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ON NUMERICAL MODELING OF COUPLED MAGNETOELASTIC PROBLEM

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Summary. This paper focuses on modeling of a coupled magnetoelastic problem. The model for magnetostriction is developed for isotropic ferromagnetic materials commonly used in electrical machines. The constitutive equations of the model are written on the basis of the Helmholtz free energy in which the strain tensor and the magnetic flux density vector are chosen as the basic variables. Numerical solution is performed by using FEM.

1 INTRODUCTION

Theories aimed to determine the interaction of the strain-stress field with the electromagnetic field in deformable bodies are generally called electrodynamics of moving media or simply electrodynamics. There are many coexisting theories on the interaction of electromagnetism with thermoelastic fields. This may sound very suprising since fallacious theories can usually be sorted out by carefully designed experiments. However, in the theories of electrodymamics the principal difficulty is that the electromagnetic fields inside matter are expressed in terms of field variables which cannot be directly measured in laboratories [1, 2, 3].

In the traditional models for magnetostrictive materials, the constitutive relations were decoupled [4]. Recently, several coupled models have been introduced for a large class of magnetoelastic solids [5] but there is still a lack of both theoretical and experimental work in the development of the constitutive equations [6].

This paper focuses on modeling a coupled magnetoelastic problem. The model for magnetostriction is developed for isotropic ferromagnetic materials used in electrical machines. The constitutive equations of the model are written on the basis of the Helmholtz free energy in which the strain tensor and the magnetic flux density vector are chosen as the basic variables [7]. It is a modified version of the model presented in [8].

2 MAGNETOELASTICITY

In stationary case the balance equations for a magnetoelastic solid are

$$-\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{f} + \mathbf{f}_{\mathrm{em}}, \qquad \mathrm{curl}\,\mathbf{H} = \mathbf{J}, \qquad \mathrm{div}\,\mathbf{B} = 0, \qquad \mathrm{div}\,\mathbf{J} = 0, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{f}$ and $\mathbf{f}_{\mathrm{em}}$ are the mechanical and electromagnetic forces per unit volume, respectively. In the Ampère's circuital law $(1)_2$ $\mathbf{H}$ is the magnetic field strength and $\mathbf{J}$ is the current density. The condition $(1)_3$ of solenoidal magnetic flux density $\mathbf{B}$ expresses the physical fact of nonexisting magnetic monopoles. Equation $(1)_4$ is the charge conservation law.

To complete the system (1) the constitutive equations have to be specified. The magnetic field is related to the magnetic flux density as

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{2}$$

where μ_0 is the magnetic permeability in the vacuum ($4\pi \cdot 10^{-7}$ N/A^2) and $\mathbf{M}$ is the magnetization. For an isotropic magnetoelastic solid, the Helmholtz free energy, ψ, is a tensor function depending on the symmetric second order strain tensor $\boldsymbol{\varepsilon}$ and the magnetic flux density vector $\mathbf{B}$, the integrity basis thus consists of the following six invariants

$$I_1 = \mathrm{tr}\,\boldsymbol{\varepsilon}, \quad I_2 = \tfrac{1}{2}\mathrm{tr}\,\boldsymbol{\varepsilon}^2, \quad I_3 = \tfrac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}^3, \quad I_4 = \mathbf{B}\cdot\mathbf{B}, \quad I_5 = \mathbf{B}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{B}, \quad I_6 = \mathbf{B}\cdot\boldsymbol{\varepsilon}^2\cdot\mathbf{B}. \tag{3}$$

From the Clausius-Duhem inequality, the constitutive equations of the material can be expressed as (ρ is the density of material)

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}, \qquad \mathbf{M} = -\rho\frac{\partial\psi}{\partial\mathbf{B}}. \tag{4}$$

It is customary to submerge the electromagnetic force into the stress tensor by defining the total stress tensor $\boldsymbol{\tau}$ as [1, 2, 3]

$$\boldsymbol{\tau} = \boldsymbol{\sigma} + \mu_0^{-1}[\mathbf{BB} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}] + (\mathbf{M}\cdot\mathbf{B})\mathbf{I} - \mathbf{BM}, \tag{5}$$

where $\mathbf{BB}$ denotes the dyadic product of two vectors. The force equilibrium equation $(1)_1$ thus reduces to the usual form: $-\mathrm{div}\,\boldsymbol{\tau} = \mathbf{f}$.

