On the numerical solution of a micropolar continuum model

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Equilibrium equations

Force balance

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = 0,$$

Moment balance

$$\frac{\partial \mu_{ji}}{\partial x_j} + \rho c_i + \epsilon_{ijk}\sigma_{jk} = 0,$$

$\boldsymbol{\sigma}$ force stress tensor
$\boldsymbol{\mu}$ moment stress tensor
$\boldsymbol{b}$ body force per unit mass
$\boldsymbol{c}$ body moment per unit mass
ϵ_{ijk} alternating tensor
ρ the mass density

Constitutive equations

Linear material model

$$\sigma_{ij} = C^{(\gamma)}_{ijkl}\gamma_{kl}, \qquad \mu_{ij} = C^{(\kappa)}_{ijkl}\kappa_{kl},$$

the Cosserat's first strain tensor γ_{ij} and microcurvature tensor κ_{ij} are

$$\gamma_{ij} = u_{j,i} - \epsilon_{kij}\varphi_k, \quad \text{and} \quad \kappa_{ij} = \varphi_{j,i},$$

φ is the microrotation field

The material stiffness tensors for an isotropic solid can be expressed as

$$C^{(\gamma)}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu_{\mathrm{c}}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$
$$C^{(\kappa)}_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Comprehensible set of material parameters

the characteristic length in torsion	$\ell_{\rm t} = \sqrt{\dfrac{\beta}{\mu}},$
the characteristic length in bending	$\ell_{\rm b} = \sqrt{\dfrac{\beta+\gamma}{4\mu}},$
the coupling number	$N = \sqrt{\dfrac{\mu_{\rm c}}{\mu+\mu_{\rm c}}},$
and the polar ratio	$\psi = \dfrac{\beta}{\beta+\frac{1}{2}\alpha}.$

$0 \leq N \leq 1,$ and $0 \leq \psi \leq \dfrac{3}{2},$

Conformally invariant curvature state

Neff et al. 2009, 2010

$$\mu_{ij} = (\alpha + \tfrac{2}{3}\beta)\kappa_{kk}\delta_{ij} + 2\beta\kappa'_{(ij)} + 2\gamma\kappa_{[ij]}$$

$$\gamma_{(ij)} = \tfrac{1}{2}(\gamma_{ij} + \gamma_{ji}), \qquad \gamma_{[ij]} = \tfrac{1}{2}(\gamma_{ij} - \gamma_{ji}).$$

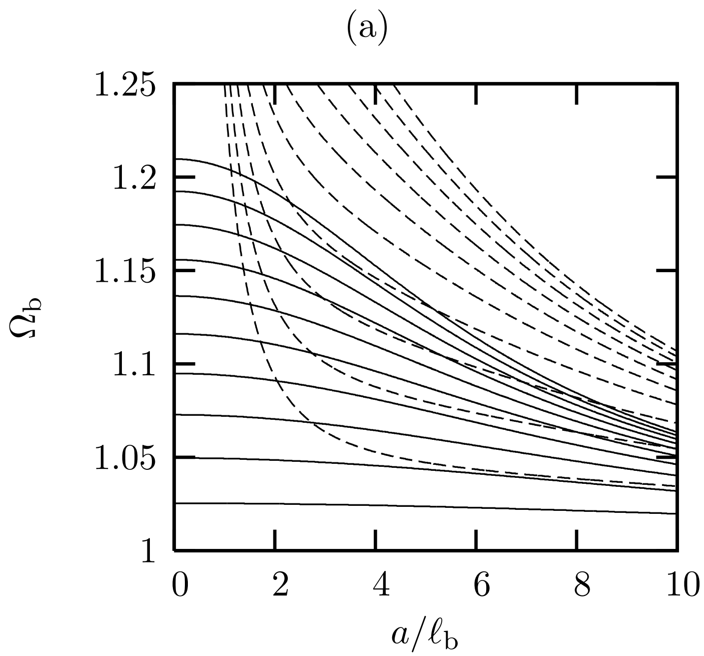
The curvature tensor is symmetric and deviatoric

