On numerical modeling of coupled magnetoelastic problem

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MAGNETOELASTICITY

- Balance equations

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{f} + \mathbf{f}_{\mathrm{em}}$$

$$\operatorname{curl}\mathbf{H} = \mathbf{J}, \qquad \operatorname{div}\mathbf{B} = 0, \qquad \operatorname{div}\mathbf{J} \qquad = 0$$

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$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \qquad \mathbf{M} = -\rho \frac{\partial \psi}{\partial \mathbf{B}}$$

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$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mu_0^{-1}[\mathbf{B}\mathbf{B} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}] + (\mathbf{M}\cdot\mathbf{B})\mathbf{I} - \mathbf{B}\mathbf{M}$$

$$-\operatorname{div}\boldsymbol{\tau} = \mathbf{f}$$

CONSTITUTIVE MODEL - Helmholtz free energy

- For isotropic magnetoelastic solid: $\psi(\boldsymbol{\varepsilon}, \mathbf{B})$
- Integrity basis

$$I_1 = \mathrm{tr}\,\boldsymbol{\varepsilon}, \qquad I_2 = \tfrac{1}{2}\mathrm{tr}\,\boldsymbol{\varepsilon}^2, \qquad I_3 = \tfrac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}^3$$

$$I_4 = \mathbf{B}\cdot\mathbf{B}, \qquad I_5 = \mathbf{B}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{B}, \qquad I_6 = \mathbf{B}\cdot\boldsymbol{\varepsilon}^2\cdot\mathbf{B}$$

- Suitable choice for $\rho\psi =$

$$\tfrac{1}{2}\lambda I_1^2 + 2GI_2 + \tfrac{1}{2}\left(\sum_{i=0}^{4}\frac{1}{i+1}g_i(I_1)I_4^{i+1}\right) + \tfrac{1}{2}\gamma_5 I_5 + \tfrac{1}{2}\gamma_6 I_6$$

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CONSTITUTIVE MODEL - functions $g_i(I_1)$

- Deformation isochoric under pure magnetic excitation $\Longrightarrow g_i(I_1)$
- Result

$$\gamma_6 = 2\gamma_5$$

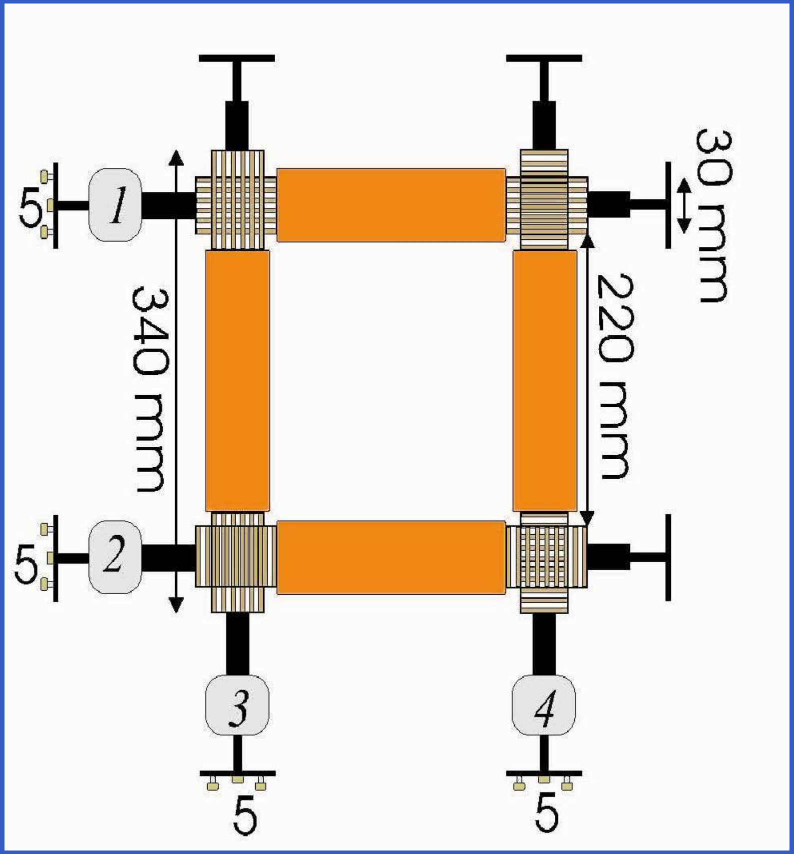
$$g_0 = (\gamma_4^{(0)} + \tfrac{1}{4}\mu_0^{-1} - \tfrac{1}{4}\gamma_5)\exp(\tfrac{4}{3}I_1) - \tfrac{1}{4}\mu_0^{-1} + \tfrac{1}{4}\gamma_5$$

