A time discontinuous Petrov-Galerkin method for the integration of inelastic constitutive equations

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OUTLINE

- Motivation
- Integration algorithms
  - Amplification factors of a model problem
  - Discontinuous Galerkin approach
- Example
- Concluding remarks

MOTIVATION

SIMO & HUGHES, *Computational Inelasticity*, Remark 3.3.2.2 on page 125:

"The overall superiority of the radial return method relative to other return schemes is conclusively established in Krieg and Krieg [1977]; Schreyer, Kulak and Kramer [1979] and Yoder and Whirley [1984]."

WHY ??

SCALAR MODEL PROBLEM

Maxwell creep model $\dot{\epsilon}^{\text{in}} = \gamma(\sigma/\sigma_{\text{ref}})$ $\dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^{\text{in}})$

$$\dot{\sigma} + (E\gamma/\sigma_{\text{ref}})\sigma = E\dot{\epsilon}$$

$$\boxed{\dot{y} + \lambda y = f, \;\; y(0) = y_0}$$

$$\lambda = E\gamma/\sigma_{\text{ref}} \geq 0, \qquad f = \eta\lambda\sigma_{\text{ref}}, \qquad \eta = \dot{\epsilon}/\gamma$$

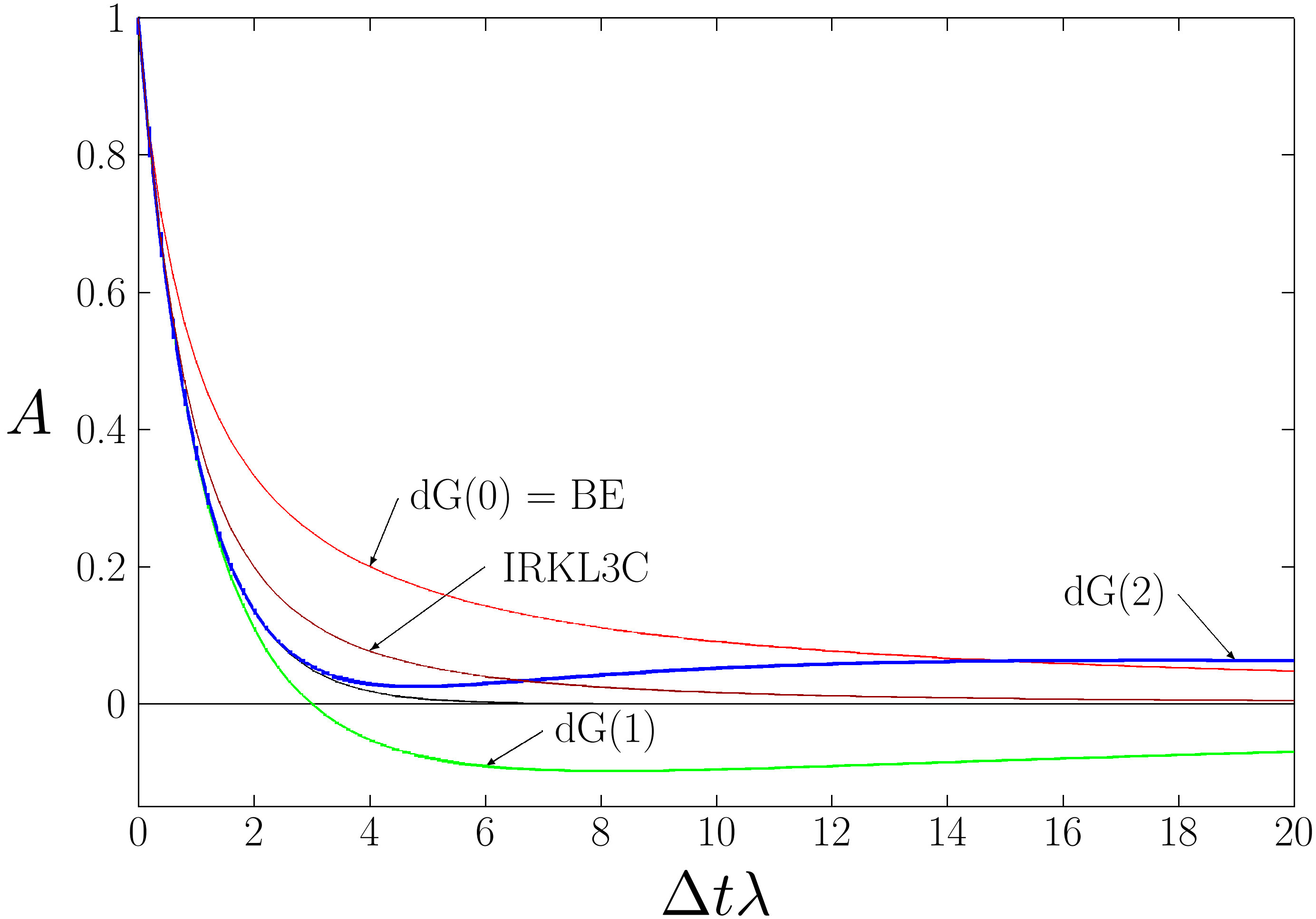
AMPLIFICATION FACTOR ($f = 0$ and λ constant)