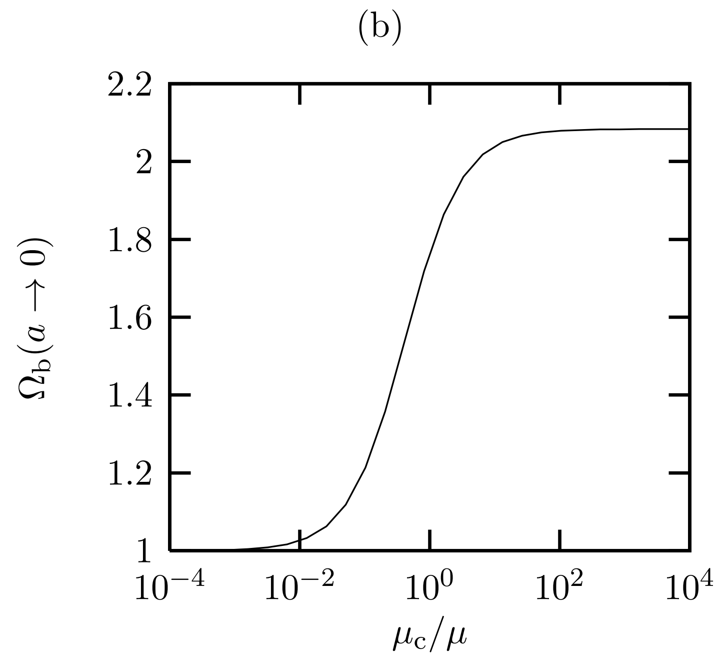
$$\gamma = 0, \qquad \beta = \mu\ell_{\mathrm{t}}^2, \qquad \alpha = -\tfrac{2}{3}\mu\ell_{\mathrm{t}}^2.$$

Four material parameters $\lambda, \mu, \mu_{\mathrm{c}}, \ell_{\mathrm{t}}$.

Size effect - pure bending of a beam

Circular cross-section with radius a





(a) From bottom to top $\mu_{\mathrm{c}}/\mu = 0.01, 0.02, 0.03, \ldots, 0.09, 0.1$. Solid lines: conformally invariant curvature case ($\beta = 4\mu\ell_{\mathrm{b}}^2, \gamma = 0$); dashed lines $\gamma = 0.01\beta$.

(b) $\Omega_{\mathrm{b}}(a \to 0)$ for the conformally invariant curvature state.

Numerical solution

The virtual work expression

$$\int_V \left[\left(-\nabla\cdot\boldsymbol{\sigma}^T-\rho\boldsymbol{b}\right)\cdot\delta\boldsymbol{u}+\left(-\nabla\cdot\boldsymbol{\mu}^T-\boldsymbol{\epsilon}:\boldsymbol{\sigma}-\rho\boldsymbol{c}\right)\cdot\delta\boldsymbol{\phi}\right]\,\mathrm{d}V=0$$

$$\int_V \left[\boldsymbol{\sigma}:\left((\nabla\delta\boldsymbol{u})^T-\operatorname{skew}(\delta\boldsymbol{\varphi})\right)+\boldsymbol{\mu}:(\nabla\delta\boldsymbol{\varphi})^T\right]\,\mathrm{d}V \\ -\int_V(\rho\boldsymbol{b}\cdot\delta\boldsymbol{u}+\rho\boldsymbol{c}\cdot\delta\boldsymbol{\varphi})\,\mathrm{d}V-\int_S(\boldsymbol{t}\cdot\delta\boldsymbol{u}+\boldsymbol{m}\cdot\delta\boldsymbol{\varphi})\,\mathrm{d}S=0$$

Standard C_0-interpolation for both $\boldsymbol{u}$ and $\boldsymbol{\varphi}$.

