Lecture notes of the course
Introduction to materials modelling

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I am very pleased to have a note of any error and any kind of idea on improving the text is welcome. E-mail address is reijo.kouhia@tuni.fi.

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Chapter 1
Introduction

1.1 The general structure of continuum mechanics

In principle, the general structure of equations in continuum mechanics is threefold. First, there is a balance equation (or balance equations) stating the equilibrium or force balance of the system considered. These equations relate e.g. the stress with external forces. Secondly, the stress is related to some kinematical quantity, such as strain, by the constitutive equations. Thirdly, the strain is related to displacements by the kinematical equations.

Balance equations are denoted as $B^*\sigma = f$, where B^* is the equilibrium operator, usually a system of differential operators. In the constitutive equations $\sigma = C\varepsilon$ the operator C can be either an algebraic or differential operator. Finally, the geometrical relation, i.e. the kinematical equations, are denoted as $\varepsilon = Bu$. These three equations form the system to be solved in continuum mechanics and it is illustrated in figure 1.1. In geometrically linear problems, the equilibrium operator B^* is the adjoint operator of the kinematical operator B. Therefore, there are only two independent operators in the system.

Example - axially loaded bar. The equilibrium equation in terms of the axial force N is

$$-\frac{\mathrm{d}N}{\mathrm{d}x} = f, \tag{1.1}$$

where f is the distributed load [force/length] in the direction of the bar's axis. Thus, the equilibrium operator B^* is

$$B^* = -\frac{\mathrm{d}}{\mathrm{d}x}. \tag{1.2}$$

The axial force is related to the strain via the elastic constitutive equation (containing the cross-section area as a geometric quantity)

$$N = EA\varepsilon. \tag{1.3}$$

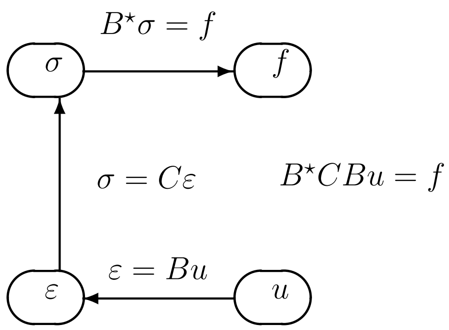


Figure 1.1: The general structure of equations in mechanics.

In this case the constitutive operator C is purely algebraic constant $C = EA$. The kinematical relation is

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}, \tag{1.4}$$

thus, the kinematic operator

$$B = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.5}$$

for which B^* is clearly the adjoint. The equilibrum equation expressed in terms of axial displacement is

$$B^*CBu = -\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f. \tag{1.6}$$

Example - thin beam bending. The equilibrium equation in terms of the bending moment M is

$$-\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = f, \tag{1.7}$$

where f is the distributed transverse load [force/length]. Thus, the equilibrium operator B^* is

$$B^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \tag{1.8}$$

The bending moment is related to the curvature via the elastic constitutive equation (containing the inertia of the cross-section as a geometric quantity)

$$M = EI\kappa. \tag{1.9}$$

Again, the constitutive operator C is purely algebraic constant $C = EI$. The kinematical relation is

$$\kappa = -\frac{\mathrm{d}^2v}{\mathrm{d}x^2}. \tag{1.10}$$

The kinematical operator is

$$B = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \tag{1.11}$$

for which B^* is clearly the adjoint and also in this case $B^* = B$. The equilibrum equation expressed in terms of the axial displacement is

$$B^*CBu = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) = f. \tag{1.12}$$

Example - linear 3-D elasticity. The equilibrium, constitutive and kinematical equations are

$$-\operatorname{div}\boldsymbol{\sigma}^T = \rho\boldsymbol{b}, \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{1.13}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}, \tag{1.14}$$

$$\boldsymbol{\varepsilon} = \operatorname{sym}\operatorname{grad}\boldsymbol{u}, \tag{1.15}$$

where $\boldsymbol{\sigma}$ is the symmetric stress tensor, ρ is the material density, $\boldsymbol{b}$ is the body force per unit mass, $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{C}$ is the elasticity tensor. Thus the operators B^*, B and C are

$$B^* = -\operatorname{div}, \tag{1.16}$$

$$B = \operatorname{grad}, \tag{1.17}$$

$$C = \boldsymbol{C}. \tag{1.18}$$

The formal adjoint of the $B^* = -\operatorname{div}$ operator is the gradient operator.

1.2 Constitutive models

Constitutive equations, the operator C, describe the response of a material to applied loads. In continuum mechanics, distinction between fluids and solids can be characterized in this stage. It is important to notice that the balance equations and the kinematical relations described in the previous sections are equally valid both for fluids and solids. In this lecture notes only *macroscopic*[1] models will be introduced, which roughly means that mathematical expressions are fitted to experimental data. Macroscopic models are not capable to relate the actual physical mechanisms of deformation to the underlying mcroscopic physical structure of the material.

The constitutive equations should obey the thermodynamic principles, (i) the conservation of energy and (ii) the dissipation inequality, i.e. the nonnegativity of the entropy rate.

Excellent texts for materials modelling are [26, 31, 41].

[1] Macroscopic models are often called as *phenomenological* models in contrast to *micromechanical* models where the physical mechanisms can be more directly modelled. However, in micromechanical models the phenomenology is only a level or some levels deeper.