$$A_{\mathrm{bE}} = A_{\mathrm{dG}(0)} = \frac{1}{1 + \lambda \Delta t}$$

$$A_{\mathrm{dG}(1)} = A_{\mathrm{IRKR2A-3}} = \frac{1 - \frac{1}{3}\lambda \Delta t}{1 + \frac{2}{3}\lambda \Delta t + \frac{1}{6}(\lambda \Delta t)^2}$$

$$A_{\mathrm{dG}(2)} = A_{\mathrm{IRKR2A-5}} = \frac{1 - \frac{2}{5}\lambda \Delta t + \frac{1}{20}(\lambda \Delta t)^2}{1 + \frac{3}{5}\lambda \Delta t + \frac{3}{20}(\lambda \Delta t)^2 + \frac{1}{60}(\lambda \Delta t)^3}$$

$$A_{\mathrm{IRKL3C-2}} = \frac{1}{1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2}$$



IRKL3C method is the most accurate when Δt is large enough !!!!

BE also good when $\lambda \Delta t > 15$

SOME REQUIREMENTS

Ideal integrator for inelastic constitutive models should be:

1. L-stable

2. For $\dot{y} + \lambda y = 0$ (λ constant) the amplification factor should be
   (a) strictly positive
   (b) monotonous (convex)

Padé $(0, q)$-approximations of $\exp(-\lambda t)$ are positive and monotonous.

IRKL3C-2 = Padé-(0,2) for $\dot{y} + \lambda y = 0$

QUESTION

Can we design a discontinuous Galerkin method with these properties ??

ANSWER

yes, if using discontinuous Petrov-Galerkin approach or underintegration

$$\int_{t_n}^{t_{n+1}} (\dot{y} + \lambda y) w \; dt + [y_n] \, w_n^+ = \int_{t_n}^{t_{n+1}} f w \; dt$$

where $\quad [y_n] = y_n^+ - y_n^-$

PETROV-GALERKIN

$$y = N_1 y_n^+ + N_2 y_{n+1}^-, \quad w = N_1^w \omega_1 + N_2^w \omega_2 \quad \text{and} \quad \lambda = N_1 \lambda_n + N_2 \lambda_{n+1}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}^T \left[\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} y_n^+ \\ y_{n+1}^- \end{pmatrix} + (y_n^+ - y_n^-) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right] = 0$$

$$A_{ij} = m_{ij} + k_{ij}, \quad m_{ij} = \int_{t_n}^{t_{n+1}} N_i^w \dot{N}_j \, dt, \quad k_{ij} = \int_{t_n}^{t_{n+1}} \lambda N_i^w N_j \, dt$$

$$y_{n+1}^- = \frac{-A_{21}}{(1 + A_{11}) A_{22} - A_{12} A_{21}} y_n^- + \frac{(1 + A_{11}) f_2 - A_{21} f_1}{(1 + A_{11}) A_{22} - A_{12} A_{21}}$$

UNDERINTEGRATION

Using 2-point Lobatto integration for the dG(1)-scheme we get the two stage IRKL3C-method

MODEL PROBLEM

$$\dot{y} + \lambda y = f, \quad y(0) = y_0$$

$$(f = \eta \lambda \sigma_{\mathrm{ref}}, \qquad \eta = \dot{\epsilon}/\gamma, \qquad \lambda = E\gamma/\sigma_{\mathrm{ref}})$$