To model magnetostriction of ferromagnetic solids, the following choice for the Helmholtz free energy seems to be suitable:

$$\rho\psi = \tfrac{1}{2}\lambda I_1^2 + 2GI_2 + \tfrac{1}{2}\left[\sum_{i=0}^{4}\frac{1}{i+1}g_i(I_1)\left(\frac{I_4}{B_{\mathrm{ref}}^2}\right)^i I_4\right] + \tfrac{1}{2}\gamma_5 I_5 + \tfrac{1}{2}\gamma_6 I_6, \tag{6}$$

where λ, G are the Lamé coefficients, γ_5, γ_6 are constants and B_{ref} is an arbitrary reference value. Assuming isochoric deformation under pure magnetic loading, allows determination of the functions g_i, depending on the invariant I_1, as

$$g_0 = (\gamma_4^{(0)} + \tfrac{1}{4}\mu_0^{-1} - \tfrac{1}{4}\gamma_5)\exp(\tfrac{4}{3}I_1) - \tfrac{1}{4}\mu_0^{-1} + \tfrac{1}{4}\gamma_5, \tag{7}$$

$$g_i = \gamma_4^{(i)}\exp(\tfrac{4}{3}I_1) \quad \text{when} \quad i = 1, \dots, 4 \tag{8}$$

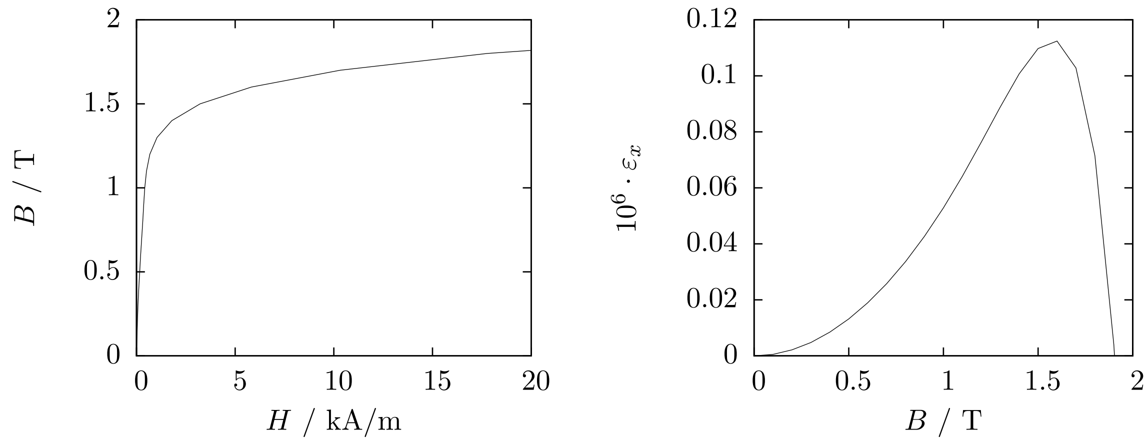

Figure 1: Magnetization curve (l.h.s.) and magnetostrictive strain (r.h.s).

and where $\gamma_4^{(i)} = g_i(0)$ are unknown material parameters.

In Fig. 1 the model predictions are presented in uniaxial magnetic excitation $\mathbf{B} = (B, 0, 0)^T$. The parameters are chosen as in [8], i.e. the Young's modulus $E = 183.62$ GPa, the Poisson's ratio $\nu = 0.34$, $\gamma_4^{(0)} = -0.99974858\,\mu_0^{-1}, \gamma_4^{(1)} = 0.00076054\,\mu_0^{-1}, \gamma_4^{(2)} = -0.00089881\,\mu_0^{-1}, \gamma_4^{(3)} = 0.00008964\,\mu_0^{-1}, \gamma_4^{(4)} = 0.00012688\,\mu_0^{-1}, B_{\text{ref}} = 1$ T, except γ_5 which here has the value $-0.028\,\mu_0^{-1}$.

3 FINITE ELEMENT PROCEDURE

Discretization of the mechanical part of the balance equations (1) is standard. In magnetostatics it is quite common to use vector potential $\mathbf{A}$, such that $\mathbf{B} = \operatorname{curl}\mathbf{A}$. As $\operatorname{curl}(\operatorname{grad} p) = \mathbf{0}$, the vector potential $\mathbf{A}$ is not uniquely determined. To make $\mathbf{A}$ unique a common practice is to add an additional gauging condition $\operatorname{div}\mathbf{A} = 0$ in the weak sense by using a Lagrange multiplier p. The mixed type weak form is