$$g_i = \gamma_4^{(i)}\exp(\tfrac{4}{3}I_1) \quad \text{when} \quad i = 1, \ldots, 4$$

where $\gamma_4^{(i)} = g_i(0)$ are unknown parameters

CONSTITUTIVE MODEL - experimental verification

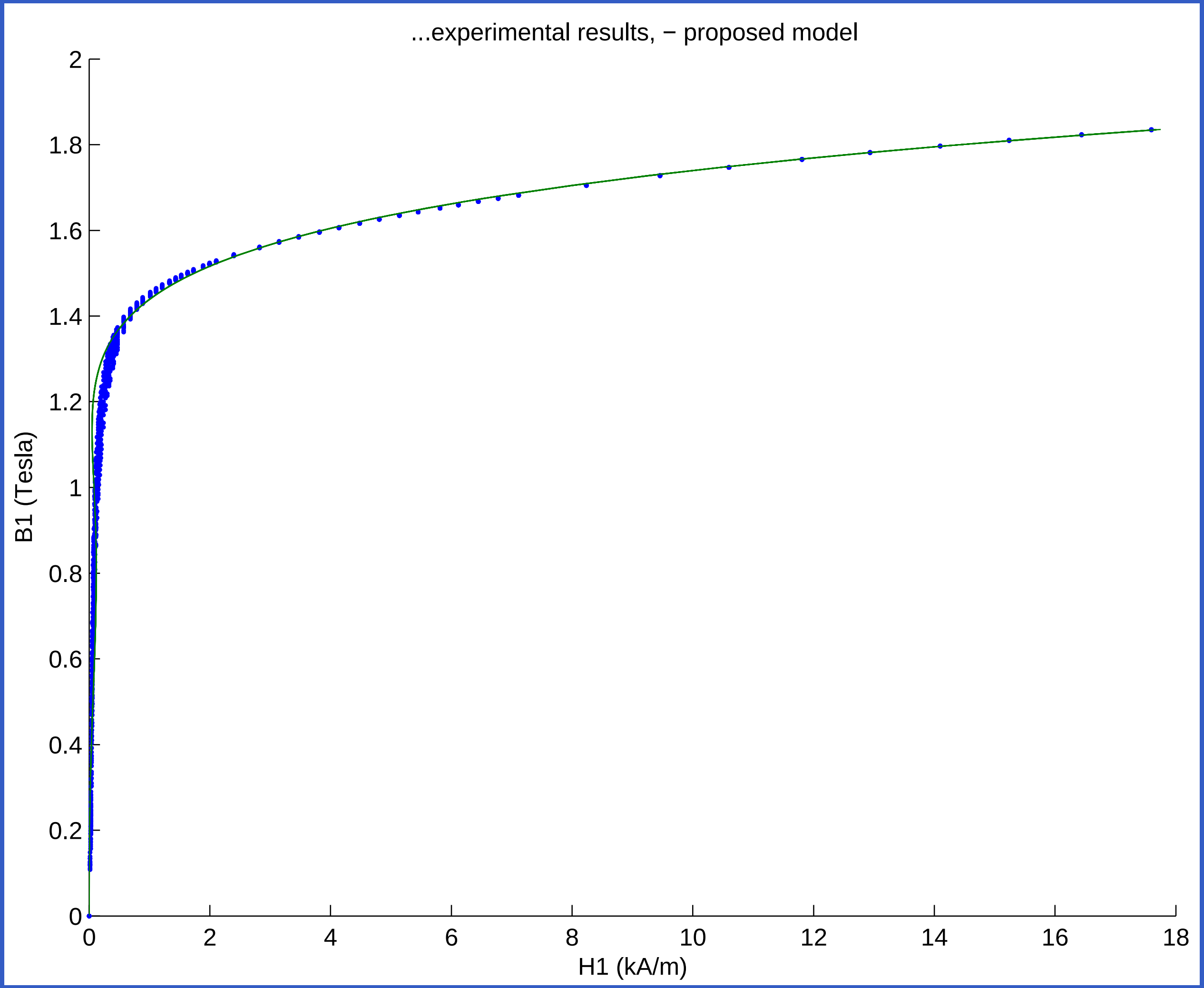
Epstein frame (Belahcen 2004)



