On Determinant as Singularity Test Function

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Introduction

Determinant is probably the most used singularity test function in static non-linear finite element computations. The purpose of this note is to point out the shortcomings of determinant based test functions. To some extent these difficulties can be avoided by scaling. A possible form of scaling is also introduced.

Singularity test functions

Detection and estimation of singular i.e. limit and bifurcation points are usually based on some singularity indicator, which is also called as test function [1], [2]. Probably the most reliable singularity test function is the lowest eigenvalue of the tangent stiffness matrix. However, determinant of the tangent matrix is commonly used, while if not redefined, it is highly unreliable. Its polularity is perhaps due to practical aspects; determinant is an easy byproduct of factorization of the tangent matrix if direct linear solvers are used.

A proper singularity test function should fulfill at least two basic requirements. Firstly, the sign of the test function should change whenever changes in the inertia of the tangent stiffness matrix occur. Secondly, the behaviour of the test function should be independent of the mesh size. The determinat does not satisfy these requirements without modifications. This is due to the fact that determinant is a product of all eigenvalues, which can result in high variations during the path ruining the predictive quality of the test function and making it mesh size dependent. In addition, determinant cannot locate singular points with even multiplicity.

A choice which succeeds in removing the above mentioned shortcomings for determinant based singularity test function (dbstf) is defined as

$$\mathrm{dbstf} = \mathrm{ci}(\boldsymbol{K}_n, \boldsymbol{K}_{n-1}) \frac{\mathrm{sdet}(\boldsymbol{K}_n)}{\mathrm{sdet}(\boldsymbol{K}_1)}, \quad \text{where} \quad \mathrm{sdet}(\boldsymbol{K}) = \prod_{i=1}^{N} |d_{ii}|^{1/N^\gamma},$$

where d_{ii} are the diagonals of the root free Cholesky decomposition, γ is a parameter ($\gamma \in [0,1]$), which should reflect the average rate of change in the eigenvalue spectrum and N is the dimension of the problem. The subscript refers to the increment number. The "change in inertia"-function: $\mathrm{ci}(\boldsymbol{K}_n, \boldsymbol{K}_{n-1})$ is defined to be ± 1, and changes its sign when a change in the inertia of the tangent stiffness matrix occurs along the path between the increments $n-1$ and n.

Other singularity test functions proposed in the literature are the smallest pivot [3] and the current stiffness parameter [4]. The latter can only be used to locate limit points, however, it is easily computed also when iterative linear equation solvers are used.

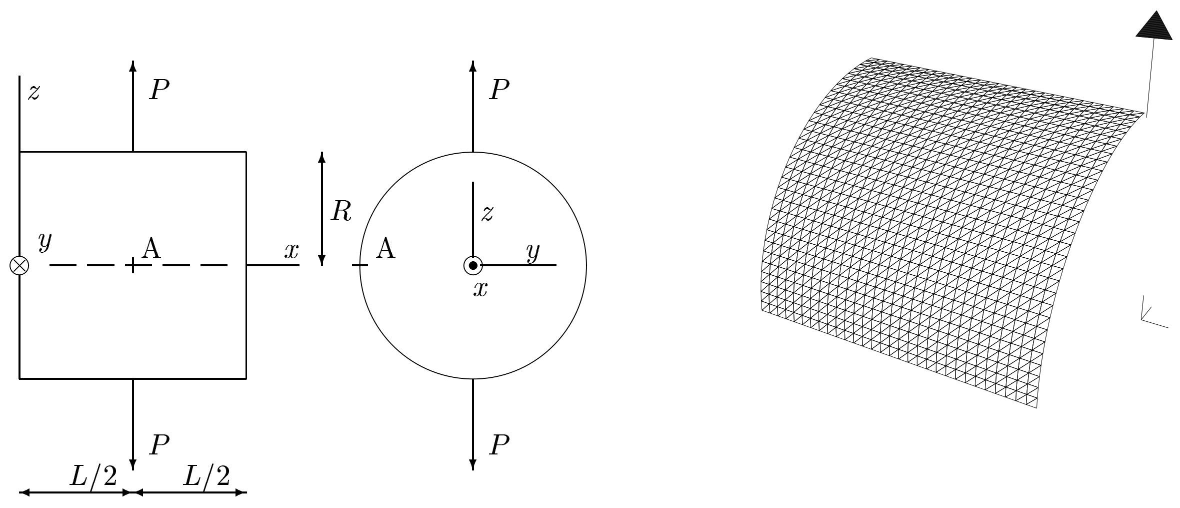

Figure 1: Uniform 30×30 mesh for an octant of a pulled cylindrical shell.

