On the integration of inelastic constitutive models coupled to damage

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  - Kinematic hardening model
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  - Conclusions

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MOTIVATION

SIMO & HUGHES, *Computational Inelasticity*, Remark 3.3.2.2 on page 125:

"The overall superiority of the radial return method relative to other return schemes is conclusively established in Krieg and Krieg [1977]; Schreyer, Kulak and Kramer [1979] and Yoder and Whirley [1984]."

All the ref's above consider only rate-independent small strain plasticity models

What can be said for finite strain models / rate-dependent plasticity and coupling with damage?

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INTRODUCTION – Scalar model problems

Maxwell creep model $\quad \dot{\epsilon}^{\mathrm{i}} = \tau_{\mathrm{vp}}^{-1}(\sigma/\sigma_{\mathrm{r}}) \qquad \dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^{\mathrm{i}})$

$$\dot{\sigma} + \frac{E}{\tau_{\mathrm{vp}}}\frac{\sigma}{\sigma_{\mathrm{r}}} = E\dot{\epsilon}, \qquad \Longrightarrow \qquad \dot{\hat{\sigma}} + \frac{1}{\tau_{\mathrm{vp}}\epsilon_{\mathrm{r}}}\hat{\sigma} = \frac{\dot{\epsilon}}{\epsilon_{\mathrm{r}}}, \quad \hat{\sigma}(t_0) = \hat{\sigma}_0$$

where $\hat{\sigma} = \sigma/\sigma_{\mathrm{r}}$ and $\epsilon_{\mathrm{r}} = \sigma_{\mathrm{r}}/E$.

Kachanov/Rabotnov type damage model

$$\sigma = (1 - D)E\epsilon, \quad \dot{D} = \frac{1 + D}{\tau_{\mathrm{d}}}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r} \quad Y_{\mathrm{r}} = \frac{\sigma_{\mathrm{r}}^2}{2E}, \quad D(t_0) = D_0.$$

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Solutions

For constant strain-rate loading $\dot{\epsilon} = \dot{\epsilon}_{\mathrm{c}}$.

The creep model

$$\hat{\sigma}(t) = \tau_{\mathrm{vp}}\dot{\epsilon}_{\mathrm{c}}\left[1 + \left(\frac{\hat{\sigma}_0}{\tau_{\mathrm{vp}}\dot{\epsilon}_{\mathrm{c}}} - 1\right)\exp\left(-\frac{t - t_0}{\tau_{\mathrm{d}}\epsilon_{\mathrm{r}}}\right)\right].$$

$\hat{\sigma}(t) \longrightarrow \tau_{\mathrm{d}}\dot{\epsilon}_{\mathrm{c}}$ as $t \longrightarrow \infty$.

The damage model

$$D = (1 + D_0)\exp\left\{\frac{\epsilon_{\mathrm{r}}}{(2r+1)\tau_{\mathrm{d}}\dot{\epsilon}_{\mathrm{c}}}\left[\left(\frac{\dot{\epsilon}_{\mathrm{c}}t}{\epsilon_{\mathrm{r}}}\right)^{2r+1} - \left(\frac{\dot{\epsilon}_{\mathrm{c}}t_0}{\epsilon_{\mathrm{r}}}\right)^{2r+1}\right]\right\} - 1.$$

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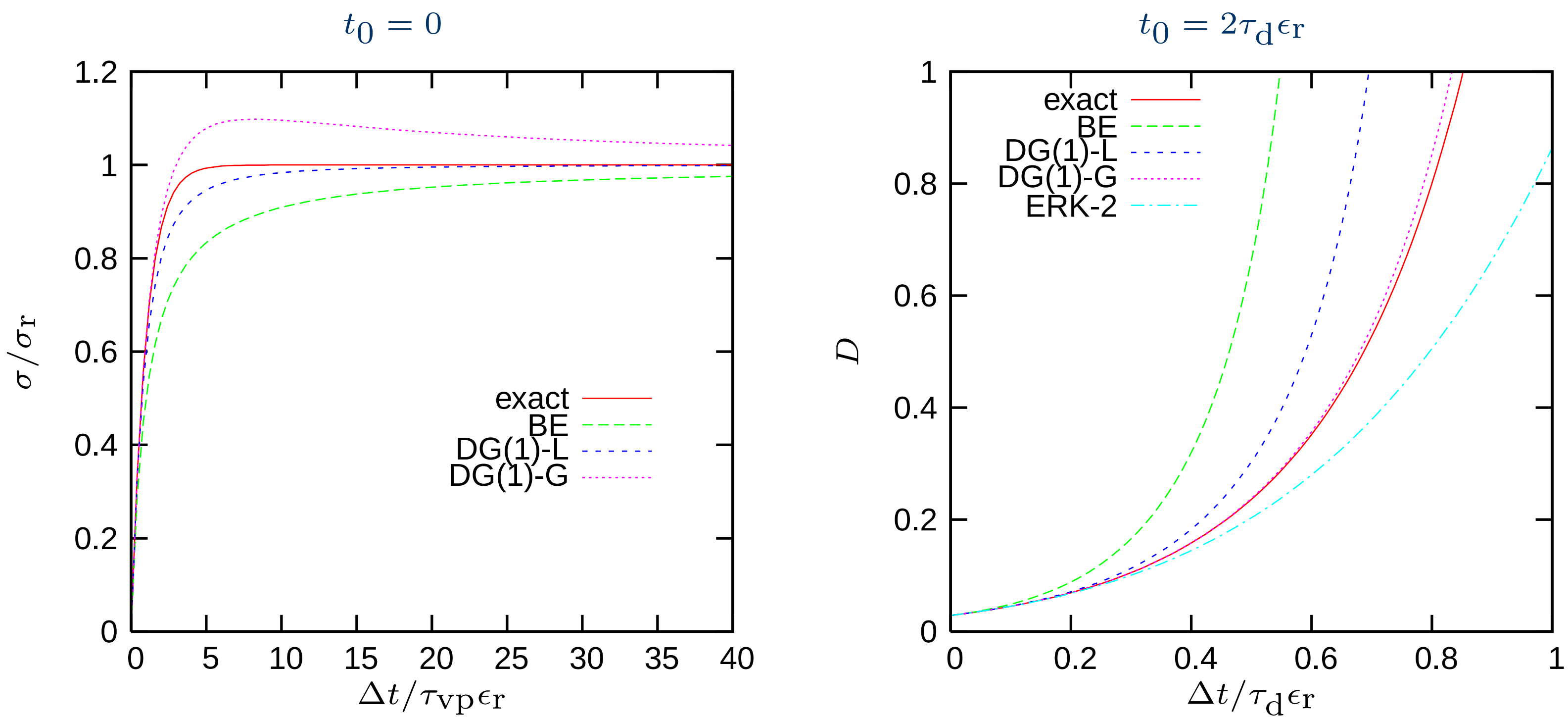
Amplification factors = one-step solution