1–4: load cells
5: screw system for loading

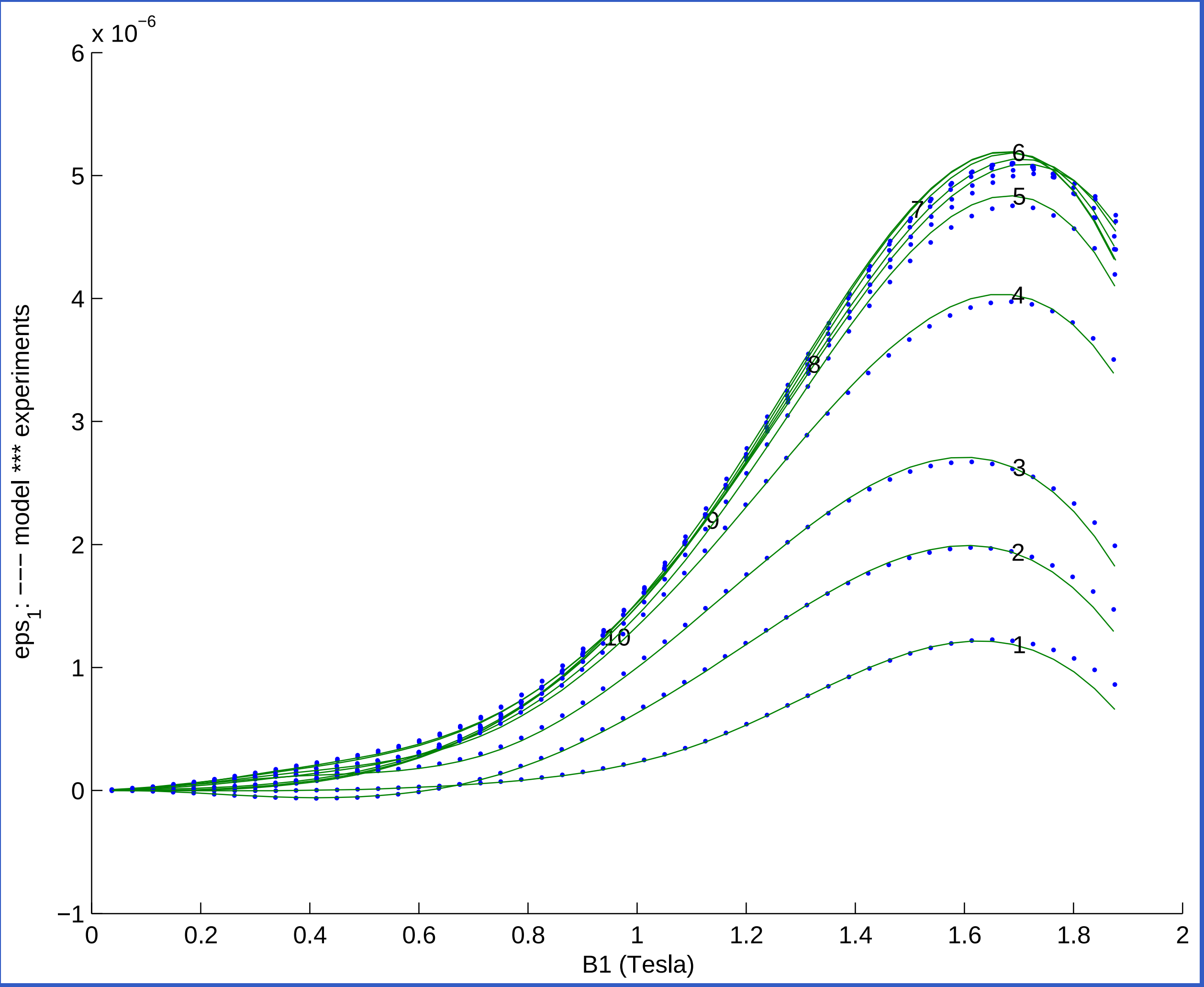
CONSTITUTIVE MODEL - experimental verification

