An adaptive integration scheme in computational inelasticity

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MOTIVATION

SIMO & HUGHES, *Computational Inelasticity*, Remark 3.3.2.2 on page 125:

"The overall superiority of the radial return method relative to other return schemes is conclusively established in Krieg and Krieg [1977]; Schreyer, Kulak and Kramer [1979] and Yoder and Whirley [1984]."

WHY ?

SCALAR MODEL PROBLEM

Maxwell creep model $\dot{\epsilon}^{\mathrm{in}} = \gamma(\sigma/\sigma_{\mathrm{ref}})$ $\quad \dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^{\mathrm{in}})$

$$\dot{\sigma} + (E\gamma/\sigma_{\mathrm{ref}})\sigma = E\dot{\epsilon}$$

$$\boxed{\dot{\sigma} + \lambda\sigma = f, \;\; \sigma(0) = \sigma_0}$$

$$\lambda = E\gamma/\sigma_{\mathrm{ref}} \geq 0, \qquad f = \eta\lambda\sigma_{\mathrm{ref}}, \qquad \eta = \dot{\epsilon}/\gamma$$

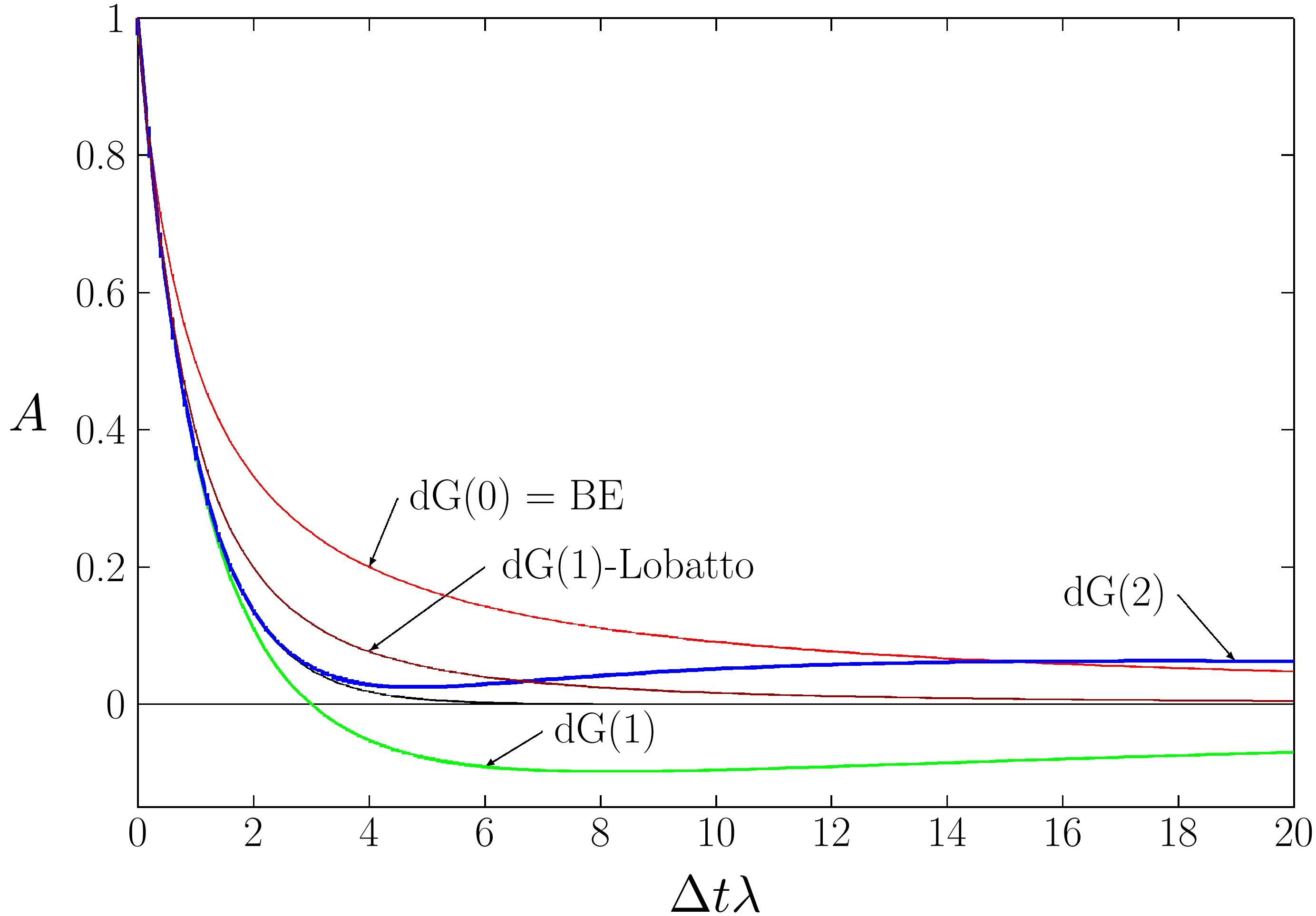
AMPLIFICATION FACTOR ($f = 0$ and λ constant)