1.3 Vectors and tensors

1.3.1 Motivation

In any physical science physical phenomena are described by mathematical models, which should be independent of the position and orientation of the observer. If the equations of a particular model are expressed in one coordinate system, they have to be able describe the same behaviour also in another coordinate system too. Therefore, the equations of mathematical models describing physical phenomena are vecor or tensor equations, since vectors and tensors transform from one coordinate system to another coordinate system in such a way that if a vector or tensor equation holds in one coordinate system, it holds in any other coordinate system not moving relative to the first one [28, p. 7].

1.3.2 Vectors

In three-dimensional space a vector can be visualized as a an arrow having a length and a direction. In mathematics a vector can have a more abstract meaning.

1.3.3 Second order tensors

A second order tensor, denoted e.g. by $\boldsymbol{A}$ can be understood as a general linear transformation that acts on a vector $\boldsymbol{u}$ and producing a vector $\boldsymbol{v}$.

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}. \tag{1.19}$$

In many texts, especially older ones, the dot indicating the multiplication, $\cdot$, is is used and the equation (1.19) is written as

$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u}. \tag{1.20}$$

In this lecture notes, only cartesian rectangular coordinate system is used, and the orthonormal unit base vectors of an arbitrary coordinate system are denoted as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Since the tensor equation (1.19), or (1.20) embraces information of the the underlying coordinate system, it can be expressed in a *dyadic* form

$$\begin{aligned}\boldsymbol{A} =& A_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ &+ A_{21}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ &+ A_{31}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3,\end{aligned} \tag{1.21}$$

which can be written shortly as

$$\boldsymbol{A} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{1.22}$$

In the last form of (1.22) the *Einstein's summation convention* is used.[2] The summation convention states that whenever the same letter subscript occurs twice in a term, a summation over the range of this index is implied unless othetwise indicated. That index is called a *dummy* index and the symbol given for a dummy index is irrelevant. The tensor product[3], or dyad, $\boldsymbol{u} \otimes \boldsymbol{v}$ of the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is a second order defined as a linear transformation

$$\boldsymbol{u} \otimes \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}, \tag{1.23}$$

i.e. it transforms a vector $\boldsymbol{w}$ in the direction of the vector $\boldsymbol{u}$. In the older literature the $\otimes$ symbol is often omitted and notation $\boldsymbol{uv}$ for the tensor product is also used. In index notation it is written as $u_i v_j$ and in matrix form as $\boldsymbol{u}\boldsymbol{v}^T$.

As an example, a scalar product between two vectors is defined as

$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3)\cdot(b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3) \\ &= a_i\boldsymbol{e}_i \cdot b_j\boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = a_i b_j \delta_{ij} = a_i b_i,\end{aligned} \tag{1.24}$$

where the Kronecker delta-symbol, which is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \end{cases} \tag{1.25}$$

which defines the second order unit tensor.

Using the summation convention the equation (1.19) can be written as

$$v_i\boldsymbol{e}_i = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot u_k\boldsymbol{e}_k = A_{ij}u_k\delta_{jk}\boldsymbol{e}_i = A_{ij}u_j\boldsymbol{e}_i, \tag{1.26}$$

from which we can deduce

$$(v_i - A_{ij}u_j)\boldsymbol{e}_i = \boldsymbol{0}, \tag{1.27}$$

and the relation between the components is

$$v_i = A_{ij}u_j. \tag{1.28}$$

Since in this lecture notes only cartesian coordinate systems are used, the tensor equations can be written simply either in the absolute notation, like equation (1.19), or in the index notation without the base vectors, like in equation (1.28). The cartesian second-order tensor operates just like a matrix. An index which is not dummy is called *free*, like the index i in eq. (1.28).

The *dot product* of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted as $\boldsymbol{AB}$ (in the older literature also denoted as $\boldsymbol{A} \cdot \boldsymbol{B}$) and is defined as

$$(\boldsymbol{AB})\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{Bu}) \tag{1.29}$$

[2]The summation convention appeared first time in Albert Einstein's (1879-1955) paper on general relativity in 1916.

[3]The tensor product is also known as a direct or matrix product.

for all vectors $\boldsymbol{u}$. The result of a dot product between two second-order tensors is also a second-order tensor. In general, the dot product is *not commutative*, i.e. $\boldsymbol{AB} \neq \boldsymbol{BA}$. The components of the dot product $\boldsymbol{C} = \boldsymbol{AB}$ between cartesian tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ are given as

$$C_{ij} = A_{ik}B_{kj}. \tag{1.30}$$

The transpose of a tensor is defined as

$$\boldsymbol{b} \cdot \boldsymbol{A}^T\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{Ab} = (\boldsymbol{Ab}) \cdot \boldsymbol{a}, \tag{1.31}$$

for all vectors $\boldsymbol{a}, \boldsymbol{b}$. Note that $(\boldsymbol{A}^T)^T = \boldsymbol{A}$.