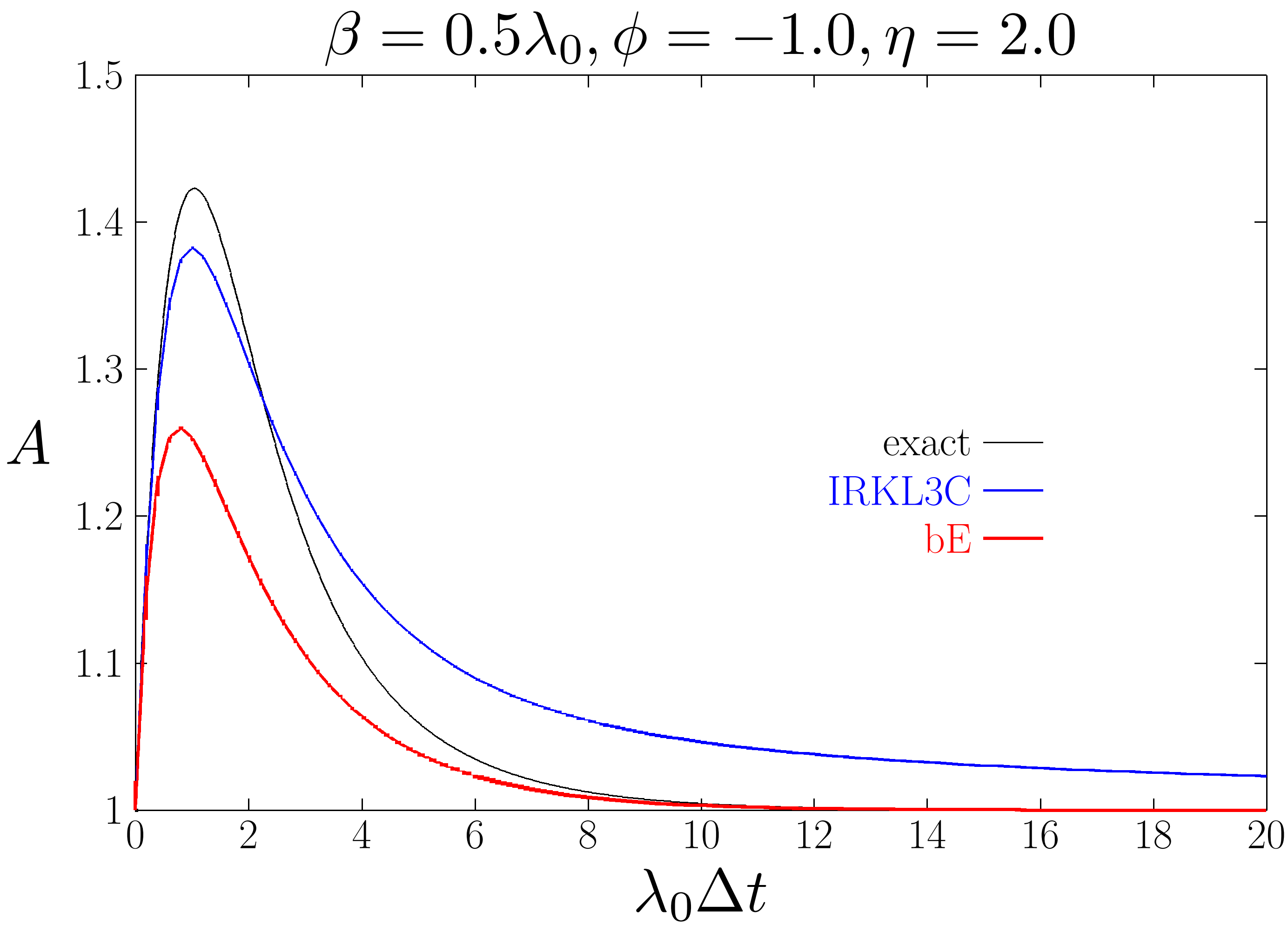
$$\boxed{\lambda(t) = \lambda_0 \left[1 - \phi + \phi \exp(-\beta t)\right]}$$