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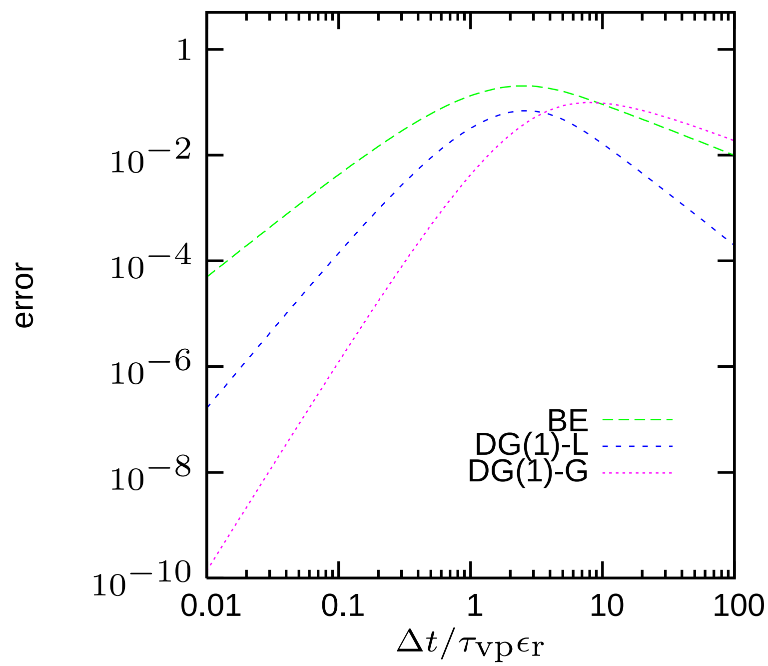
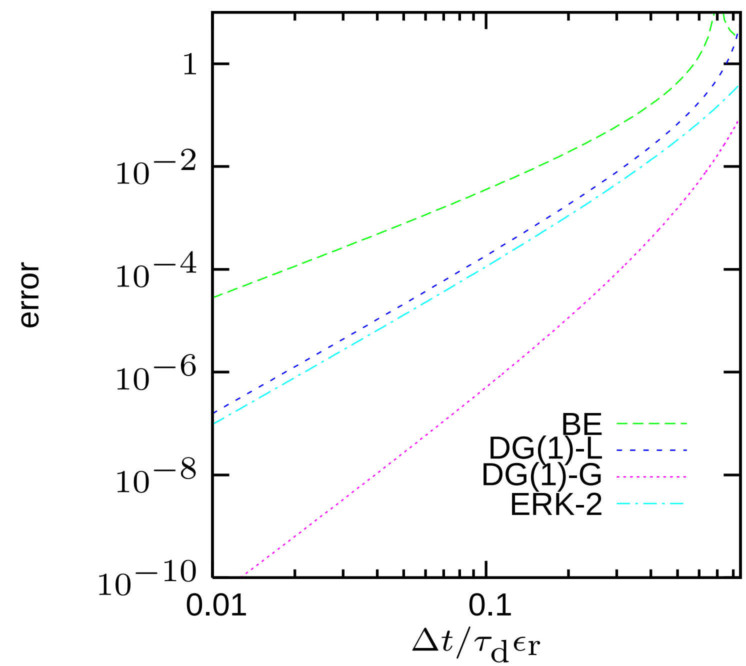
Amplification factors = one-step solution

Creep model



Damage model



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An ideal integrator

for rate-dependent inelastic constitutive models should be:

1. L-stable
2. For $\dot{y} + \lambda y = 0$ (λ constant) the amplification factor should be
   (a) strictly positive
   (b) monotonous with respect to time step

Padé $(0, q)$-approximations of $\exp(-\lambda t)$ are positive and monotonous.
DG(1)-Lobatto = IRKL3C-2 = Padé-(0,2) for $\dot{y} + \lambda y = 0$

What can be said when the model is coupled to damage?