The trace of a dyad $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined as

$$\text{tr}\,(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i. \tag{1.32}$$

For a second-order tensor $\boldsymbol{A}$, expressed in an orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, the trace is thus given as

$$\text{tr}\,\boldsymbol{A} = \text{tr}(A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\,\text{tr}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\boldsymbol{e}_i \cdot \boldsymbol{e}_j = A_{ij}\delta_{ij} = A_{ii}. \tag{1.33}$$

A double dot product of two second-order tensors is defined as

$$\boldsymbol{A} : \boldsymbol{B} = \text{tr}(\boldsymbol{A}^T\boldsymbol{B}) = \text{tr}(\boldsymbol{B}^T\boldsymbol{A}) = \text{tr}(\boldsymbol{AB}^T) = \text{tr}(\boldsymbol{BA}^T) = \boldsymbol{B} : \boldsymbol{A}, \tag{1.34}$$

which in index notation and for cartesian tensors can be written as

$$A_{ij}B_{ij} = B_{ij}A_{ij}. \tag{1.35}$$

A second-order tensor $\boldsymbol{A}$ can be written as a sum of it's eigenvalues λ_i and eigenvectors $\boldsymbol{\phi}$ as

$$\boldsymbol{A} = \boldsymbol{A}\boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_i = \sum_{i=1}^{3} \lambda_i \boldsymbol{\phi}_i \otimes \boldsymbol{\phi}_i, \tag{1.36}$$

which is known as the *spectral decomposition* or *spectral representation* of $\boldsymbol{A}$.

1.3.4 Higher-order tensors

In these lecure notes, the *permutation tensor* $\mathcal{E}$ is the only third order tensor to be used. It is expressed as

$$\mathcal{E} = \epsilon_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k, \tag{1.37}$$

where $\epsilon_{ijk} = (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k$ are the 3^3 components of $\mathcal{E}$. The components ϵ_{ijk} can be expressed as[4]

$$\epsilon_{ijk} = \begin{cases} +1, & \text{for even permutations of } (i,j,k), \text{i.e. 123, 231, 312,} \\ -1, & \text{for odd permutations of } (i,j,k), \text{i.e. 132, 213, 321,} \\ 0, & \text{if there are two or more equal indexes.} \end{cases} \tag{1.38}$$

[4]The permutation symbol ϵ_{ijk} is also known as alternating or Levi-Civita-ϵ symbol.

Fourth-order tensors are used in constitutive models. As an example of a fourth-order tensor is a tensor product of two second-order tensors

$$\mathbb{C} = \boldsymbol{A}\boldsymbol{B}, \quad \text{or in index notation} \quad C_{ijkl} = A_{ij}B_{kl}. \tag{1.39}$$

There are two different fourth-order unit tensors $\mathbb{I}$ and $\bar{\mathbb{I}}$, defined as

$$\boldsymbol{A} = \mathbb{I} : \boldsymbol{A}, \quad \text{and} \quad \boldsymbol{A}^T = \bar{\mathbb{I}} : \boldsymbol{A}, \tag{1.40}$$

for any second-order tensor $\boldsymbol{A}$. In index notation for cartesian tensors the identity tensors have the forms

$$I_{ijkl} = \delta_{ik}\delta_{jl}, \quad \bar{I}_{ijkl} = \delta_{il}\delta_{jk}. \tag{1.41}$$

1.3.5 Summary

Some hints to access the validity of a tensor equation expressed in the index notation:

1. identify the dummy and free indexes,
2. if three or more same indexes appear in a single term, there is an error,
3. perform contactions (dot products) and replacements (identity tensor) if possible.

Recommendable modern treatment of tensor analysis for engineers is [19].

1.4 Nomenclature

Strain and stress

$\boldsymbol{e}, e_{ij}$	=	deviatoric strain tensor
$\boldsymbol{s}, s_{ij}$	=	deviatoric stress tensor
s_1, s_2, s_3	=	principal values of the deviatoric stress
γ	=	shear strain
γ_{oct}	=	octahedral shear strain
ε_{ij}	=	strain tensor
ε_{oct}	=	octahedral strain
ε_{v}	=	volumetric strain
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	=	principal strains
σ	=	normal stress
$\boldsymbol{\sigma}, \sigma_{ij}$	=	stress tensor
σ_{m}	=	mean stress
σ_{oct}	=	octahedral stress
$\sigma_1, \sigma_2, \sigma_3$	=	principal stresses
τ	=	shear stress
τ_{m}	=	mean shear stress
τ_{oct}	=	octahedral shear stress

Invariants

$I_1(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} = A_{ii}$	=	the first invariant of tensor $\boldsymbol{A}$
$I_2(\boldsymbol{A}) = \frac{1}{2}[\operatorname{tr}(\boldsymbol{A}^2) - (\operatorname{tr} \boldsymbol{A})^2]$	=	second invariant
$I_3 = \det \boldsymbol{A}$	=	third invariant
$J_2(\boldsymbol{s}) = \frac{1}{2} \operatorname{tr} \boldsymbol{s}^2$	=	second invariant of the deviatoric tensor $\boldsymbol{s}$
$J_3 = \det \boldsymbol{s}$	=	third invariant of a deviatoric tensor
ξ, ρ, θ	=	the Heigh-Westergaard stress coordinates
ξ	=	hydrostatic length
ρ	=	the length of the stress radius on the deviatoric plane
θ	=	the Lode angle on the deviatoric plane

Material parameters

E	=	Young's modulus
G	=	shear modulus
G_{f}	=	fracture energy
K	=	bulk modulus, hardening parameter
k	=	shear strength
m	=	$f_{\mathrm{c}} = m f_{\mathrm{t}}$
α, β	=	parameters in the Drucker-Prager yield condition
ν	=	Poisson's ratio
ϕ	=	internal friction angle of the Mohr-Coulomb criterion

1.5 On the references

This lecture notes is mostly based on the following excellent books:

1. L.E. Malvern: *Introduction to the Mechanics of a Continuous Medium*. Beautifully written treatease on the topic.

2. G.A. Holzapfel: *Nonlinear Solid Mechanics, A Continuum Approach for Engineers*. A modern treatment of some basic material in Malvern's book. Contains usefull material for understanding nonlinear finite element methods.