$$A_{\mathrm{bE}} = A_{\mathrm{dG(0)}} = \frac{1}{1 + \lambda \Delta t}$$

$$A_{\mathrm{dG(1)}} = A_{\mathrm{IRKR2A-3}} = \frac{1 - \frac{1}{3}\lambda \Delta t}{1 + \frac{2}{3}\lambda \Delta t + \frac{1}{6}(\lambda \Delta t)^2}$$

$$A_{\mathrm{dG(2)}} = A_{\mathrm{IRKR2A-5}} = \frac{1 - \frac{2}{5}\lambda \Delta t + \frac{1}{20}(\lambda \Delta t)^2}{1 + \frac{3}{5}\lambda \Delta t + \frac{3}{20}(\lambda \Delta t)^2 + \frac{1}{60}(\lambda \Delta t)^3}$$

$$A_{\mathrm{dG(1)-Lobatto}} = A_{\mathrm{IRKL3C-2}} = \frac{1}{1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2}$$



dG(1)-Lobatto = IRKL3C method is the most accurate when Δt is large enough !!!!

bE also good when $\lambda\Delta t > 15$

SOME REQUIREMENTS

An ideal integrator for inelastic constitutive models should be:

1. L-stable

2. For $\dot{y} + \lambda y = 0$ (λ constant) the amplification factor should be

   (a) strictly positive
   (b) monotonous (convex)

Padé $(0, q)$-approximations of $\exp(-\lambda t)$ are positive and monotonous.
dG(1)-Lobatto = IRKL3C-2 = Padé-(0,2) for $\dot{\sigma} + \lambda\sigma = 0$

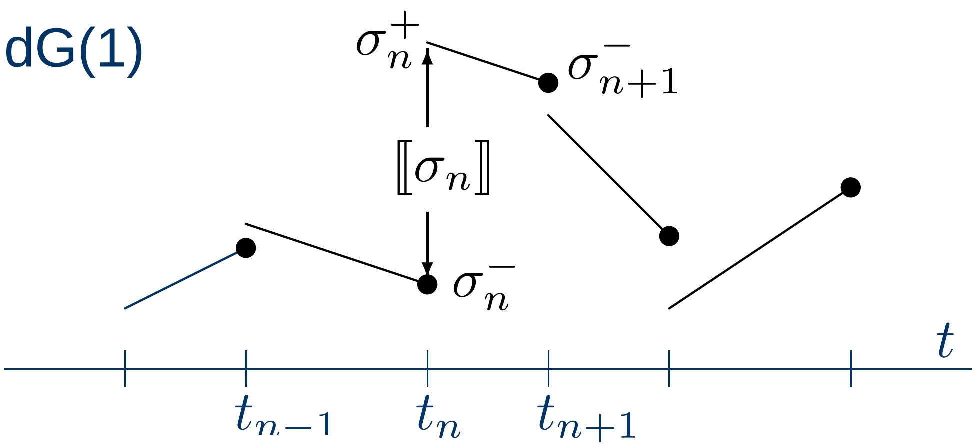
DISCONTINUOUS GALERKIN METHOD

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{f}(\boldsymbol{\sigma}),$$

dG(q): find $\boldsymbol{\sigma}$ (polynomial of degree q, as test functions $\boldsymbol{\tau}$) such that