Magnetization curves



CONSTITUTIVE MODEL - experimental verification

Magnetostriction
different compressive pre-stress 2 - 7 MPa



NUMERICAL SOLUTION - potential approach

- Vector potential $\mathbf{A}$, such that $\mathbf{B} = \operatorname{curl}\mathbf{A}$.
- FE-problem: find $\{\mathbf{A}_h, p_h\} \in R_0^h \times G_0^h$
  $R_0^h \subset H_0(\operatorname{curl}, \Omega)$ and $G_0^h \subset H_0^1(\Omega)$

$$(\operatorname{curl}\hat{\mathbf{A}}_h, \mathbf{H}_h) + (\hat{\mathbf{A}}_h, \operatorname{grad} p_h) = (\hat{\mathbf{A}}_h, \mathbf{J}) - [\hat{\mathbf{A}}_h, \bar{\mathbf{H}}]; \quad \forall\, \hat{\mathbf{A}}_h \in R_0^h$$

$$(\operatorname{grad}\hat{p}_h, \mathbf{A}_h) = 0; \quad \forall\, \hat{p}_h \in G_0^h$$

Magnetic field strength from the constitutive model

$$\mathbf{H} = (\mu_0^{-1} + 2f_4)\mathbf{B} + \gamma_5 \mathbf{B}\cdot\boldsymbol{\varepsilon} + \gamma_6 \mathbf{B}\cdot\boldsymbol{\varepsilon}^2, \qquad f_4 = \rho\frac{\partial\psi}{\partial I_4}$$

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- Interelement continuity for the tangential component required for fields in $H(\operatorname{curl}, \Omega)$
- Nedelec edge elements

NUMERICAL SOLUTION - least-square approach

- B-formulation

$$\mathcal{L}_B = \frac{1}{2}(\operatorname{div}\mathbf{B}, \operatorname{div}\mathbf{B}) - (\mathbf{p}, \operatorname{curl}\mathbf{H} - \mathbf{J})$$

FEM: $\mathbf{B}_h \in D_0^h \subset H_0(\operatorname{div}, \Omega), \quad \mathbf{p} \in R_0^h \subset H_0(\operatorname{curl}, \Omega)$

Continuity for the normal component of $\mathbf{B}_h$ required

Raviart-Thomas elements for $\mathbf{B}_h$

NUMERICAL SOLUTION - least-square approach

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  Continuity for the normal component of $\mathbf{B}_h$ required
  Raviart-Thomas elements for $\mathbf{B}_h$

- H-formulation

$$\mathcal{L}_H = \frac{1}{2}(\operatorname{curl}\mathbf{H} - \mathbf{J}, \operatorname{curl}\mathbf{H} - \mathbf{J}) - (p, \operatorname{div}\mathbf{B})$$

  FEM: $\mathbf{H}_h \in R^h \subset H(\mathrm{curl}, \Omega), \quad p \in G^h \subset H^1(\Omega)$
  Continuity for the tangential component of $\mathbf{H}_h$ required
  Nedelec elements for $\mathbf{H}_h$

NUMERICAL SOLUTION - least-square approach

- Minimizing the error in the constitutive model $\mathbf{B} = \mu\mathbf{H}$:

$$\mathcal{L}_{BH} = \int_{\Omega} \left(\tfrac{1}{2}|\mathbf{B} - \mu\mathbf{H}|^2 + q\,\mathrm{div}\,\mathbf{B} + \mathbf{p} \cdot (\mathrm{curl}\,\mathbf{H} - \mathbf{J}) \right) dV$$