3. W.F. Chen, D.J. Han: *Plasticity for Structural Engineers*.

4. J. Lemaitre, J.-L. Chaboche: *Mechanics of Solid Materials*.

5. N.S. Ottosen, M. Ristinmaa: *Mechanics of Constitutive Modelling*.

6. J.N. Reddy: *An Introduction to Continuum Mechanics with Applications*.

Chapter 2
Stress

2.1 Stress tensor and the theorem of Cauchy

Consider a body $\mathcal{B}$ in a 3-dimensional space occupying a volume domain Ω, see figure 2.1. If the body $\mathcal{B}$ is divided into two parts by a surface $\mathcal{S}$ and the parts separated from each other. The force acting on a small surface ΔS is denoted by $\Delta \boldsymbol{f}$. A traction vector $\boldsymbol{t}$ is defined as

$$\boldsymbol{t} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{f}}{\Delta S} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}S}. \tag{2.1}$$

The traction vector depends on the position $\boldsymbol{x}$ and also on the normal direction $\boldsymbol{n}$ of the surface, i.e.

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}), \tag{2.2}$$

a relationship, which is called as the postulate of Cauchy.[1]

In the rectangular cartesian coordinate system, the traction vectors acting in three perpendicular planes, parallel to the coordinate axes are denoted as $\boldsymbol{t}_1, \boldsymbol{t}_2$ and $\boldsymbol{t}_3$, see figure 2.2. The components of the traction vectors are shown in the figure and expressed in terms of the unit vectors parallel to the coordinate axes $\boldsymbol{e}_i$ the traction vectors are

$$\boldsymbol{t}_1 = \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3, \tag{2.3}$$

$$\boldsymbol{t}_2 = \sigma_{21}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_3, \tag{2.4}$$

$$\boldsymbol{t}_3 = \sigma_{31}\boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3. \tag{2.5}$$

To obtain the expression of the traction vector in terms of the components σ_{ij}, let us consider a tetrahedron where the three faces are parallel to the coordinate planes and the remaining one is oriented in an arbitrary direction, see figure 2.3. In each of the faces, the average traction is denoted as $\boldsymbol{t}_i^*$, where $i = 1, 2, 3$, and the area of the triangle $A_1A_2A_3$ is denoted as ΔS and $\Delta S_1, \Delta S_2, \Delta S_3$ are the areas of triangles OA_2A_3, OA_3A_1 and OA_1A_2, respectively. The body force acting on the tetrahedron is $\rho^*\boldsymbol{b}^*\Delta V$, where the volume element $\Delta V = \frac{1}{3}h\Delta S$, and h is the distance ON.

[1] Sometimes traction vector $\boldsymbol{t}$ is also called as a stress vector. However, in this lecture notes this naming is not used since the stress has a tensor character.

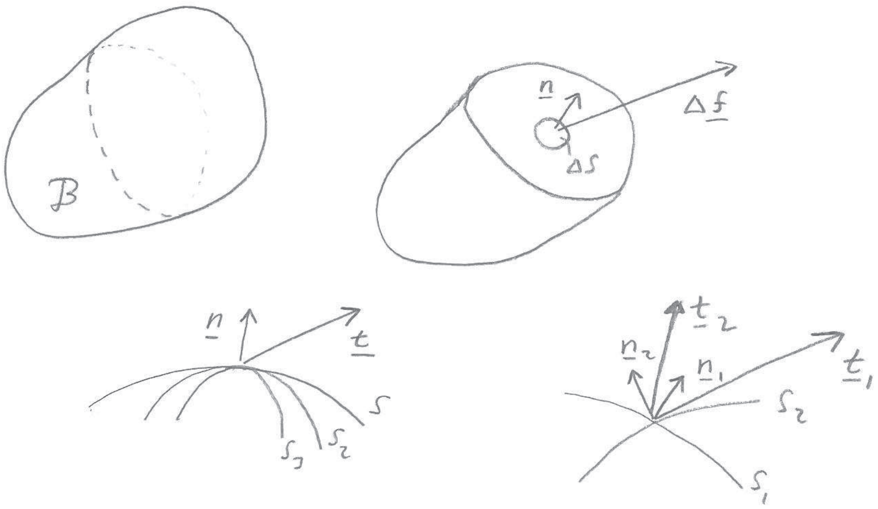


Figure 2.1: A continuum body and the traction vector.

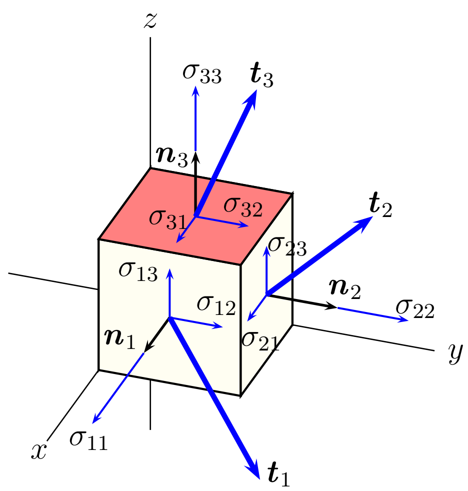


Figure 2.2: Traction vectors in three perpendicular directions.