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An ideal (good) integrator

for rate-dependent inelastic models coupled to damage

- based on discontinuous Galerkin methods,
- an explicit-implicit split type methods
- DIRK, LIRK or ????
- ATS-tensor should be derivable

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RATE-DEPENDENT MODEL

Helmholtz free energy

$$\rho\psi = \tfrac{1}{2}(1 - D)\epsilon_\mathrm{e} : C_\mathrm{e} : \epsilon_\mathrm{e} = \tfrac{1}{2}\omega\epsilon_\mathrm{e} : C_\mathrm{e} : \epsilon_\mathrm{e}$$

Dissipation potential

$$\varphi(\sigma, Y) = \varphi_\mathrm{d}(Y)\varphi_\mathrm{tr}(\sigma) + \varphi_\mathrm{vp}(\sigma)$$

$\varphi_\mathrm{tr} \geq 0$ $\quad\varphi_\mathrm{tr} \approx 0$ when $||\dot{\epsilon}_\mathrm{i}|| < \eta$ and $\varphi_\mathrm{tr} > 1$ when $||\dot{\epsilon}_\mathrm{i}|| > \eta$

$$\varphi_\mathrm{d} = \frac{1}{r+1}\frac{Y_\mathrm{r}}{\tau_\mathrm{d}\omega}\left(\frac{Y}{Y_\mathrm{r}}\right)^{r+1}$$

$$\varphi_\mathrm{tr} = \frac{1}{pn}\left[\frac{1}{\tau_\mathrm{vp}\eta}\left(\frac{\bar{\sigma}}{\omega\sigma_\mathrm{r}}\right)^{p}\right]^{n} \qquad \sim \frac{||\dot{\epsilon}_\mathrm{i}||}{\eta}$$

$$\varphi_\mathrm{vp} = \frac{1}{p+1}\frac{\sigma_\mathrm{r}}{\tau_\mathrm{vp}}\left(\frac{\bar{\sigma}}{\omega\sigma_\mathrm{r}}\right)^{p+1}$$

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Evolution equations

$$\begin{cases} \dot{\boldsymbol{\sigma}} & = \boldsymbol{f}_\sigma(\boldsymbol{\sigma}, \omega) = \omega \boldsymbol{C}_\mathrm{e}(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_\mathrm{i}) + \dfrac{f_\omega}{\omega}\boldsymbol{\sigma} = \omega \boldsymbol{C}_\mathrm{e}\left(\dot{\boldsymbol{\epsilon}} - g(\bar{\sigma}, \omega)\dfrac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}\right) + \dfrac{f_\omega}{\omega}\boldsymbol{\sigma}, \\ \dot{\omega} & = f_\omega(\boldsymbol{\sigma}, \omega) = -\dfrac{\varphi_\mathrm{tr}}{\tau_\mathrm{d}\omega}\left(\dfrac{Y}{Y_\mathrm{r}}\right)^r . \end{cases}$$

Non-dimensional form $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$ where $\boldsymbol{y} = \left[(\boldsymbol{\sigma}/\sigma_\mathrm{r})^T, \omega\right]^T$

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Jacobian matrix

In a uniaxial case the Jacobian of $\boldsymbol{f}$ is a 2×2 matrix
$\boldsymbol{y} = [\sigma/\sigma_{\mathrm{r}}, \omega]^T = [z, \omega]^T$ and $\boldsymbol{f} = [f_z, f_\omega]^T$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{bmatrix} \dfrac{\partial f_z}{\partial z} & \dfrac{\partial f_z}{\partial \omega} \\ \dfrac{\partial f_\omega}{\partial z} & \dfrac{\partial f_\omega}{\partial \omega} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \widetilde{J}_{11} - J_{22} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad \text{if} \quad \varphi_{\mathrm{tr}} \equiv 1$$

Properties

$$\widetilde{J}_{11} \leq 0, \quad J_{12} > 0 \quad \text{when} \quad \dot{\epsilon}_{\mathrm{c}} > 0, \quad J_{21} < 0, \quad J_{22} \geq 0$$

Initially, when $z = \sigma/\sigma_{\mathrm{r}} \ll 1$ and $\omega \approx 1$ then

$$|J_{12}| \gg |J_{11}| \gg |J_{22}| \sim |J_{21}|$$

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Eigenvalues of the Jacobian

$$\lambda_{1,2} = \tfrac{1}{2}\widetilde{J}_{11} \pm \sqrt{(\tfrac{1}{2}\widetilde{J}_{11} - J_{22})^2 + J_{12}J_{21}}$$

Three possible cases:

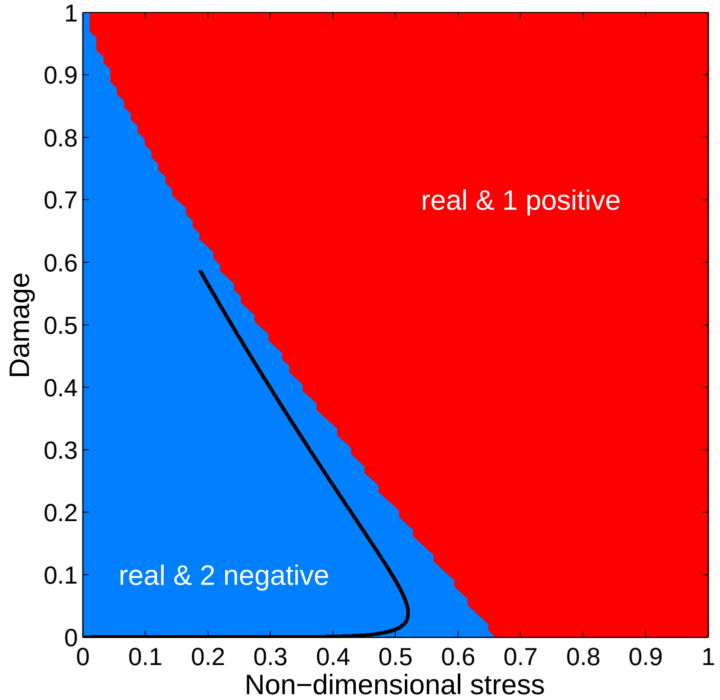
1. both eigenvalues are real and negative (BE-good)
2. eigenvalues are complex conjugates with negative real part ($\widetilde{J}_{11} < 0$) (LIRK-good)
3. eigenvalues are real, but of opposite sign (explicit-implicit split?)

dG(q)-schemes for $q \geq 1$ are accurate also for large timesteps but expensive

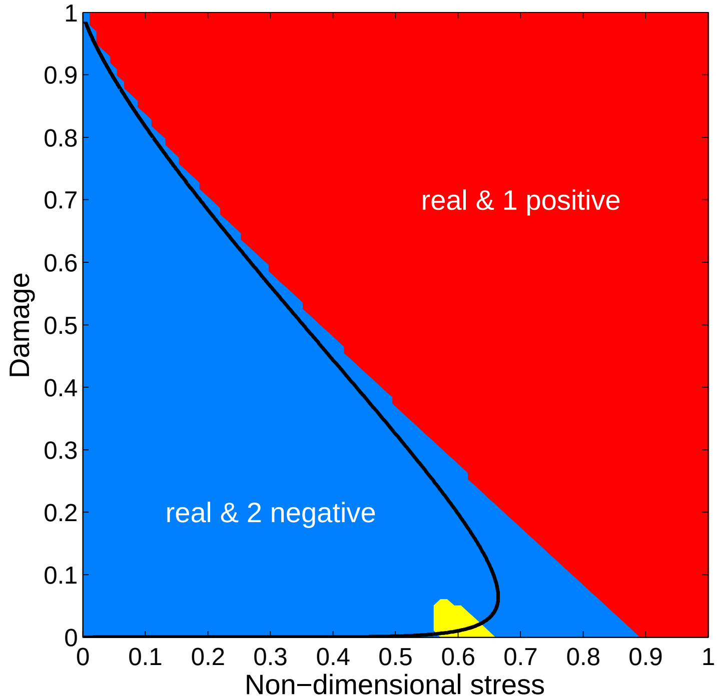
Eigenvalues of the Jacobian

$$\varphi_{\mathrm{tr}} \equiv 1$$

$$\dot{\epsilon}_{\mathrm{c}} = 10^{-4}$$



$$\dot{\epsilon}_{\mathrm{c}} = 4.05 \cdot 10^{-4}$$



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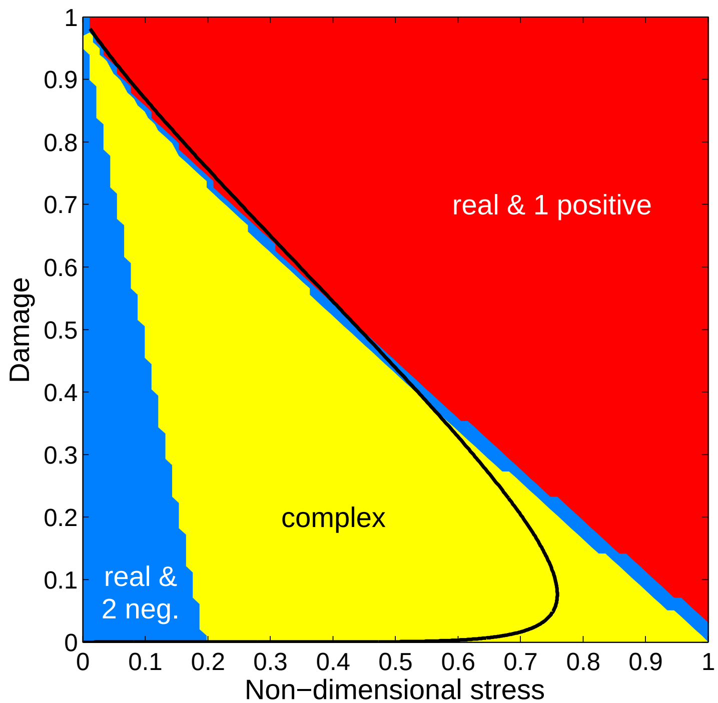
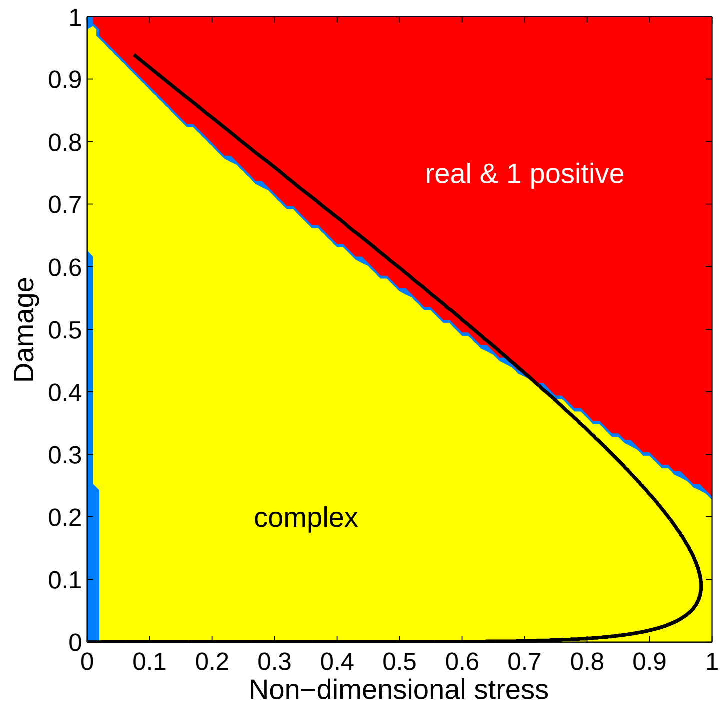
Eigenvalues of the Jacobian