- Piecewise continuous interpolation for $q, \mathbf{p}$
- Penalty approach $\Rightarrow$ element level elimination of $q, \mathbf{p}$

EXAMPLE

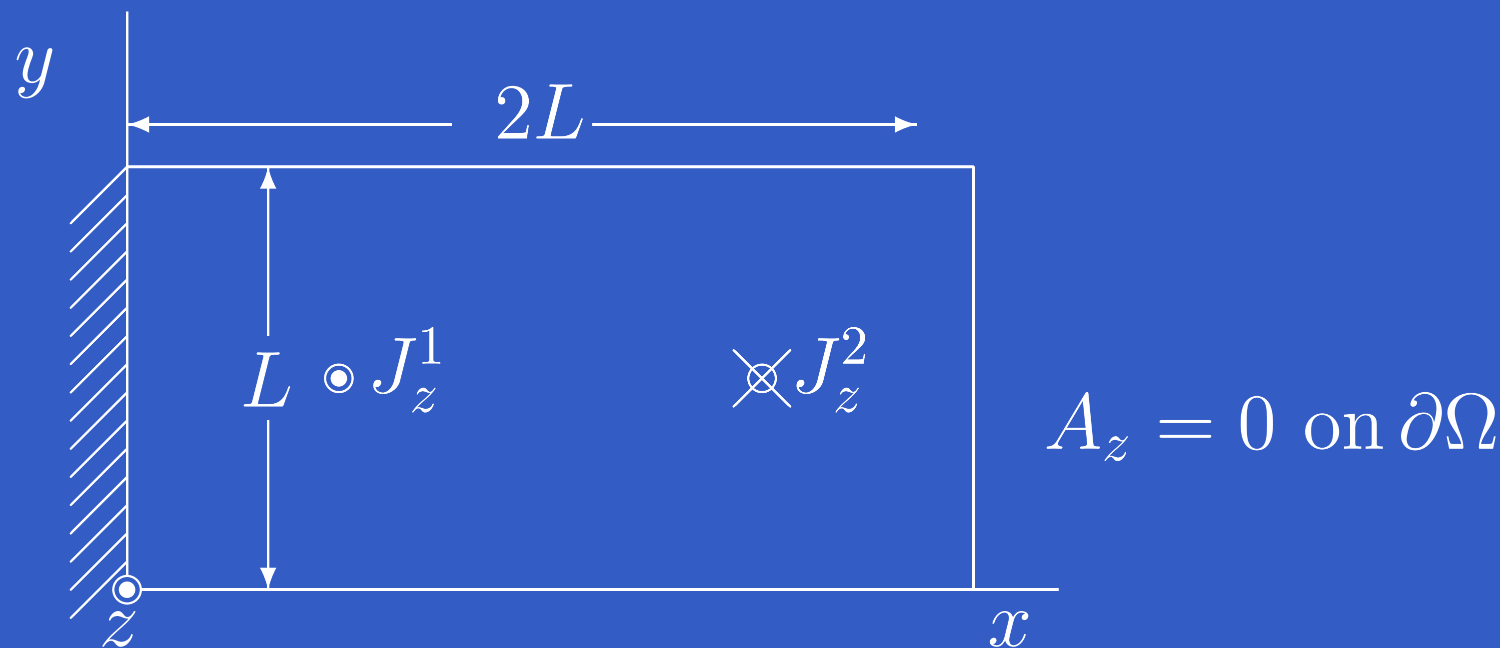
Plane-strain problem, 36×18 bilinear element mesh

$J_z^1 = 100\,\mathrm{A}, \quad J_z^2 = -100\,\mathrm{A},\ L = 1\,\mathrm{m}$

$E = 183.62\,\mathrm{GPa}, \nu = 0.34,\ \mu_0 = 4\pi \cdot 10^{-7}\mathrm{N/A^2}$