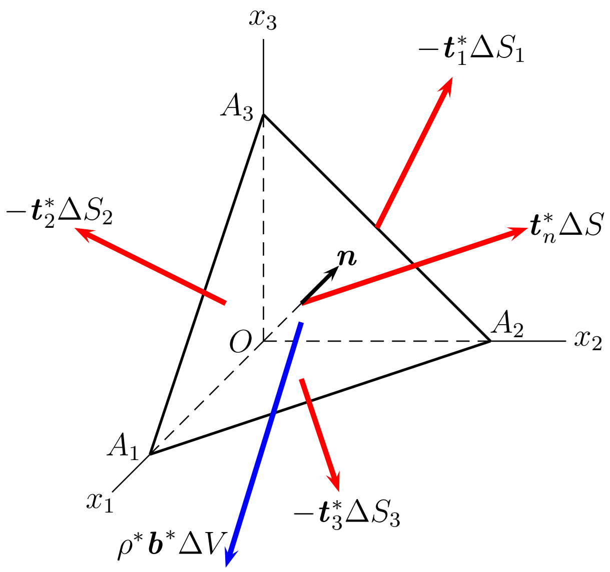


Figure 2.3: Traction vectors acting on te faces of the Cauchy's tetrahedron.

Equilibrium equation for the tetrahedron is

$$\boldsymbol{t}_n^*\Delta S + \tfrac{1}{3}\rho^*\boldsymbol{b}^*h\Delta S - \boldsymbol{t}_1^*\Delta S_1 - \boldsymbol{t}_2^*\Delta S_2 - \boldsymbol{t}_3^*\Delta S_3 = 0, \tag{2.6}$$

which can be written as

$$\Delta S(\boldsymbol{t}_n^* + \tfrac{1}{3}\rho^*\boldsymbol{b}^*h - n_1\boldsymbol{t}_1^* - n_2\boldsymbol{t}_2^* - n_3\boldsymbol{t}_3^*) = 0. \tag{2.7}$$

Now, letting $h \to 0$, we get $\boldsymbol{t}_i^* \to \boldsymbol{t}_i$ and

$$\begin{aligned}\boldsymbol{t}_n &= \sum_{i=1}^{3} n_1\boldsymbol{t}_i = n_i\boldsymbol{t}_i \\ &= n_1(\sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3) + n_2(\sigma_{21}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_3) \\ &\qquad + n_3(\sigma_{31}\boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3),\end{aligned} \tag{2.8}$$

or

$$\boldsymbol{t}_n = \begin{pmatrix} n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31} \\ n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32} \\ n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} \end{pmatrix} = \boldsymbol{n}^T\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\boldsymbol{n}. \tag{2.9}$$

Notice the transpose in the stress tensor $\boldsymbol{\sigma}$ in the last expression. The stress tensor $\boldsymbol{\sigma}$, expressed in rectangular cartesian coordinate system is

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}. \tag{2.10}$$

The form of the right hand side is know as von Kármán notation and the σ-symbol in it describes the normal component of the stress and τ the shear stresses. Such notation is common in engineering literature.

The equation (2.9) is called the Cauchy stress theorem and it can be written as

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = [\boldsymbol{\sigma}(\boldsymbol{x})]^T \boldsymbol{n}, \tag{2.11}$$

expressing the dependent quantities explicitly. It says that the traction vector depends linearly on the normal vector $\boldsymbol{n}$.

2.2 Coordinate transformation

If the stress tensor (or any other tensor) is known in a rectangular Cartesian coordinate system (x_1, x_2, x_3) with unit base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and we would like to know its components in other recangular Cartesian coordinate system (x_1', x_2', x_3') with unit base vectors $\boldsymbol{e}_1'$, $\boldsymbol{e}_2'$, $\boldsymbol{e}_3'$, a coordinate transformation tensor is needed. Let us write the stress tensor $\boldsymbol{\sigma}$ in the x_i-coordinate system as

$$\begin{aligned}\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ + \sigma_{21}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ + \sigma_{31}\boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3.\end{aligned} \tag{2.12}$$

This kind of representation is called the dyadic form and the base vector part $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be written either in matrix notation $\boldsymbol{e}_i\boldsymbol{e}_j^T$. It underlines the fact that a tensor contains not only the components but also the base in which it is expressed. Using the Einstein's summation convention it is briefly written as

$$\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sigma_{ij}'\boldsymbol{e}_i' \otimes \boldsymbol{e}_j', \tag{2.13}$$

indicating the fact that the tensor is the same irrespectively in which coordinate system it is expressed.

Taking a scalar product by parts with the vector $\boldsymbol{e}_k'$ from the left and with $\boldsymbol{e}_p'$ from the right, we obtain

$$\sigma_{ij}\underbrace{\boldsymbol{e}_k' \cdot \boldsymbol{e}_i}_{\beta_{ki}} \otimes \underbrace{\boldsymbol{e}_j \cdot \boldsymbol{e}_p'}_{\beta_{jp}} = \sigma_{ij}'\underbrace{\boldsymbol{e}_k' \cdot \boldsymbol{e}_i'}_{\delta_{ki}} \otimes \underbrace{\boldsymbol{e}_j' \cdot \boldsymbol{e}_p'}_{\delta_{jp}}. \tag{2.14}$$

It can be written in the index notation as

$$\sigma_{kp}' = \beta_{ki}\beta_{pj}\sigma_{ij} \quad \text{or in matrix notation} \quad [\sigma'] = [\beta][\sigma][\beta]^T, \tag{2.15}$$

where the compnents of the transformation matrix are $\beta_{ij} = \boldsymbol{e}_i' \cdot \boldsymbol{e}_j$. Notice that $\boldsymbol{\beta}$ is the transformation from x_i-system to x_i'-coordinate system.