$$\varphi_{\mathrm{tr}} \equiv 1$$

$$\dot{\epsilon}_{\mathrm{c}} = 10^{-3}$$



$$\dot{\epsilon}_{\mathrm{c}} = 7.5 \cdot 10^{-3}$$



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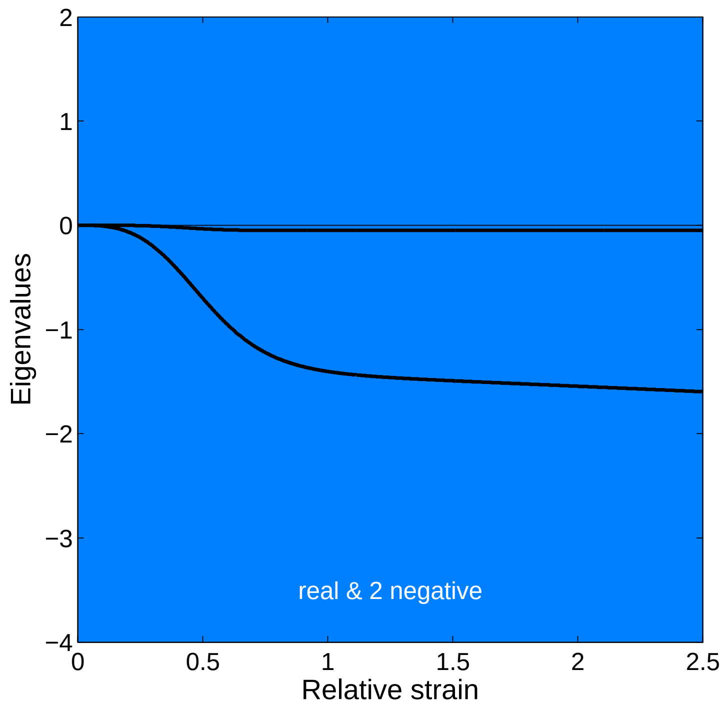
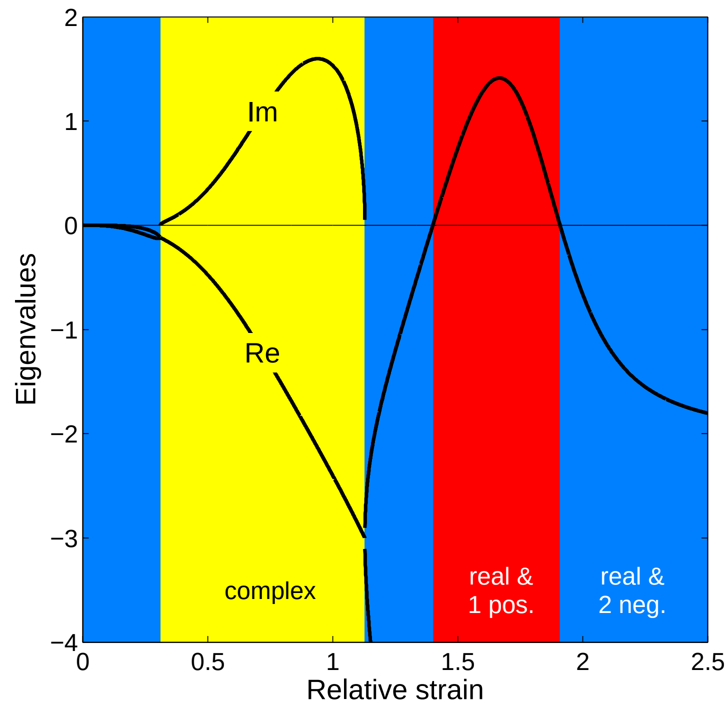
Eigenvalues of the Jacobian