$$\int_{I_n} (\dot{\boldsymbol{\sigma}} - \boldsymbol{f}(\boldsymbol{\sigma})) : \boldsymbol{\tau} \, \mathrm{d}t + [\![\boldsymbol{\sigma}_n]\!] : \boldsymbol{\tau}_n^+ = 0$$

dG(1)



dG(q) – abstract notation

Find $\boldsymbol{\sigma}_{n+1}^{-}$ such that

$$b_n(\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + \boldsymbol{\sigma}_n^{+} : \boldsymbol{\tau}_n^{+} = -c_n(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \boldsymbol{\sigma}_n^{-} : \boldsymbol{\tau}_n^{+}$$

with the bilinear form

$$b_n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_{I_n} \boldsymbol{\sigma} : \boldsymbol{\tau} \,\mathrm{d}t$$

and the semi-linear form

$$c_n(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_{I_n} \lambda \boldsymbol{n} : \boldsymbol{C}^{\mathrm{e}} : \boldsymbol{\tau} \,\mathrm{d}t$$

Case $q = 0$

$$\boldsymbol{\sigma}_n^+ = -\int_{I_n} \lambda \boldsymbol{n} : \boldsymbol{C}^{\mathrm{e}} \,\mathrm{d}t + \boldsymbol{\sigma}_n^- \quad \Rightarrow \quad \boldsymbol{\sigma}_{n+1}^- = -\Delta t_n \lambda \boldsymbol{n} : \boldsymbol{C}^{\mathrm{e}} + \boldsymbol{\sigma}_{n+1}^{\mathrm{trial},-}$$

Backward Euler scheme

$$\boldsymbol{\sigma}_{n+1}^- = -\Delta t_n \lambda_{n+1} \boldsymbol{n}_{n+1} : \boldsymbol{C}^{\mathrm{e}} + \boldsymbol{\sigma}_{n+1}^{\mathrm{trial},-}$$

Case $q \geq 1$, linearisation:

$$b_n(\boldsymbol{\Delta}\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + c_{n,\sigma}(\boldsymbol{\Delta}\boldsymbol{\sigma}, \boldsymbol{\tau}) = -b_n(\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}) - c_n(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \boldsymbol{\sigma}_n^+ : \boldsymbol{\tau}_n^+ + \boldsymbol{\sigma}_{n+1}^{\mathrm{trial},-} : \boldsymbol{\tau}_n^+$$

where

$$c_{n,\sigma}(\boldsymbol{\Delta}\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_{I_n} \left\{ \boldsymbol{\Delta}\boldsymbol{\sigma} : \left[\left(\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \right) \otimes (\boldsymbol{n} : \boldsymbol{C}^{\mathrm{e}}) \right] : \boldsymbol{\tau} + \lambda \boldsymbol{\Delta}\boldsymbol{\sigma} : \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C}^{\mathrm{e}} : \boldsymbol{\tau} \right\} \mathrm{d}t$$

UNIAXIAL EXAMPLES

$$\dot{\sigma} = E\left[\dot{\varepsilon} - f^* \exp\left(-Q/R\theta\right) \sinh^m\left(\sigma/\sigma_{\mathrm{r}}\right)\right], \tag{1}$$