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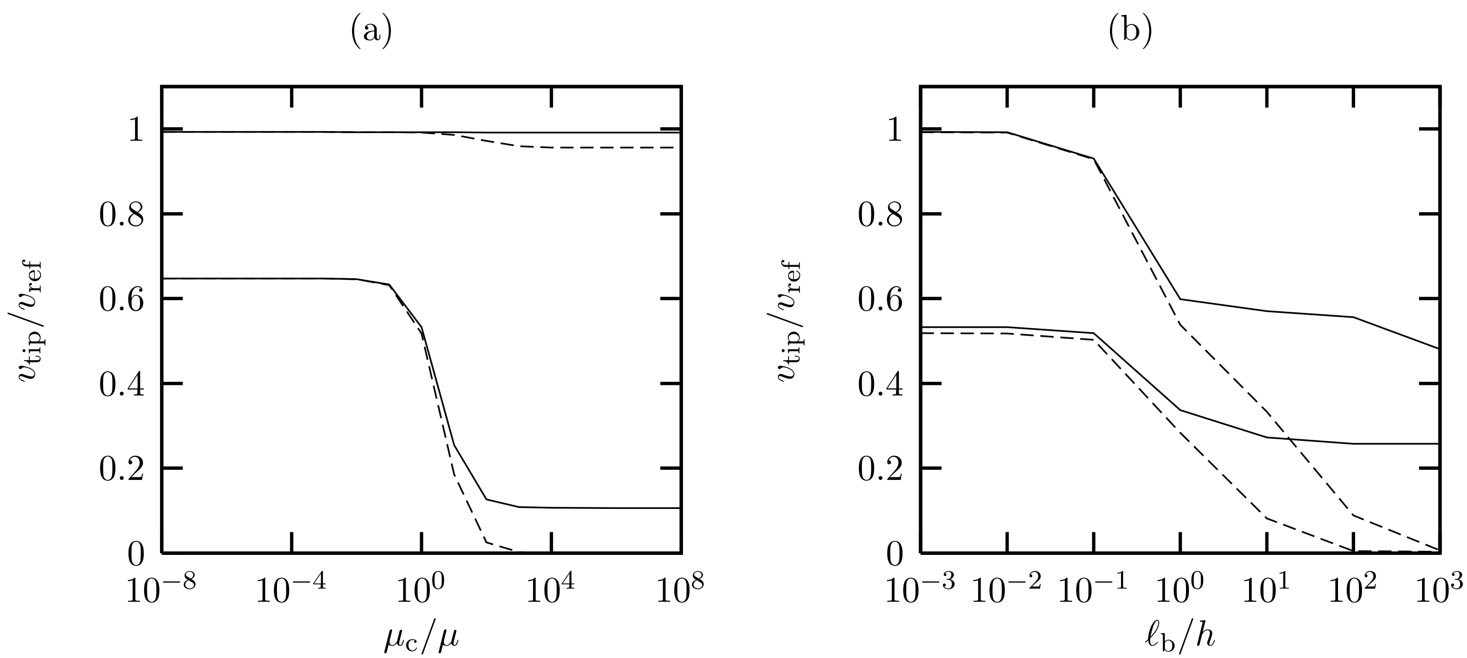
Example - cantilever beam



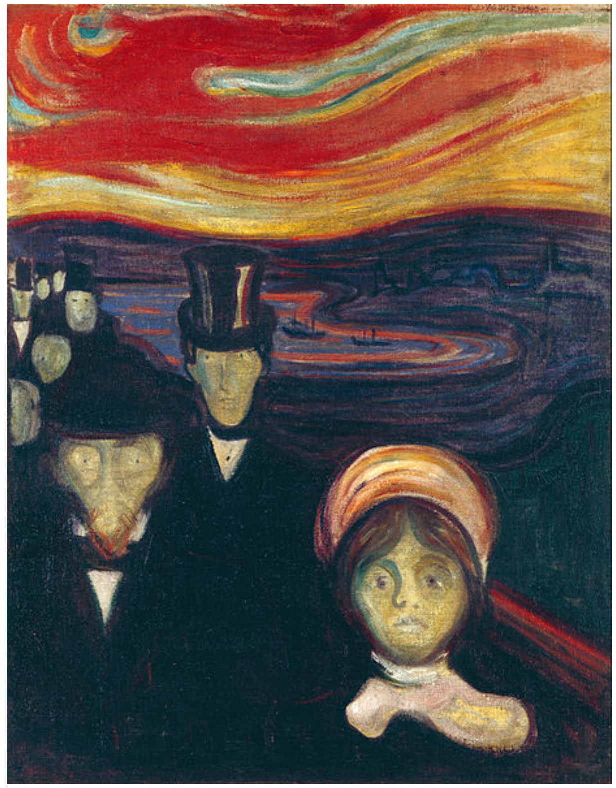
(a)$\ell_{\mathrm{b}} = h/200$. The upper two curves triquadratic elements and the lower ones with trilinear elements. Solid lines: rotations are free at the clamped edge, dashed lines: rotations are supressed.