$$\varphi_{\mathrm{tr}} \equiv 1$$

$$\dot{\epsilon}_{\mathrm{c}} = 10^{-4}$$



$$\dot{\epsilon}_{\mathrm{c}} = 6.5 \cdot 10^{-4}$$



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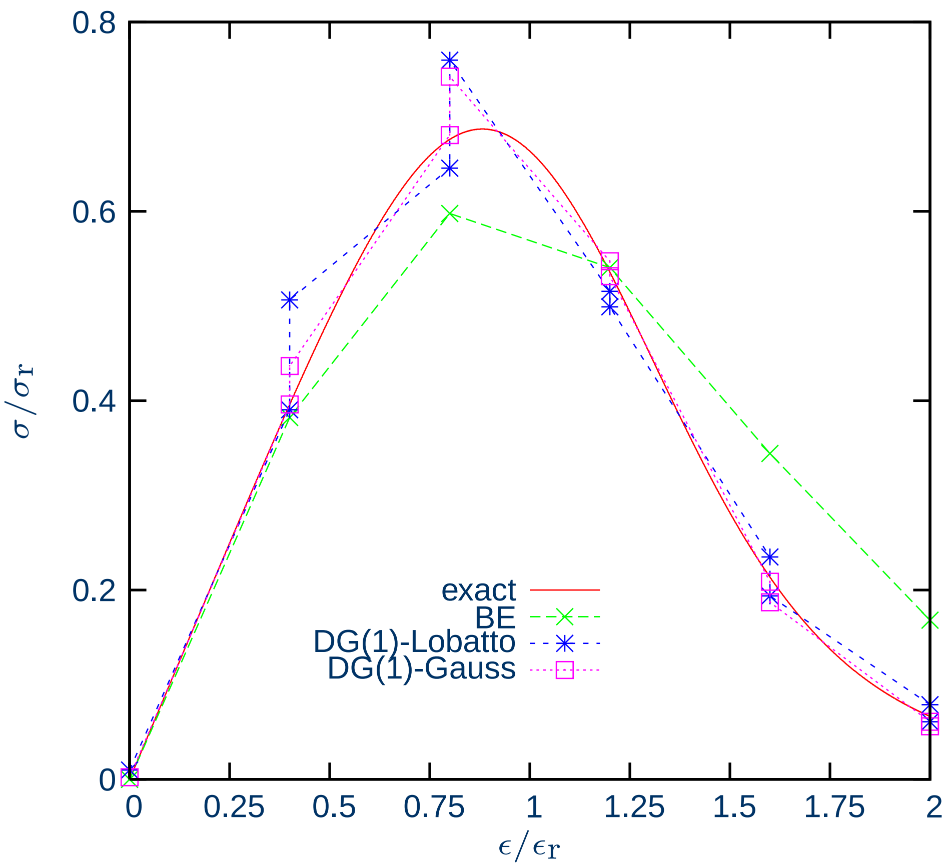
Numerical example

Uniaxial constant strain-rate loading $\dot{\epsilon}_{\mathrm{c}} = 5 \cdot 10^{-4}$ 1/s, and $\varphi_{\mathrm{tr}} \equiv 1$



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Uniaxial constant strain-rate loading $\dot{\epsilon}_{\mathrm{c}} = 5 \cdot 10^{-4}$ 1/s, and $\varphi_{\mathrm{tr}} \equiv 1$

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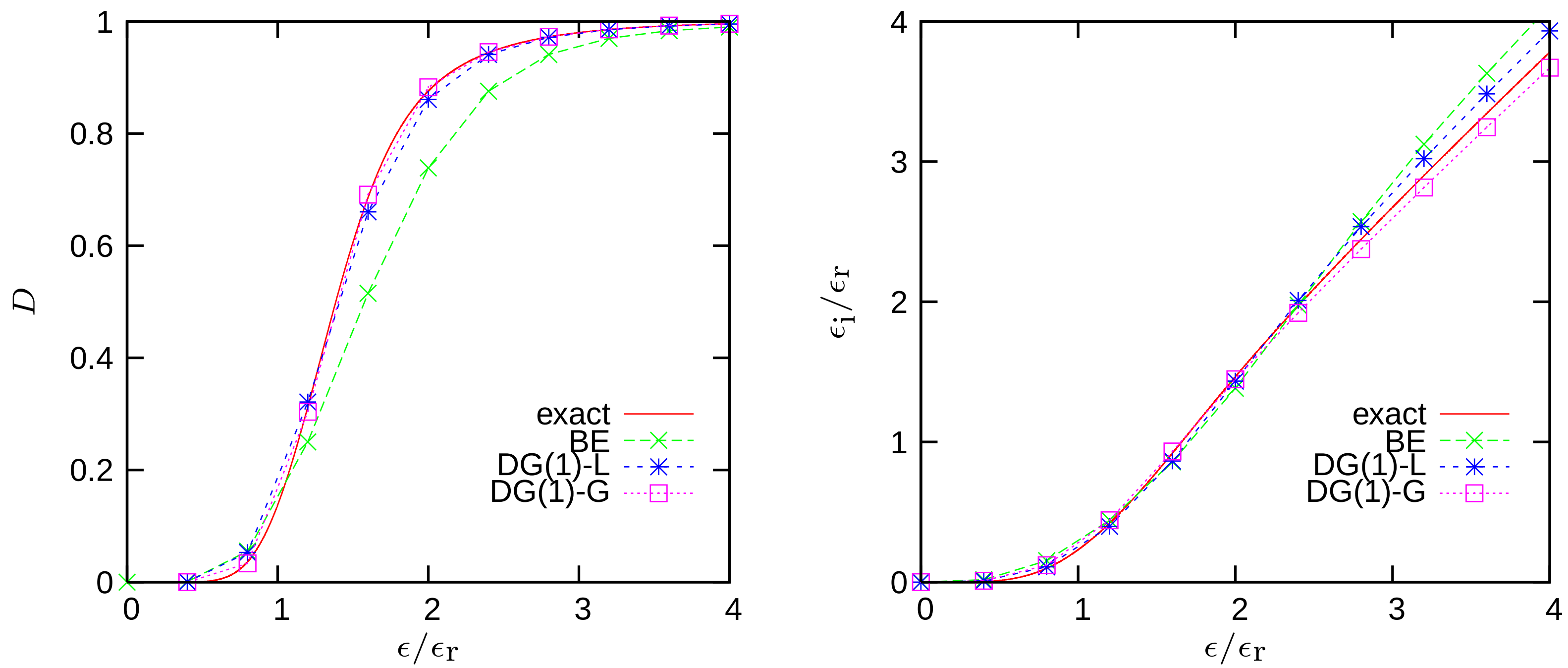