Two special cases

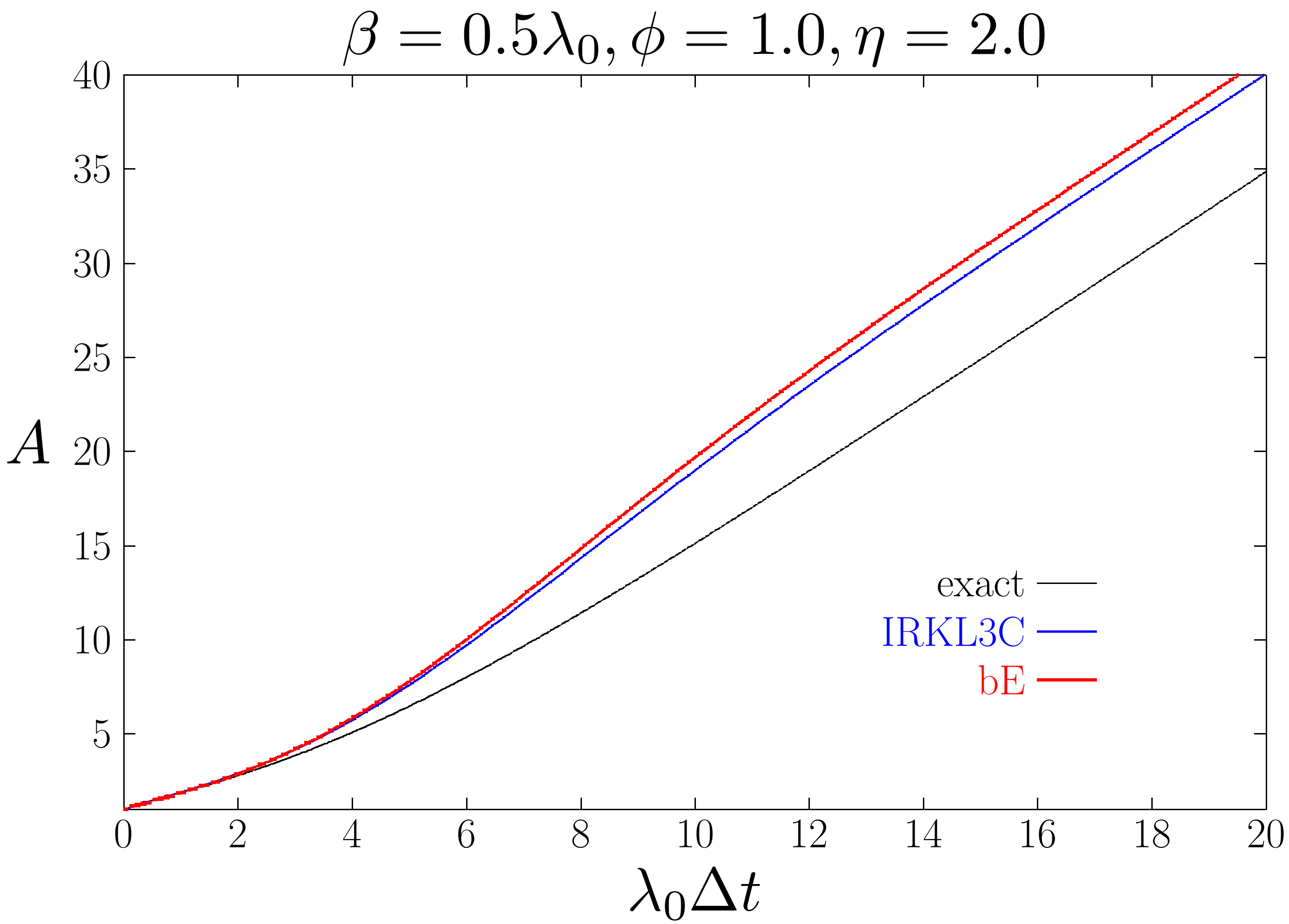
increasing diffusivity $\phi = -1$

vanishing diffusivity $\phi = 1$, then $\lambda \to 0$ when $t \to \infty$

Increasing diffusivity (strain softening)



Vanishing diffusivity (strain hardening)



