \mathbf{Z}_e + \mathbf{Z}_d, \quad X = X_e + X_d, \quad (14)$$

where the energetic parts are

$$Y_e = \rho \frac{\partial \psi}{\partial D}, \quad \mathbf{Z}_e = \rho \frac{\partial \psi}{\partial \text{grad } D}, \quad X_e = \rho \frac{\partial \psi}{\partial \Delta D}. \quad (15)$$

The dual form of the dissipation potential

$$\varphi^*(Y_d, \mathbf{Z}_d, X_d; D, \text{grad } D, \Delta D) \quad (16)$$

and the dissipation power can be expressed in the forms

$$\gamma = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + Y \dot{D} + \mathbf{Z} \cdot \text{grad } \dot{D} + X_e \Delta \dot{D} - \rho \dot{\psi} = \frac{\partial \varphi^*}{\partial Y_d} Y_d + \frac{\partial \varphi^*}{\partial \mathbf{Z}_d} \cdot \mathbf{Z}_d + \frac{\partial \varphi^*}{\partial X_d} X_d \geq 0. \quad (17)$$

Gradient enhancement - specification

If we assume the simplest form of the free energy as

$$\begin{aligned}\rho\psi &= \frac{1}{2}(1-D)\mathbf{C}_e\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} + \frac{1}{2}k_1E\ell_1^2(\text{grad } D)^2 + \frac{1}{2}k_2E\ell_1^4(\Delta D)^2 \\ &= (1-D)W_e + \frac{1}{2}k_1E\ell_1^2(\text{grad } D)^2 + \frac{1}{2}k_2E\ell_1^4(\Delta D)^2.\end{aligned}\quad (18)$$

Further assume that in addition to stress also $\mathbf{Z}$ and X are energetic, thus

$$\mathbf{Z}_e = k_1\text{grad } D, \quad X_e = k_2\ell_1^2\Delta D \quad (19)$$

and

$$Y_d = -\Delta X_e + \text{div } \mathbf{Z}_e - Y_e = -\Delta(k_2\ell_1^2\Delta D) + \text{div}(k_1\text{grad } D) + W_e. \quad (20)$$

Now we consider the dual form of the dissipation potential in the form

$$\varphi^*(Y_d; D, \text{grad } D, \Delta D) \quad (21)$$

Gradient enhancement - specification (cont'd)

Assume now

$$\varphi^*(Y_d; D, \text{grad } D, \Delta D) = \frac{Y_r}{r+1} \frac{\langle 1 + \ell_2^2 \|\text{grad } D\|^2 + \ell_3^2 \Delta D \rangle}{t_d (1-D)^p} \left(\frac{Y_d}{Y_r} \right)^{r+1}, \quad (22)$$

then the damage rate is

$$\dot{D} = \frac{1}{t_d (1-D)^p} \left(\frac{Y_d}{Y_r} \right)^r \langle 1 + \ell_2^2 \|\text{grad } D\|^2 + \ell_3^2 \Delta D \rangle, \quad (23)$$

with

$$Y_d = W_e - \Delta (k_2 \ell_1^2 \Delta D) + \text{div} (k_1 \text{grad } D). \quad (24)$$

Gradient enhancement - example 1 - only first order gradients

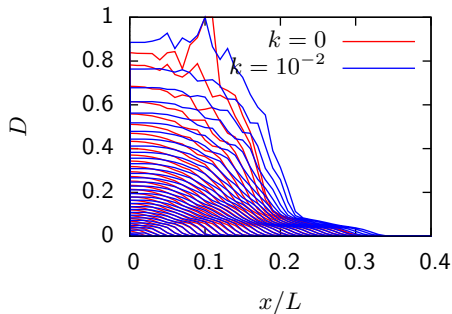
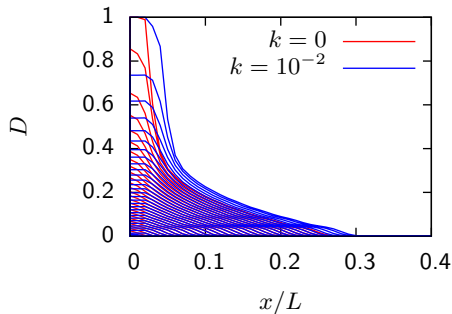
$$\rho\psi(\varepsilon, D, \text{grad } D) = (1 - D)W_e + \frac{1}{2}k_1 E \ell_1^2 (\text{grad } D)^2, \quad \varphi^*(Y_d; D, \text{grad } D) = \frac{Y_r}{r+1} \frac{\langle 1 + \ell_2^2 \|\text{grad } D\|^2 \rangle}{t_d (1 - D)^p} \left(\frac{Y_d}{Y_r} \right)^{r+1}$$