(b) The effect of internal length scale ℓ_{b} on the FE solution: $\mu_{\mathrm{c}} = \mu$.

Concluding remarks

- Stable and locking free FE-formulation needed for the micropolar model.

Thanks!



Edvard Munch, Anxiety, 1894

Equilibrium equations in terms of displacements and rotations

$$(\lambda + \mu - \mu_{\rm c})\text{grad}\,\text{div}\boldsymbol{u} + (\mu + \mu_{\rm c})\Delta\boldsymbol{u} + 2\mu_{\rm c}\text{curl}\boldsymbol{\varphi} + \rho\boldsymbol{b} = \boldsymbol{0}$$

$$(\alpha + \beta - \gamma)\text{grad}\,\text{div}\boldsymbol{\varphi} + (\beta + \gamma)\Delta\boldsymbol{\varphi} - 4\mu_{\rm c}\boldsymbol{\varphi} + 2\mu_{\rm c}\text{curl}\boldsymbol{u} + \rho\boldsymbol{c} = \boldsymbol{0}$$

$$\Delta\boldsymbol{u} = \text{div}\,\text{grad}\boldsymbol{u}$$

Constitutive parameters

$$\boldsymbol{e} = [\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{21}, \gamma_{23}, \gamma_{32}, \gamma_{31}, \gamma_{13}, \kappa_{11}, \kappa_{22}, \kappa_{33}, \kappa_{12}, \kappa_{21}, \kappa_{23}, \kappa_{32}, \kappa_{31}, \kappa_{13}]^T,$$

$$\boldsymbol{C} = \mathrm{diag}\left[\begin{array}{cccccccc} \boldsymbol{C}_1 & \boldsymbol{C}_2 & \boldsymbol{C}_2 & \boldsymbol{C}_2 & \boldsymbol{C}_3 & \boldsymbol{C}_4 & \boldsymbol{C}_4 & \boldsymbol{C}_4 \end{array}\right],$$

$$\boldsymbol{C}_1 = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda \\ \lambda & \lambda+2\mu & \lambda \\ \lambda & \lambda & \lambda+2\mu \end{bmatrix}, \qquad \boldsymbol{C}_3 = \begin{bmatrix} \alpha+2\beta & \alpha & \alpha \\ \alpha & \alpha+2\beta & \alpha \\ \alpha & \alpha & \alpha+2\beta \end{bmatrix},$$

$$\boldsymbol{C}_2 = \begin{bmatrix} \mu+\mu_c & \mu-\mu_c \\ \mu-\mu_c & \mu+\mu_c \end{bmatrix}, \qquad \boldsymbol{C}_4 = \begin{bmatrix} \beta+\gamma & \beta-\gamma \\ \beta-\gamma & \beta+\gamma \end{bmatrix}.$$

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \mu_c > 0, \quad \beta > 0, \quad 3\alpha + 2\beta > 0, \quad \gamma > 0.$$

Stiffness ratio Ω_b in pure bending of a beam with circular cross-section with radius a

$$\Omega_\mathrm{b} = 1 + \frac{8N^2}{1+\nu}\left[\frac{1-(\beta-\gamma)/(\beta+\gamma)}{(\delta a)^2} + \frac{[(\beta-\gamma)/(\beta+\gamma)+\nu]^2}{\zeta(\delta a)+8N^2(1-\nu)}\right],$$

$$\delta^2 = 4\mu_\mathrm{c}\mu/(\beta+\gamma)(\mu+\mu_\mathrm{c})$$

$$\zeta(x) = \frac{x^2(xI_0(x)-I_1(x))}{xI_0(x)-2I_1(x)},$$

I_0, I_1 the modified Bessel functions of the first kind

Krishna-Reddy and Venkatasubramanian 1978