MATERIAL MODEL

$$\dot{\boldsymbol{\epsilon}}^{\mathrm{in}} = \tfrac{3}{2}\gamma\mathbf{n}, \qquad \text{where} \qquad \mathbf{n} = \mathbf{s}/\bar{\sigma}$$

The scalar $\bar{\sigma}$ is the equivalent stress

$$\gamma = f^* \exp\left(\frac{-Q}{R\theta}\right) \sinh^m\left(\frac{\bar{\sigma}}{\sigma_y}\right)$$

Bar in uniaxial tension (strain rate 10^{-5} 1/s)

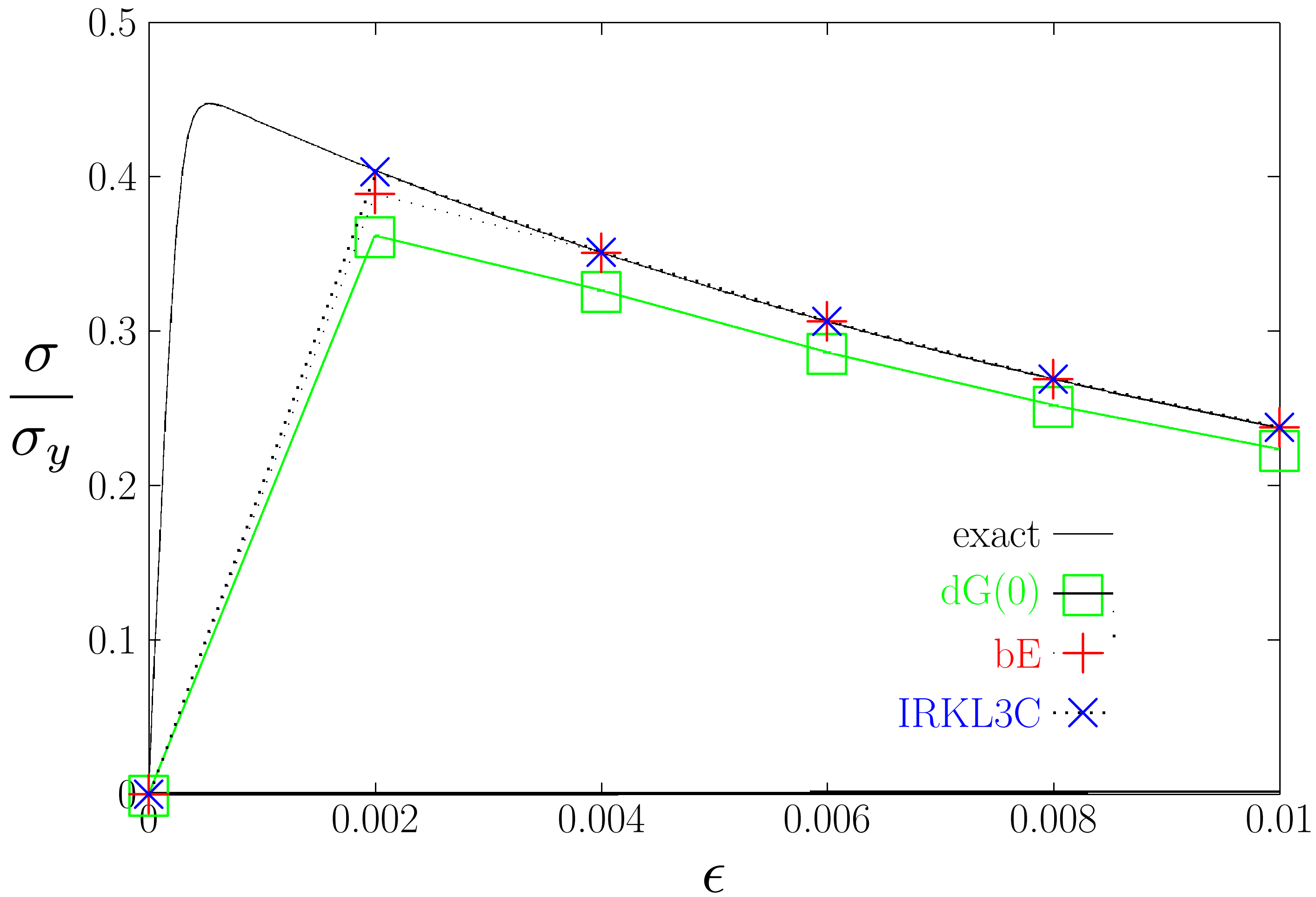
$$\theta(t) = \theta_0 \pm \Delta\theta(t/t_{\max}), \quad \text{where} \quad t_{\max} = \epsilon_{\max}/\dot{\epsilon}$$