2.3 Principal stresses and -axes

The pricipal values of the stress tensor σ are obtained from the linear eigenvalue problem

$$(\sigma_{ij} - \sigma\delta_{ij})n_j, \tag{2.16}$$

where the vector n_i defines the normal of the plane where the principal stress acts. The homogeneous system (2.16) has solution only if the coefficient matrix is singular, thus the determinant of it has to vanish, and we obtain the characteristic equation

$$-\sigma^3 + I_1\sigma^2 + I_2\sigma + I_3 = 0. \tag{2.17}$$

The coefficients $I_i, i = 1, \dots, 3$ are

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}, \tag{2.18}$$
$$I_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\sigma}^2) - (\mathrm{tr}\boldsymbol{\sigma})^2] = \tfrac{1}{2}(\sigma_{ij}\sigma_{ji} - I_1^2), \tag{2.19}$$
$$I_3 = \det(\sigma_{ij}). \tag{2.20}$$

Solution of the characteristic equation gives the principal values of the stress tensor, i.e. principal stresses σ_1, σ_2 and σ_3, which are often numbered as: $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

The coefficients I_1, I_2 and I_3 are independent of the chosen coordinate system, thus they are called *invariants*.[2] Notice, that the principal stresses are also independent of the chosen coordinate system. Invariants have a central role in the development of constitutive equations, as we will see in the subsequent chapters.

If the cordinate axes are chosen to coincide to the principal directions $\boldsymbol{n}_i$ (2.16), the stress tensor will be diagonal

$$\boldsymbol{\sigma} = [\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{2.21}$$

The invariants $I_1, \dots, I_3$ have the following forms expressed in terms of the principal stresses

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \tag{2.22}$$
$$I_2 = -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1, \tag{2.23}$$
$$I_3 = \sigma_1\sigma_2\sigma_3. \tag{2.24}$$

[2]The invariants appearing in the characteristic equation are usually called as *principal invariants*. Notice that in this note the second invariant is often defined as of opposite sign. However, we would like to define the principal invariants of the tensor and its deviator in a similar way. This convention is also used e.g. in [27]

2.4 Deviatoric stress tensor

The stress tensor can be additively decomposed into a deviatoric part, describing a pure shear state and an isotropic part describing hydrostatic pressure

$$\sigma_{ij} = s_{ij} + \sigma_{\mathrm{m}}\delta_{ij}, \tag{2.25}$$

where $\sigma_{\mathrm{m}} = \frac{1}{3}I_1 = \frac{1}{3}\sigma_{kk}$ is the mean or hydrostatic stress and s_{ij} the deviatoric stress tensor, for which the notation $\boldsymbol{\sigma}'$ is also often used in the literature. From the decomposition (2.25) it is observed that the trace of the deviatoric stress tensor will vanish

$$\mathrm{tr}\boldsymbol{s} = 0. \tag{2.26}$$

The principal values s of the deviatoric stress tensor $\boldsymbol{s}$ can be solved from

$$|s_{ij} - s\delta_{ij}| = 0, \tag{2.27}$$

giving the characteristic equation

$$-s^3 + J_1 s^2 + J_2 s + J_3 = 0, \tag{2.28}$$

where $J_1, \ldots, J_3$ are the invariants of the deviatoric stress tensor. They can be expressed as

$$J_1 = \mathrm{tr}\boldsymbol{s} = s_{ii} = s_x + s_y + s_z = 0, \tag{2.29}$$
$$J_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{s}^2) - (\mathrm{tr}\boldsymbol{s})^2] = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{s}^2) = \tfrac{1}{2}s_{ij}s_{ji} \tag{2.30}$$
$$= \tfrac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \tag{2.31}$$
$$= \tfrac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2], \tag{2.32}$$
$$J_3 = \det \boldsymbol{s} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{s}^3) = \tfrac{1}{3}(s_1^3 + s_2^3 + s_3^3) = s_1 s_2 s_3. \tag{2.33}$$

The deviatoric stress tensor is obtained from the stress tensor by substracting the isotropic part, thus the principal directions of the deviatoric stress tensor coincide to the principal directions of the stress tensor itself. Also the principal values of the deviatoric stress tensor are related to those of the stress tensor as

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} - \begin{bmatrix} \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \end{bmatrix}. \tag{2.34}$$

The deviatoric invariants expressed in terms of the principal values are

$$J_2 = \tfrac{1}{2}(s_1^2 + s_2^2 + s_3^2), \tag{2.35}$$
$$J_3 = \tfrac{1}{3}(s_1^3 + s_2^3 + s_3^3) = s_1 s_2 s_3. \tag{2.36}$$

In general, the characteristic equation (2.28) for the deviator, i.e.