Numerical example

A common test problem for non-linear shell analysis is a pulled cylidrical shell, see Fig. 1. Boundaries of the shell are free and the following dimensions are used in the computations: $R = 1, E = 10^6, \nu = 0.3125, R/t = 52.69$. Three different cases are computed with different length to radius ratio: $L/R = 2.08964, 3$ and 3.5. The first of these corresponds to the common test case, see e.g. Refs. [5], [6]. The loading is imposed by two normal and equal point loads applied centrally at the opposite sides of the cylindrical surface.

One octant of the shell is discretized by uniform 30×30 mesh of three noded triangular flat facet type shell elements, resulting in 5489 unknowns. Drilling degrees of freedom are included by using the Hughes-Brezzi formulation. The plate bending part of the element is formulated using the classical discrete Kirchhoff condensation technique.

In Fig. 2 the displacement of point A in y-axis direction is shown w.r.t. the load. At the load level $P \approx 106D/R \to 129D/R$ a snap-through happens, which is more pronounced for the longer cylinders $L/R \geq 3$. This is rather difficult example for the step lenght adaptation routines. In the beginning of the computation the the shells exhibit stiffening behaviour and the convergence is easily obtainable with the full Newton method.

To demonstrate the behaviour of some of the singularity test functions, the case $L/R = 2.08964$ has been selected. All critical point indicators show monotonous increase in their values prior the limit point, see Figs. 2b and 3. Especially the smallest pivot is almost constant, except those increments in which the tangent stiffness matrix is indefinite. Similar behaviour is also noticed with the determinant based indicator. Therefore, the limit point appears suddenly in one relatively large step and some restarts with a reduced step size are required. All the computations have been started with a load increment $\Delta P_1 = 2D/R$ and the number of desired corrector iterations has been three and 108 steps are used in the computation shown in Fig. 2 (case $L/R = 2.08964$).

The behaviour of the proposed scaling for the determinant based singularity test function is shown in Fig. 3. Without the scaling, i.e. $\gamma = 0$, the determinant based test function have huge