$$\theta_0 = 293\text{K}, \qquad \Delta\theta = 40\text{K}$$

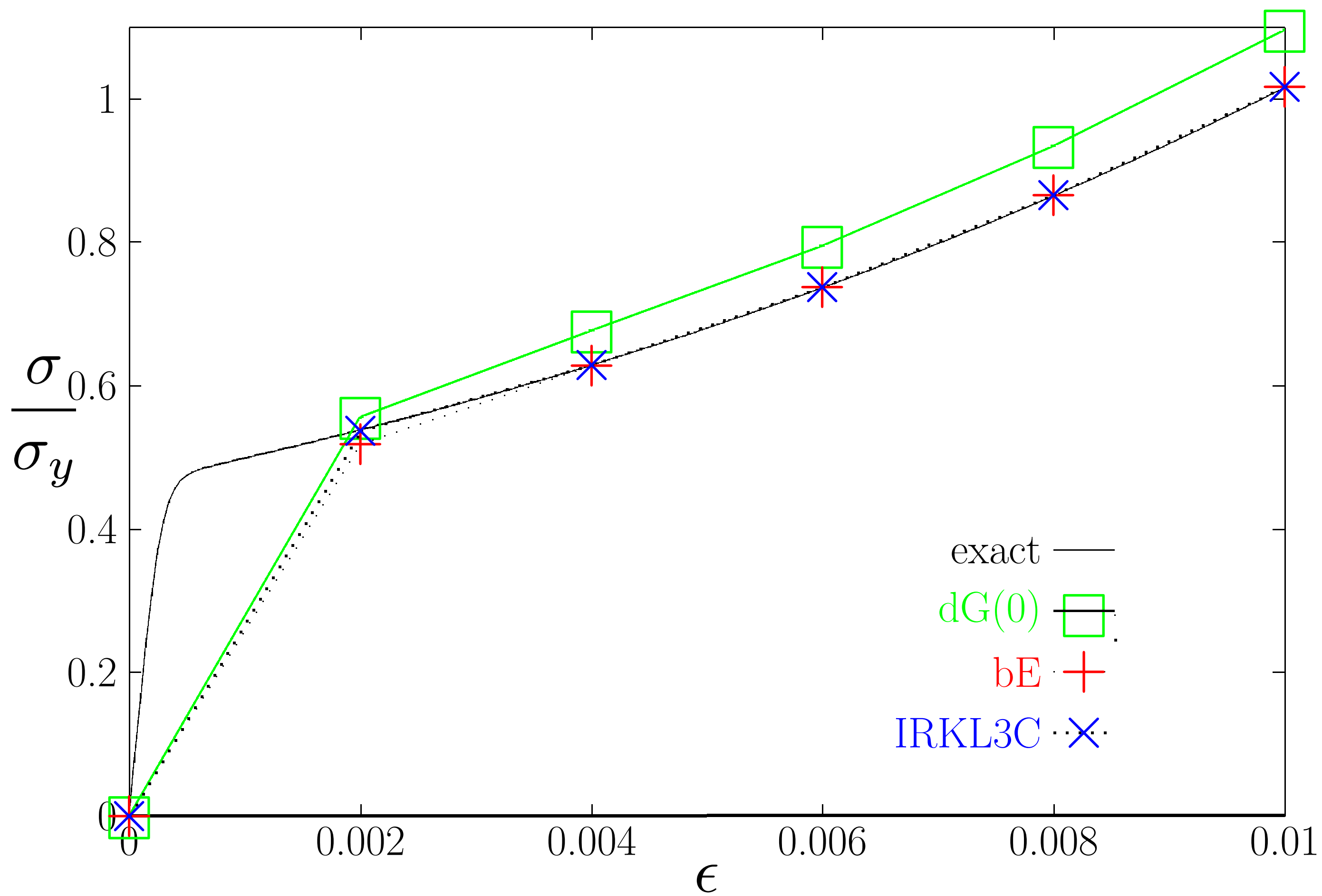
Binary near eutectic Sn40Pb solder:

$$\begin{aligned}
E &= 33 \text{ GPa} \\
Q &= 12 \text{ kcal/mol} \\
\nu &= 0.3 \\
R &= 2 \cdot 10^{-3} \text{ kcal/mol·K} \\
\sigma_y &= 20 \text{ MPa} \\
f &= 10^5 \text{ s}^{-1} \\
m &= 3.5
\end{aligned}$$