$$(\operatorname{curl}\hat{\mathbf{A}}, \mathbf{H}) + (\hat{\mathbf{A}}, \operatorname{grad} p) = (\hat{\mathbf{A}}, \mathbf{J}) - [\hat{\mathbf{A}}, \bar{\mathbf{H}}], \tag{9}$$

$$(\operatorname{grad}\hat{p}, \mathbf{A}) = 0, \tag{10}$$

where $\hat{\mathbf{A}}$ and $\hat{p}$ are the test functions. The inner products are defined as

$$(\mathbf{B}, \mathbf{H}) = \int_\Omega \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}V \quad \text{and} \quad [\hat{\mathbf{A}}, \bar{\mathbf{H}}] = \int_{\partial\Omega} (\hat{\mathbf{A}}\cdot\bar{\mathbf{H}}) \times \mathrm{d}\mathbf{s}, \tag{11}$$

where $\bar{\mathbf{H}}$ is a prescribed value of the magnetic field on the boundary and $\mathrm{d}\mathbf{s}$ denotes the vectorial surface area element of the boundary. The magnetic field $\mathbf{H}$ is obtained from the constitutive equation as

$$\mathbf{H} = (\mu_0^{-1} + 2f_4)\mathbf{B} + \gamma_5\mathbf{B}\cdot\boldsymbol{\varepsilon} + \gamma_6\mathbf{B}\cdot\boldsymbol{\varepsilon}^2, \tag{12}$$

where $f_4 = \rho\partial\psi/\partial I_4$.

To obtain a stable and convergent scheme, special care in the selection of the interpolation functions is needed [9]. For $\mathbf{A}$ and $\hat{\mathbf{A}}$ the curl-conforming edge elements should be used. The standard C^0-interpolation is suitable for the Lagrange multiplier.

In 2-D, the divergence condition is automatically satisfied and the system reduces to a standard diffusion type weak form without the Lagrange multiplier. In this case, the standard C^0-interpolation can be used to discretize the A_z-component.

An alternative formulation for the magnetic part, suitable for the present constitutive formulation, could be based on the Lagrangian functional

$$\mathcal{L} = \frac{1}{2}(\operatorname{div}\mathbf{B}, \operatorname{div}\mathbf{B}) + (\mathbf{p}, \operatorname{curl}\mathbf{H} - \mathbf{J}), \tag{13}$$

where divergence-conforming interpolation should be used for $\mathbf{B}$ and curl-conforming for the Lagrange multiplier $\mathbf{p}$.

As an example, a two-dimensional magnetoelastic plane-strain problem is solved in a rectangular domain $\Omega = \{(x, y)\,|\,0 < x < 2L,\, 0 < y < L\}$. The loading consist of two point currents $J_z^1 = 100$ A at $(\frac{1}{2}L, \frac{1}{2}L)$ and $J_z^2 = -100$ A at $(\frac{3}{4}L, \frac{1}{2}L)$. The material parameters are as in the previous section and $L = 1$ m. A uniform 36×18 bilinear element mesh is used. The magnetic potential A_z is set to zero on the whole boundary and the displacements are supressed along the line $x = 0$. The magnetic flux density vector field and deformed shape are shown in Fig.2.

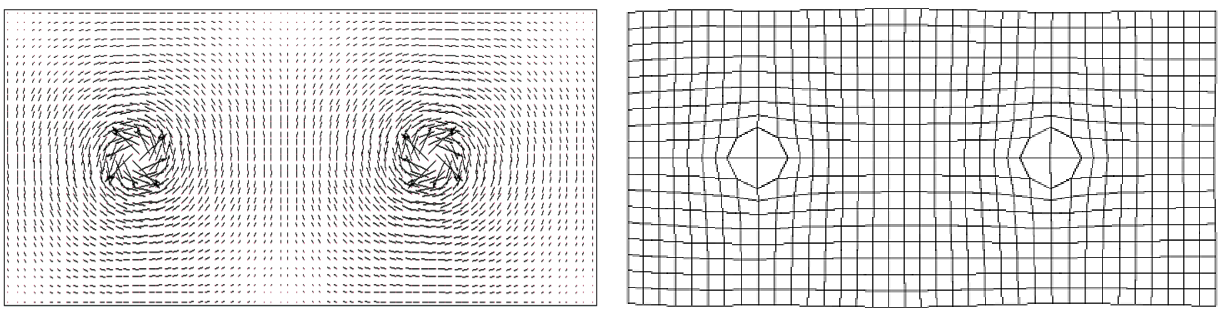

Figure 2: Magnetic flux density vector field and deformed mesh magnified by a factor of $5 \cdot 10^7$.

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