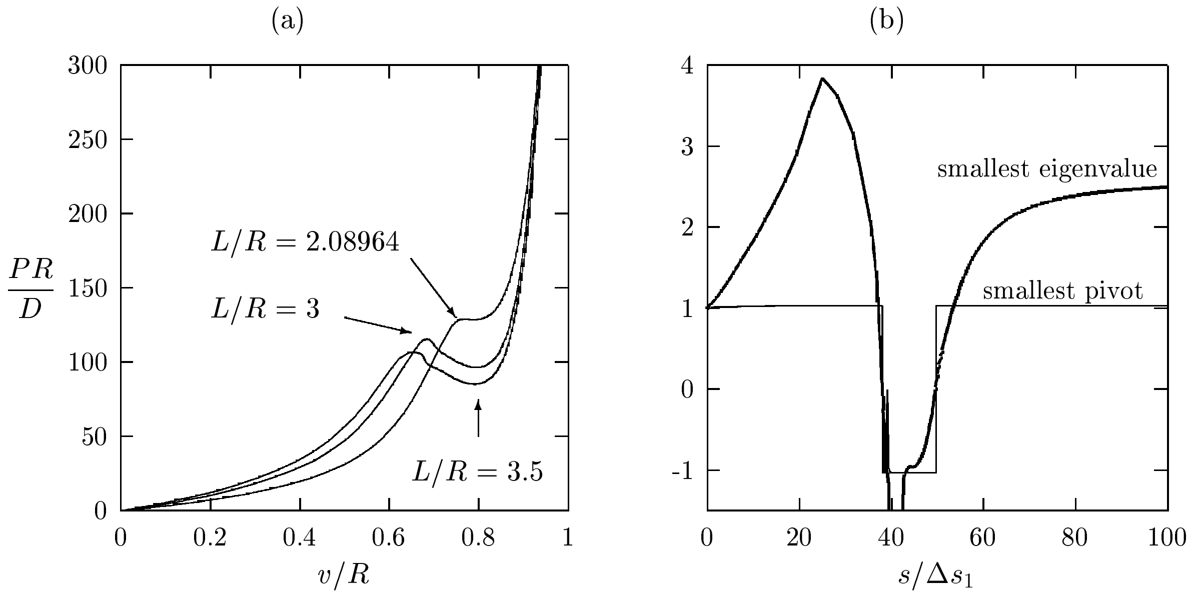

Figure 2: Pulled cylinder: (a) load-deflection at point A, (b) relative criticality indicators as a function of the path parameter, case $L/R = 2.08964$.

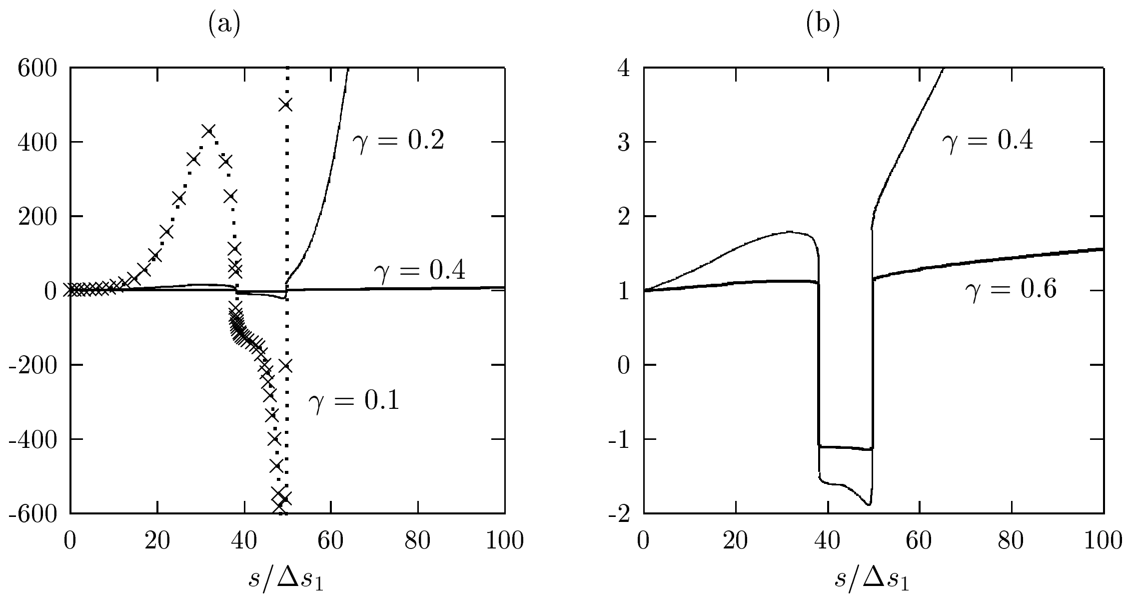

Figure 3: Determinant based singularity test function as a function of path parameter: (a) $\gamma = 0.1, 0.2, 0.4$ (b) $\gamma = 0.4, 0.6$. Aspect ratio of the cylinder $L/R = 2.08964$.

variations in its values, therefore it is not shown in the figure. Clearly, the damping introduced with a nonzero value of γ improves the predictive quality of the dbstf. However, it is still inferior in comparison to the smallest eigenvalue as a singularity indicator.

An opposite behaviour in the values of determinant is observed when a *compressed cylindrical shell* is analysed. In that case the majority of the eigenvalues are decreasing, resulting in a drop of order 10^{25} between the increments on the primary path prior to the bifurcation point, when a 120×20 mesh resulting in 13681 unknowns is used. However, when using the dbstf with $\gamma = 0.6$ results in almost linear decrease in its value on the primary path, thus making it ideal for that particular problem.

Conclusions

A scaled determinant based singularity test function for non-linear continuation algorithms is introduced. The proposed form improves the predictive quality of the determinant based singularity indicator and is almost independent of the mesh size used in the discretization. However, in view of the example shown, the predictive quality of the proposed dbstf is still inferior in comparison to the lowest eigenvalue as test function.

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