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ON COMPUTING CRITICAL EQUILIBRIUM POINTS BY A DIRECT METHOD

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Summary. Computation of critical points on an equilibrium path requires solution of a non-linear eigenvalue problem, for which solution several techniques exist. Their algorithmic treatment is usually focused for direct linear solvers and thus use the block elimination strategy. Due to the non-uniqueness of the critical eigenmode a normalizing condition is required. In addition, for bifurcation points, the Jacobian matrix of the augmented system is singular at the critical point and additional stabilization is required in order to quarantee the quadratic convergence of the Newton's method. Depending on the normalizing condition, convergence to a critical point with negative load parameter value can happen. The form of the normalizing equation is critically discussed and an alternative form is proposed.

1 INTRODUCTION

Determination of a critical point is the primary problem in structural stability analysis. Mathematically it means solution of an eigenvalue problem, which in general is non-linear, together with the equilibrium equations. However, if the pre-buckling displacements are negligible, it is usually sufficient to solve the linearized eigenvalue problem, where the linearization is performed with respect to the undeformed configuration.

The non-linear stability eigenvalue problem constitutes of solving the equilibrium equations simultaneously with the criticality condition. First appearance of this idea seems to be from 1973 by Keener and Keller[1]. In their approach the criticality condition is augmented as an eigenvalue equation, such approach has been used also in Refs.[2,3,4,5]. Another approach uses a scalar equation indicating the criticality[6,7] or expansion to a higher order polynomial eigenvalue problem[8,9].

In computational structural analysis, direct solution of the critical points along the equilibrium path requires complete kinematical description of the underlying kinematical model. In particular, for dimensionally reduced models, like beam- and shell models, the description has to be capable to handle large rotations. Development of a geometrically exact model with large rotations is not a trivial task[10,11,12].

2 STABILITY EIGENVALUE PROBLEM

The problem of finding a critical point along an equilibrium path can be stated as: find the critical values of $\mathbf{q}, \lambda$ and the corresponding eigenvector $\boldsymbol{\phi}$ such that

$$\begin{cases} \mathbf{f}(\mathbf{q},\lambda) &= \mathbf{0} \\ \mathbf{f}'(\mathbf{q},\lambda)\boldsymbol{\phi} &= \mathbf{0} \end{cases} \tag{1}$$

where $\mathbf{f}$ is a vector defining the equilibrium equations and $\mathbf{f}'$ denotes the Gateaux derivative (Jacobian matrix) with respect to the state variables $\mathbf{q}$, i.e. the stiffness matrix. At the critical point the equilibrium equations $(1)_1$ has to be satisfied at the same time with the criticality condition $(1)_2$, which states the zero stiffness in the direction of the critical eigenmode ϕ. Such a system is considered in Refs.[3,13,5].

The equilibrium equation $(1)_1$ constitutes the balance of internal forces $\mathbf{r}$ and external loads $\mathbf{p}$, which is usually parametrized by a single variable λ, the load parameter, defining the intensity of the load vector:

$$\mathbf{f}(\mathbf{q},\lambda) \equiv \mathbf{r}(\mathbf{q}) - \lambda \mathbf{p}_{\mathrm{r}}(\mathbf{q}). \tag{2}$$

If the loads does not dependent on deformations, like in dead-weight loading, the reference load vector $\mathbf{p}_{\mathrm{r}}$ is independent of the displacement field $\mathbf{q}$.

The system (1) consists of $2n+1$ unknowns, the displacement vector $\mathbf{q}$, the eigenmode $\boldsymbol{\phi}$ and the load parameter value λ at the critical state. Since the eigenvector $\boldsymbol{\phi}$ is defined uniquely up to a constant, the normalizing condition can be added to the system (1). In addition some stabilizing conditions might also be needed. In general, the full augmented system can be written as

$$\mathbf{g}(\mathbf{q},\boldsymbol{\phi},\boldsymbol{\lambda}) = \begin{cases} \hat{\mathbf{f}}(\mathbf{q},\boldsymbol{\lambda}) \equiv \mathbf{f}(\mathbf{q},\lambda) + \mathbf{f}_0(\mathbf{q},\boldsymbol{\lambda}) = \mathbf{0} \\ \mathbf{h}(\mathbf{q},\boldsymbol{\phi},\boldsymbol{\lambda}) \equiv \mathbf{f}'(\mathbf{q},\boldsymbol{\lambda})\boldsymbol{\phi} + \mathbf{h}_0(\boldsymbol{\phi},\boldsymbol{\lambda}) = \mathbf{0} \\ \mathbf{c}(\mathbf{q},\boldsymbol{\phi},\boldsymbol{\lambda}) = \mathbf{0}, \end{cases} \tag{3}$$

where $\boldsymbol{\lambda}$ is a vector of control and auxiliary parameters and $\mathbf{c}$ is a vector of constraint or stabilizing equations, dimension of these vectors is $p \geq 1$. The additional functions $\mathbf{f}_0$ and $\mathbf{h}_0$ are chosen such that $\mathbf{f}_0 = \mathbf{h}_0 = \mathbf{0}$ at the solution point. The Newton step can be written as