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Conclusions

- dG(q)-methods ($q \geq 1$) have excellent accuracy properties also for large time steps
- change from BE to explicit-implicit split algorithm when the Jacobian has a positive eigenvalue will be too late for accurate solution, if large time step is used
- 2-stage LIRK has better accuracy than BE, especially when damage evolution is dominant

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FINITE STRAIN MODEL

Constitutive relations based on the finite strain kinematic hardening hardening model of Wallin et al. (2003), Wallin and Ristinmaa (2005). Damage potential by Lemaitre (1985) or Lemaitre and Chaboche (1990).

$$\boldsymbol{\Sigma} = K_d \ln J^e \mathbf{1} + G_d \ln \boldsymbol{C}_i^e, \qquad \boldsymbol{B} = \frac{1}{2} h (\ln \boldsymbol{C}^k)^{dev}, \qquad Y = -\frac{\psi^e}{1-D}$$

Evolution laws:

$$\dot{\boldsymbol{F}}^p = \dot{\lambda} \left(\frac{3\bar{\boldsymbol{\Sigma}}}{2\sigma_y(1-D)} \right) \boldsymbol{F}^p = \dot{\lambda} \boldsymbol{N}^p, \qquad \dot{\boldsymbol{F}}^k = \dot{\lambda} \boldsymbol{F}^k \left(\frac{3\bar{\boldsymbol{\Sigma}}}{2\sigma_y} - \frac{3\boldsymbol{B}}{2B_\infty} \right) = \dot{\lambda} \boldsymbol{N}^k$$

$$\dot{D} = -\dot{\lambda} \frac{\partial g^d}{\partial Y} = \dot{\lambda} N^D, \qquad \dot{\epsilon}^p_{eff} = \frac{\dot{\lambda}}{1-D} = \dot{\lambda} N^\varepsilon, \quad \dot{\lambda} \quad \text{from} \quad \dot{f} = 0$$

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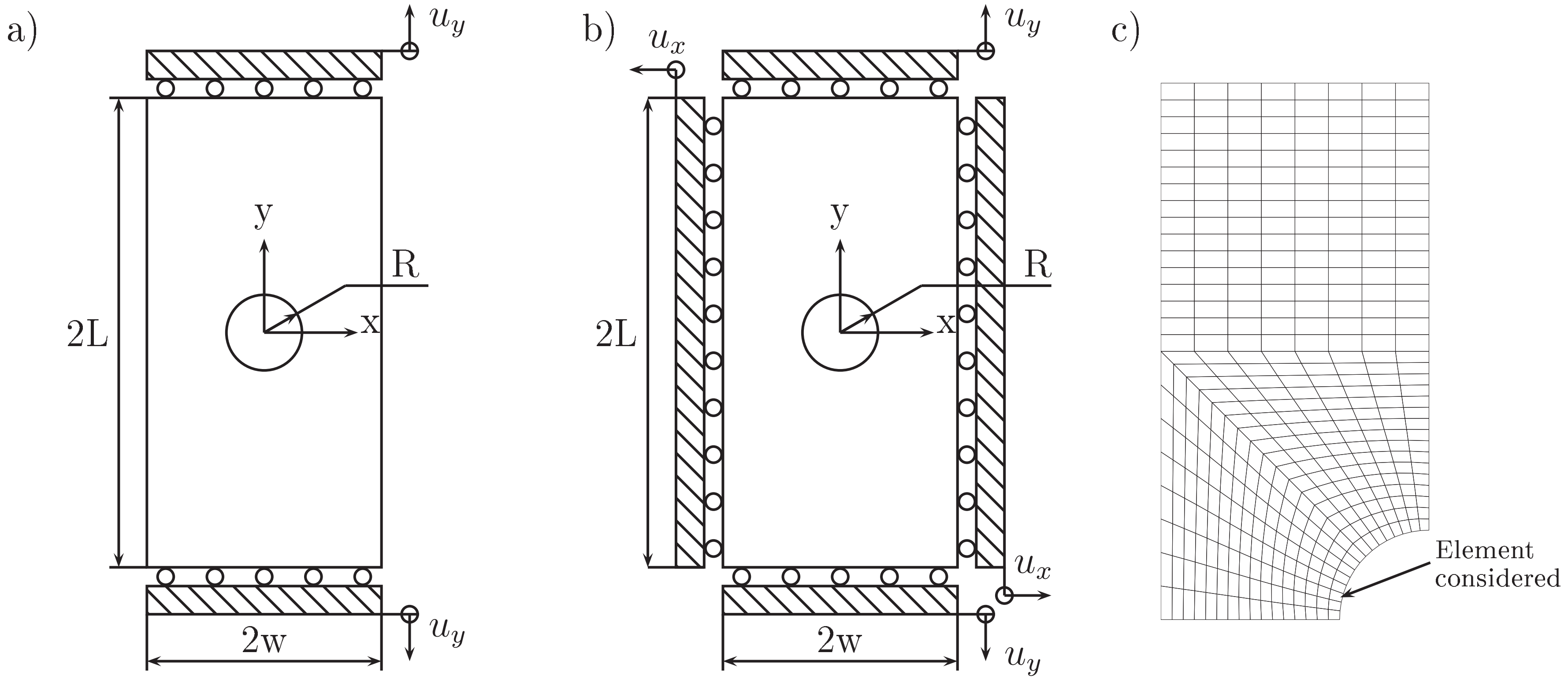
Numerical example