$$-s^3 + J_2 s + J_3 = 0, \tag{2.37}$$

facilitates the direct computation of the principal values of the deviatoric stress tensor and thus also for the stress tensor itself via equations (2.34). Substituting transformation

$$s = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta \tag{2.38}$$

to the characteristic equation (2.37) results into equation

$$-\frac{2}{3\sqrt{3}}\left(4\cos^3\theta - 3\cos\theta\right)J_2^{3/2} + J_3 = 0. \tag{2.39}$$

Since $4\cos^3\theta - 3\cos\theta = \cos 3\theta$, the angle θ can be calculated as

$$\theta = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right). \tag{2.40}$$

If the angle θ satisfies $0 \le 3\theta \le \pi$, then $3\theta + 2\pi$ and $3\theta - 2\pi$ have the same cosine. Therefore $\theta_2 = \theta + 2\pi/3$ and $\theta_3 = \theta - 2\pi/3$ and the principal values of the deviator can be computed from (2.38).

2.5 Octahedral plane and stresses

Octahedral plane is a plane, the normal of which makes equal angles with each of the principal axes of stress. In the principal stress space the normal to the octahedral plane takes the form

$$\boldsymbol{n} = [n_1, n_2, n_3]^T = \frac{1}{\sqrt{3}}[1, 1, 1]^T \tag{2.41}$$

The normal stress on the octahedral plane is thus

$$\sigma_{\text{oct}} = \sigma_{ij}n_in_j = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \sigma_{\text{m}} \tag{2.42}$$

and for the shear stress on the octahedral plane, the following equation is obtained

$$\tau_{\text{oct}}^2 = t_it_i - \sigma_{\text{oct}}^2 = \sigma_{ij}\sigma_{ik}n_jn_k - (\sigma_{ij}n_in_j)^2. \tag{2.43}$$

Expressed in terms of principal stresses, the octahedral shear stress is

$$\tau_{\text{oct}}^2 = \tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2 \tag{2.44}$$

$$= \tfrac{1}{9}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2], \tag{2.45}$$

which an be written as

$$\tau_{\text{oct}} = \tfrac{2}{3}\sqrt{\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2}. \tag{2.46}$$

If the expression (2.44) is written as $\tau_{\text{oct}}^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^3) + \frac{1}{9}\sigma_{\text{m}}^2$, and using the relationships $\sigma_i = s_i + \sigma_{\text{m}}$, the following expression is obtained

$$\tau_{\text{oct}}^2 = \tfrac{1}{3}(s_1^2 + s_2^2 + s_3^2), \tag{2.47}$$

and the octahedral shear stress can be written in terms of the second invariant of the deviatoric stress tensor as

$$\tau_{\text{oct}} = \sqrt{\tfrac{2}{3}J_2}. \tag{2.48}$$

2.6 Principal shear stresses

It is easy to see with the help of Mohr's circles that the maximun shear stress is one-half of the largest difference between any two of the principal stresses and occurs in a plane whose unit normal makes an angle of 45° with each of the corresponding principal axes. The quantities

$$\tau_1 = \tfrac{1}{2}|\sigma_2 - \sigma_3|, \quad \tau_2 = \tfrac{1}{2}|\sigma_1 - \sigma_3|, \quad \tau_3 = \tfrac{1}{2}|\sigma_1 - \sigma_2| \tag{2.49}$$

are called as principal shear stresses and

$$\tau_{\max} = \max(\tau_1, \tau_2, \tau_3) \tag{2.50}$$

or

$$\tau_{\max} = \tfrac{1}{2}|\sigma_1 - \sigma_3|, \tag{2.51}$$

if the convention $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is used.

2.7 Geometrical illustration of stress state and invariants

The six-dimensional stress space is difficult to elucidate, therefore the principal stress space is more convenient for illustration purposes. Let's consider a three-dimensional euclidean space where the coordinate axes are formed from the principal stresses σ_1, σ_2 and σ_3, see figure 2.4.

Considering the stress point $P(\sigma_1, \sigma_2, \sigma_3)$, the vector OP can be assumed to represent the stress. The hydrostatic axis is defined through relations $\sigma_1 = \sigma_2 = \sigma_3$, and it makes equal angle to each of the principal stress axes and thus the unit vector parallel to the hydrostatic axis is

$$\boldsymbol{n} = \frac{1}{\sqrt{3}}[1, 1, 1]^T. \tag{2.52}$$

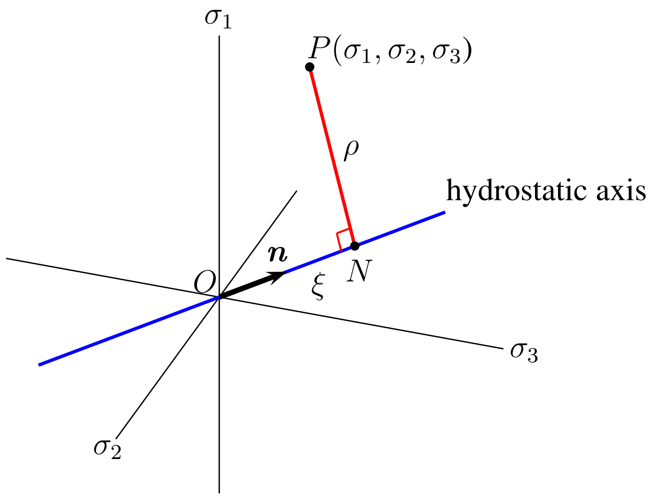


Figure 2.4: Principal stress space.