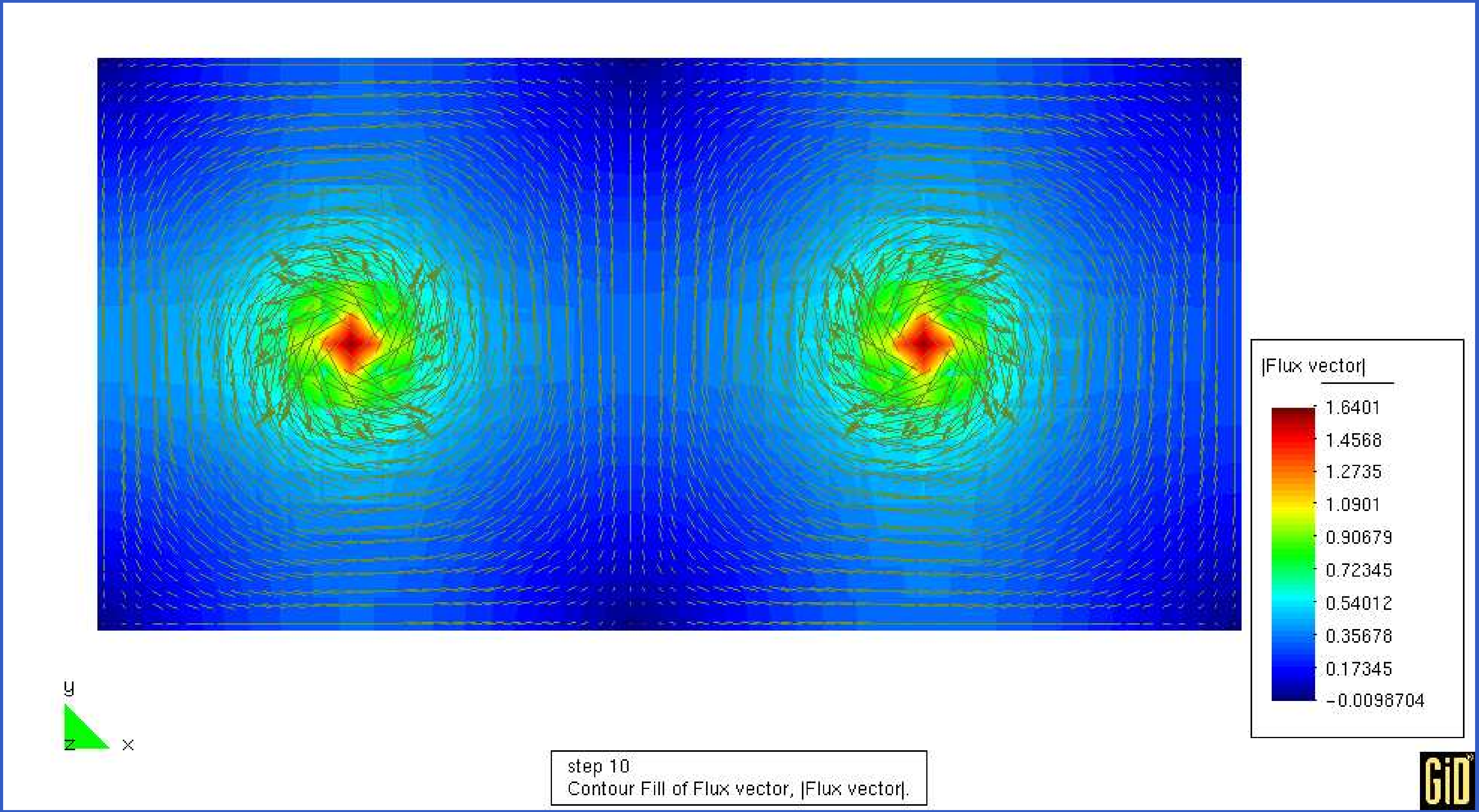
$\gamma_4^{(0)} = -0.99974858\mu_0^{-1}, \gamma_4^{(1)} = 0.00076054\mu_0^{-1}, \gamma_4^{(2)} = -0.00089881\mu_0^{-1}$

$\gamma_4^{(3)} = 0.00008964\mu_0^{-1}, \gamma_4^{(4)} = 0.00012688\mu_0^{-1}, \gamma_5 = -0.028\mu_0^{-1}$



EXAMPLE

Magnetic flux density



Deformations