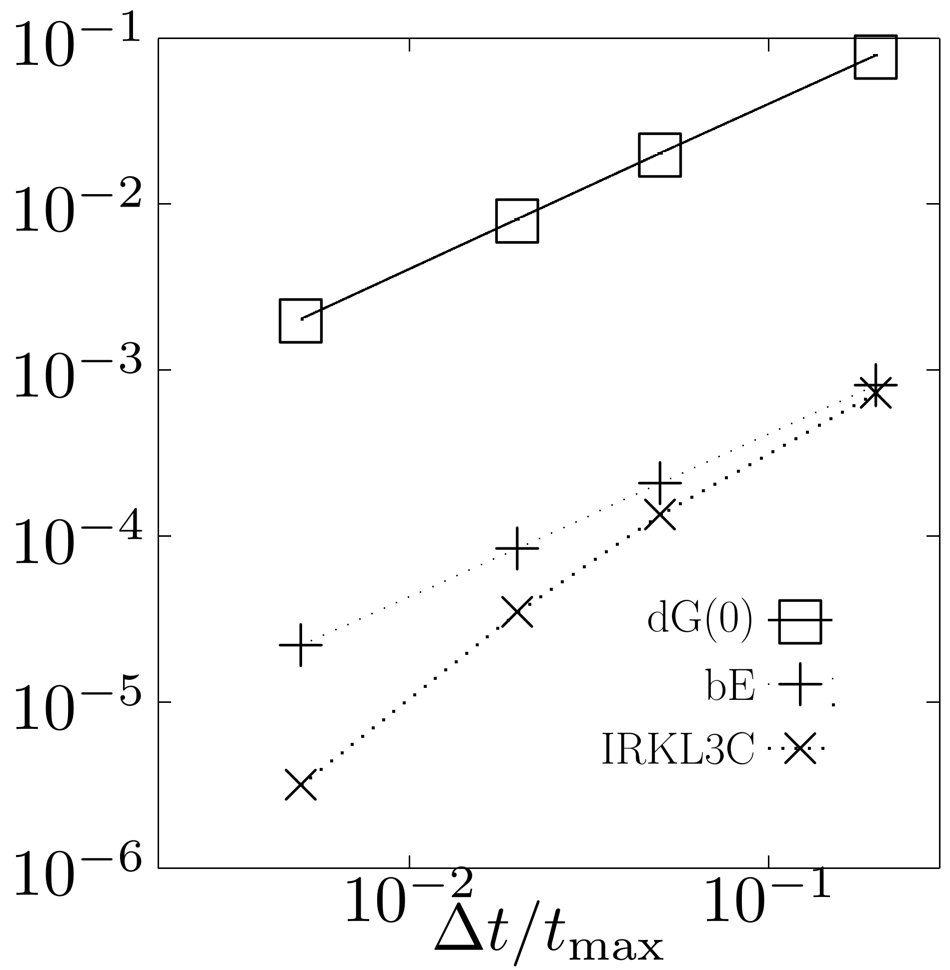
Thermally softening case ($\dot{\epsilon} = 10^{-5}$ 1/s)



Thermally hardening case ($\dot{\epsilon} = 10^{-5}$ 1/s)



Relative error ($\dot{\epsilon} = 10^{-5}$ 1/s)



CONCLUDING REMARKS

- Two-stage IRKL3C method seems to be an accurate integrator also for large time steps
- Discontinuous Petrov-Galerkin approach can produce a method similar to IRKL3C
- Underintegrated dG(1) method produces a method similar to IRKL3C
- Asymptotically third order accuracy can be obtained by switching full integration in the dG(1) scheme if the time step is small