$$\begin{bmatrix} \mathbf{K}_f & \mathbf{0} & \mathbf{P} \\ \mathbf{Z} & \mathbf{K}_h & \mathbf{N} \\ \mathbf{C}_q & \mathbf{C}_\phi & \mathbf{C}_\lambda \end{bmatrix} \begin{Bmatrix} \delta\mathbf{q} \\ \delta\boldsymbol{\phi} \\ \delta\boldsymbol{\lambda} \end{Bmatrix} = - \begin{Bmatrix} \hat{\mathbf{f}} \\ \mathbf{h} \\ \mathbf{c} \end{Bmatrix}, \tag{4}$$

where

$$\mathbf{Z} = \left[\mathbf{f}'\boldsymbol{\phi}\right]', \quad \mathbf{C}_q = \mathbf{c}' = \frac{\partial \mathbf{c}}{\partial \mathbf{q}}, \quad \mathbf{C}_\phi = \frac{\partial \mathbf{c}}{\partial \boldsymbol{\phi}}, \quad \mathbf{C}_\lambda = \frac{\partial \mathbf{c}}{\partial \boldsymbol{\lambda}}. \tag{5}$$

$$\mathbf{K}_f = \mathbf{K} + \mathbf{f}_0', \quad \mathbf{K}_h = \mathbf{K} + \frac{\partial \mathbf{h}_0}{\partial \boldsymbol{\phi}}, \quad \mathbf{P} = \frac{\partial \mathbf{f}}{\partial \boldsymbol{\lambda}} \quad \text{and} \quad \mathbf{N} = \frac{\partial \mathbf{h}}{\partial \boldsymbol{\lambda}} \tag{6}$$

Computation of the matrix $\mathbf{Z}$ requires second order derivatives of the residual. In the literature these are usually obtained by numerical differentiation. For the geometrically exact Reissner's beam model analytical derivation of the $\mathbf{Z}$-matrix is given in[14].

If the system (4) is solved by using a direct solver, the block elimination scheme is a feasible choice. Partitioning the iterative steps $\delta\mathbf{q}$ and $\delta\boldsymbol{\phi}$ as

$$\delta\mathbf{q} = \mathbf{q}_f + \mathbf{Q}_p\delta\boldsymbol{\lambda}, \quad \delta\boldsymbol{\phi} = \boldsymbol{\phi}_h + \boldsymbol{\Phi}_n\delta\boldsymbol{\lambda}, \tag{7}$$

where the vectors $\mathbf{q}_f, \boldsymbol{\phi}_h$ and the $n \times p$ matrices $\mathbf{Q}_p, \boldsymbol{\Phi}_n$ can be solved from equations

$$\mathbf{K}_f\mathbf{q}_f = -\hat{\mathbf{f}}, \qquad \mathbf{K}_f\mathbf{Q}_p = -\mathbf{P}, \tag{8}$$

$$\mathbf{K}_h\boldsymbol{\phi}_h = -\mathbf{h} - \mathbf{Z}\mathbf{q}_\mathrm{f}, \qquad \mathbf{K}_h\boldsymbol{\Phi}_n = -\mathbf{N} - \mathbf{Z}\mathbf{Q}_p. \tag{9}$$

The iterative change of the control parameters can be solved from the constraint conditions resulting in

$$\delta\boldsymbol{\lambda} = -(\mathbf{C}_\lambda + \mathbf{C}_q\mathbf{Q}_p + \mathbf{C}_\phi\boldsymbol{\Phi}_n)^{-1}(\mathbf{c} + \mathbf{C}_q\mathbf{q}_f + \mathbf{C}_\phi\boldsymbol{\phi}_h). \tag{10}$$

The specific choice by Wriggers and Simo[5] yields $\mathbf{K}_h = \mathbf{K}_f$, which is computationally very attractive. Solution of the augmented system (4) by the block elimination method requires factorization of one matrix of order n, reductions and backsubstitutions of $2(1+p)$ r.h.s.-vectors. An alternative procedure suitable for the use of preconditioned iterative linear solvers is presented in[15].

For the eigenvector normalization the following constraints can be used:

$$\|\boldsymbol{\phi}\|^2 - 1 = 0, \tag{11}$$

$$\lambda\|\boldsymbol{\phi}\|^2 - 1 = 0, \tag{12}$$

$$\mathbf{e}_i^\mathrm{T}\boldsymbol{\phi} - 1 = 0, \tag{13}$$

$$\lambda(\mathbf{e}_i^\mathrm{T}\boldsymbol{\phi})^2 - 1 = 0, \tag{14}$$

where $\mathbf{e}_i$ is a unit vector having the element 1 at i:th component. The constraint (13) is used by Wriggers and Simo[5]. Constraints (12) and (14) guarantees that the iteration will converge to a solution with a positive critical value of the load parameter λ. A proper scaling of the constraint equation is also important. For the constraint equation (12) the best numerical performance is obtained when the initial scaling of the eigenvector approximation is of order $\|\boldsymbol{\phi}\| \sim \lambda_\mathrm{cr}^{-1}$. Numerical experiments indidate that the constraint (12) results in slightly more efficient scheme than (11).

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