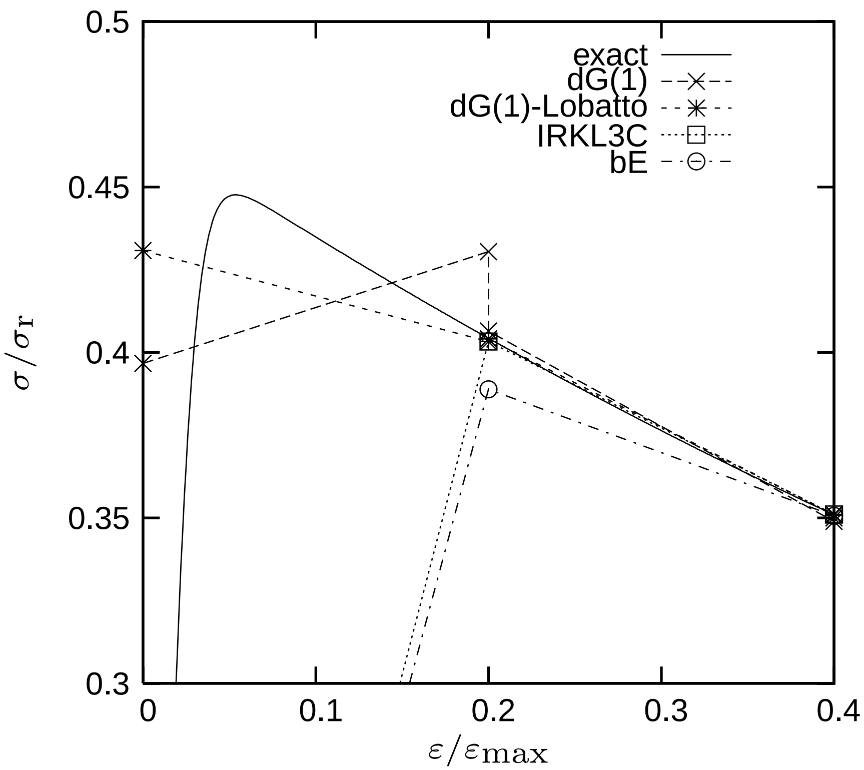
$$\theta(t) = \theta_0 + \Delta\theta(t/t_{\max}), \quad \text{where} \quad t_{\max} = \epsilon_{\max}/\dot{\epsilon}$$

$$\theta_0 = 293\,\mathrm{K}, \qquad \Delta\theta = 40\,\mathrm{K}$$

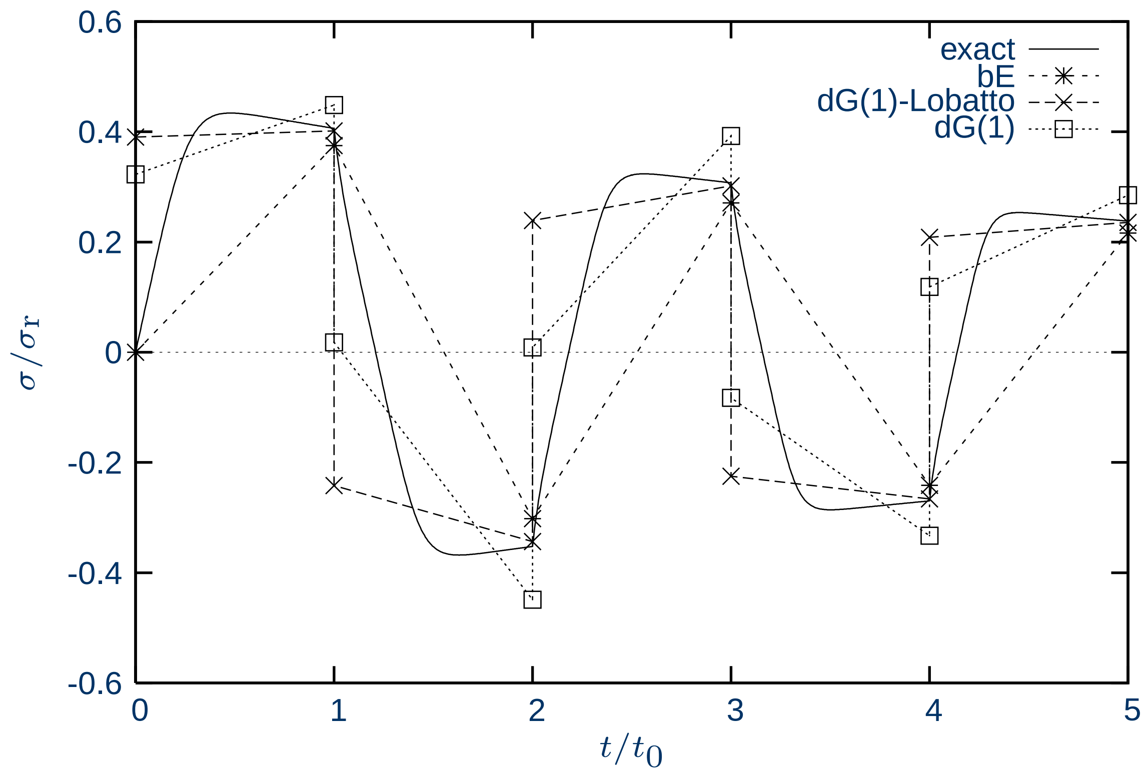
Binary near eutectic Sn40Pb solder:

$$\begin{aligned}
E &= 33\ \mathrm{GPa} \\
Q &= 12\ \mathrm{kcal/mol} \\
\nu &= 0.3 \\
R &= 2 \cdot 10^{-3}\ \mathrm{kcal/mol{\cdot}K} \\
\sigma_{\mathrm{r}} &= 20\ \mathrm{MPa} \\
f^* &= 10^5\ \mathrm{s}^{-1} \\
m &= 3.5
\end{aligned}$$

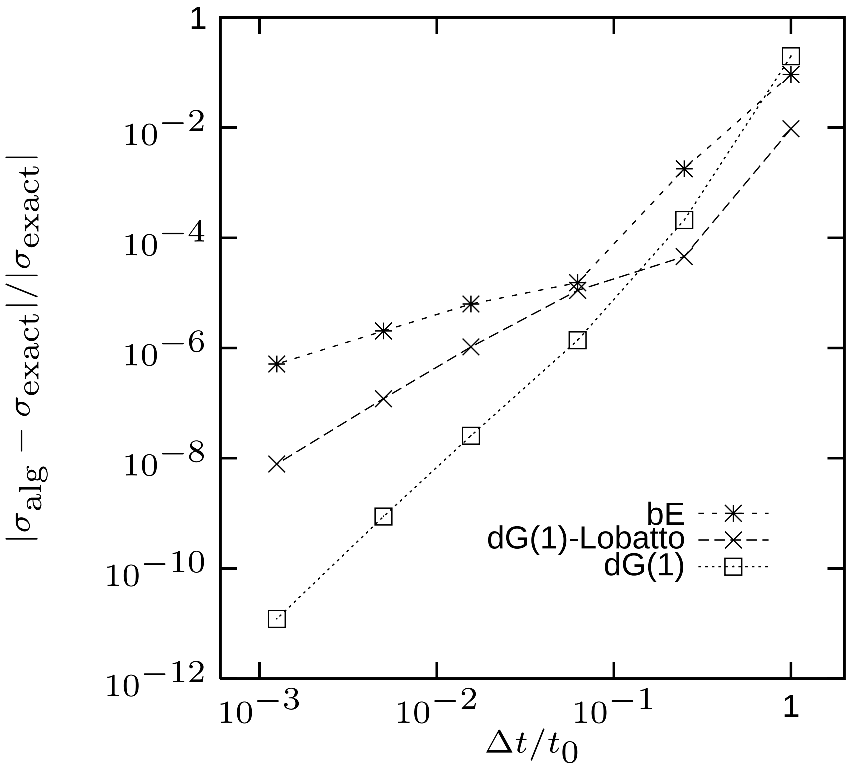
Bar in uniaxial tension (strain rate 10^{-5} 1/s)



Pulsatile uniaxial straining, strain rate $\pm 10^{-5}$ s^{-1}.



Relative error ($\dot{\epsilon} = 10^{-5}$ 1/s)



CONCLUDING REMARKS

- Underintegrated dG(1) method produces a method equivalent to IRKL3C
- Asymptotically third order accuracy can be obtained by changing Lobatto quadrature to Gauss-Legendre
- More stable in the case of damage ?