As before - **dynamic study of a semi-infinite bar** with prescribed displ. - FD also for spatial discretisation:

$$u(0, t) = \eta \varepsilon_r L t / t_d, \quad L = c_e t_d, \quad c_e = \sqrt{E/\rho}, \quad h = L/100, \quad k_1 > 0.$$

$$\ell_2 = \frac{1}{10} L$$

$$\ell_2 = \frac{2}{5} L$$



Gradient enhancement - example 2 - only second order gradients

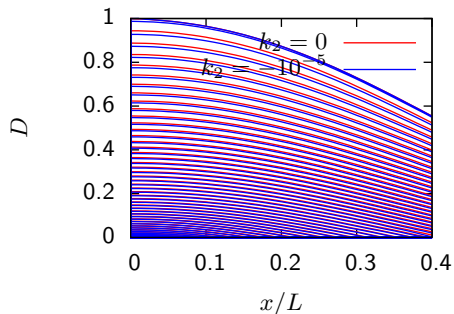
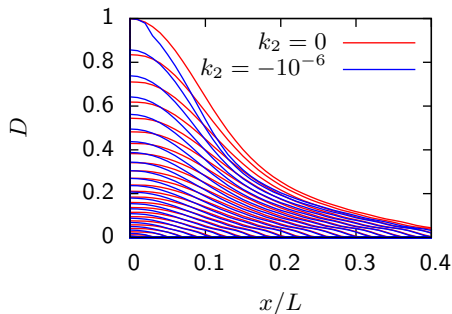
$$\rho\psi(\epsilon, D, \Delta D) = (1 - D)W_e + \frac{1}{2}k_2 E \ell_1^4 (\Delta D)^2, \quad \varphi^*(Y_d; D, \Delta D) = \frac{Y_r}{r+1} \frac{\langle 1 + \ell_3^2 \|\Delta D\|^2 \rangle}{t_d(1-D)^p} \left(\frac{Y_d}{Y_r} \right)^{r+1}$$

As before - **dynamic study of a semi-infinite bar** with prescribed displ. - FD also for spatial discretisation:

$$u(0, t) = \eta \epsilon_r L t / t_d, \quad L = c_e t_d, \quad c_e = \sqrt{E/\rho}, \quad \ell_1 = L/10, \quad h = L/100, \quad k_2^* < k_2 < 0.$$

$$\ell_3 = \frac{1}{10} L$$

$$\ell_3 = \frac{2}{5} L$$



Some references

- Askes H., Sluys, L.J. Explicit and implicit gradient series in damage mechanics. *Eur. J. Mech. A/Solids*, **21**, 2002, 379–390.
- Comi C. Computational modelling of gradient-enhanced damage in quasi-brittle materials. *Mech. Cohes.-Frict. Mater.*, **4**, 1999, 17–36.
- Comi C., Corigliano A. On uniqueness of the dynamic finite-step problem in gradient-dependent softening plasticity. *Int. J. Sol. Struct.*, **33**, 1996, 3881–3902.
- Frémond M., Nedjar B. Damage, gradient of damage and principle of virtual power. *Int. J. Sol. Struct.*, **33**, 1996, 1083–1104.
- Liebe T., Steinmann P., Benallal A. Theoretical and computational aspects of a thermodynamically consistent framework for geometrically linear damage. *Comp. Methods Appl. Mech. Engng*, **190**, 2001, 6555–6576.
- Nguyen Q.S. On standard gradient plasticity and visco-plasticity. *Int. J. Sol. Struct.*, **225**, 2021, 111038.
- Polizzotto C. Unified thermodynamic framework of nonlocal/gradient continuum theories. *Eur. J. Mech. A/Solids*, **22**, 2003, 651–668.

Concluding remarks

- Numerical study of the localisation properties of the Kachanov-Rabotnov damage model.
- Localisation width depends strongly on the loading rate and the r -parameter.
- Decreasing the loading rate increases the localisation zone width which is in contrast to strain-softening viscoplasticity.
- Gradient damage? Further studies needed, e.g. dispersion and stability analysis.

Thank you for your attention!



Riikka Soininen - Flow. Oil and gilded metal leaf on canvas, 2021