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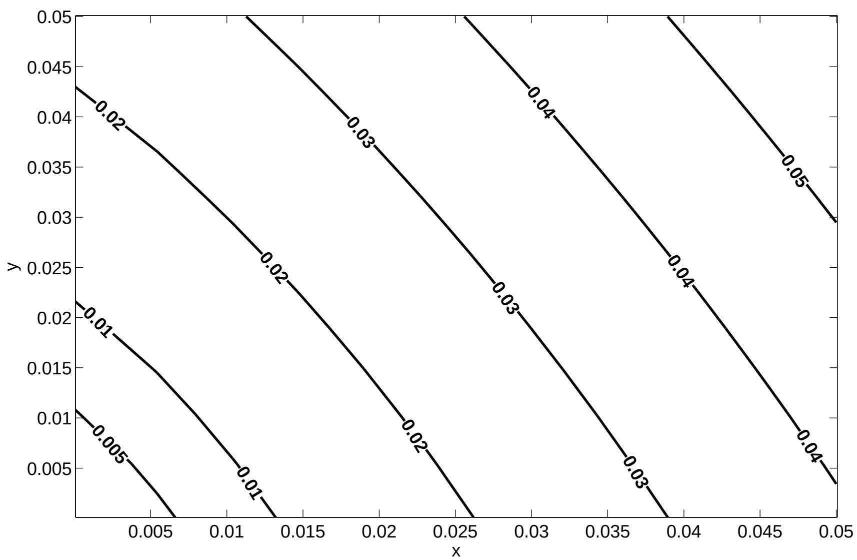
Iso-error plots

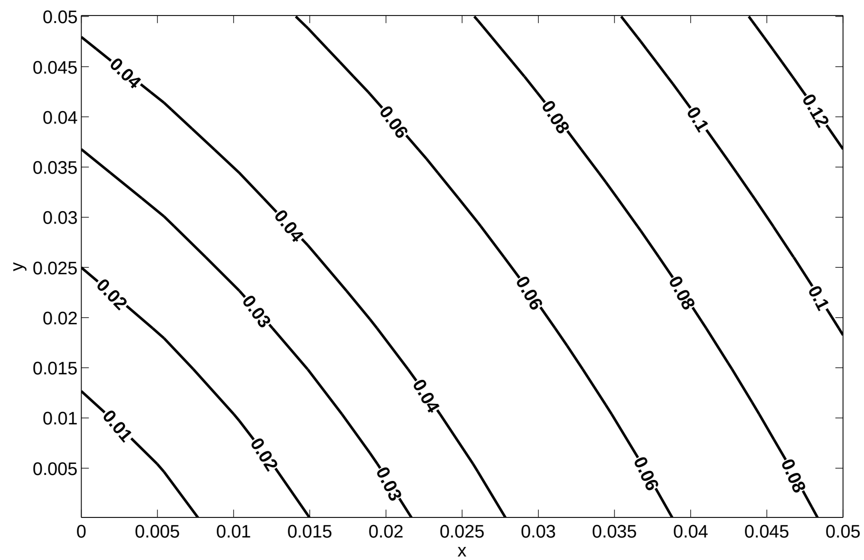
$$e_{\mathrm{err}} = \left(\int_A (\boldsymbol{y}_{xy}^{\mathrm{ref}} - \boldsymbol{y}_{xy}) \boldsymbol{D} (\boldsymbol{y}_{xy}^{\mathrm{ref}} - \boldsymbol{y}_{xy}) dA \right)^{\frac{1}{2}}$$

Displacements and internal variables stored in $\boldsymbol{y}_{xy}$. Weight matrix $\boldsymbol{D}$.

DIRK Ellsiepen (1999)



BE



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Conclusions

- two stage DIRK-scheme superior to BE for smooth damage evolution
- DIRK-scheme fits nicely in the standard FE formulation
- DIRK provides error control for the boundary value problem
- incompressibility is approximately preserved (can be controlled)
- algoritmic tangent tensor can be computed exactly the same way as in standard implicit FE formulations