Since the deviatoric stress tensor vanishes along the hydrostatic axis, the plane perpendicular to it is called the deviatoric plane. The special deviatoric plane going through the origin, i.e.

$$\sigma_1 + \sigma_2 + \sigma_3 = 0, \tag{2.53}$$

is called the π-plane. A stress state on the π-plane is a pure shear stress state.

The vector OP can be divided into a component parallel to the hydrostatic axis ON and a component lying on the deviatoric plane NP, which are thus perpendicular to each other.

The length of the hydrostatic part ON is

$$\xi = |\vec{ON}| = \vec{OP} \cdot \boldsymbol{n} = \frac{1}{\sqrt{3}} I_1 = \sqrt{3}\sigma_{\mathrm{m}} = \sqrt{3}\sigma_{\mathrm{oct}}, \tag{2.54}$$

and its component representation has the form

$$\vec{ON} = \begin{bmatrix} \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \end{bmatrix} = \tfrac{1}{3} I_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{2.55}$$

Respectively, the component NP on the devatoric plane is

$$\vec{NP} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} - \begin{bmatrix} \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \\ \sigma_{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}. \tag{2.56}$$

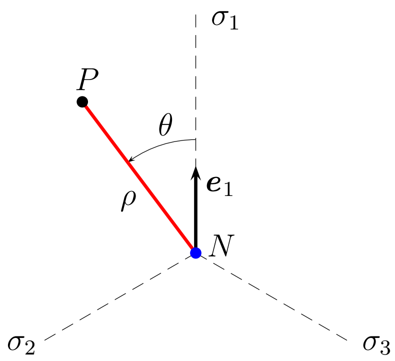


Figure 2.5: Deviatoric plane. The projections of the principal stress axes are shown with dashed line.

Square of the length $\vec{NP}$ is

$$\rho^2 = |\vec{NP}|^2 = s_1^2 + s_2^2 + s_3^2 = 2J_2 = 3\tau_{\text{oct}}^2 = 5\tau_{\text{m}}^2. \tag{2.57}$$

The invariants I_1 and J_2 have thus clear geometrical and physical interpretation. The cubic deviatoric invariant J_3 is related to the angle θ defined on the deviatoric plane as an angle between the projected σ_1-axis and the vector $\vec{NP}$, see figure 2.5. The vector $\boldsymbol{e}_1$ is a unit vector in the direction of the projected σ_1-axis and has the form

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}. \tag{2.58}$$

The angle θ can then be determined by using the dot product of vectors $\vec{NP}$ and $\boldsymbol{e}_1$ as

$$\vec{NP} \cdot \boldsymbol{e}_1 = \rho \cos\theta, \tag{2.59}$$

which gives

$$\cos\theta = \frac{1}{2\sqrt{3J_2}}(2s_1 - s_2 - s_3) = \frac{3}{2}\frac{s_1}{\sqrt{3J_2}} = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{2\sqrt{3J_2}}. \tag{2.60}$$

From the trigonometric identity, it is obtained

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{2.61}$$

and

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} = \frac{\sqrt{2}J_3}{\tau_{\text{oct}}^3}. \tag{2.62}$$

A stress space described by the coordinates ξ, ρ and θ is called the Heigh-Westergaard stress space.

2.8 Solved example problems

Example 2.1. *A stress state in a continuum at a point P is given by the following stress matrix*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_0 & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & 4\sigma_0 & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & \sigma_0 \end{bmatrix}.$$

1. *Determine the traction vector $\mathbf{t}$ on a plane, having the normal in the direction 1:-1:2.*
2. *Determine the traction vector at the point P acting on the plane $2x_1 - 2x_2 - x_3 = 0$.*
3. *Determine the normal and shear components on that plane.*
4. *Determine the principal stresses and directions.*

Solution.

1. The unit normal vector in the direction 1:-1:2 is $\boldsymbol{n} = [1, -1, 2]^T/\sqrt{6}$ and the traction vector is

$$\boldsymbol{t} = \boldsymbol{\sigma}^T \boldsymbol{n} = \begin{bmatrix} \sigma_0 & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & 4\sigma_0 & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & \sigma_0 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 2 \end{Bmatrix} \frac{1}{\sqrt{6}} = \frac{\sigma_0}{\sqrt{6}} \begin{Bmatrix} 5 \\ 10 \\ -1 \end{Bmatrix}.$$

2. The plane $2x_1 - 2x_2 - x_3 = 0$ has a normal $\boldsymbol{n} = [\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}]^T$, thus the traction vector on the plane is

$$\boldsymbol{t} = \begin{bmatrix} \sigma_0 & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & 4\sigma_0 & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & \sigma_0 \end{bmatrix} \begin{Bmatrix} 2 \\ -2 \\ -1 \end{Bmatrix} \frac{1}{3} = \frac{\sigma_0}{3} \begin{Bmatrix} -5 \\ -10 \\ -7 \end{Bmatrix}.$$

3. The normal stress action on the plane is just the projection of the traction vector on the direction of the normal

$$\sigma_n = \boldsymbol{t}^T \boldsymbol{n} = \frac{17}{9}\sigma_0 \approx 1,9\sigma_0.$$

The absolute value of the shear component action on the plane can be obtained by the Pythagoras theorem

$$\tau_n = \sqrt{\boldsymbol{t}^T \boldsymbol{t} - \sigma_n^2} = \sqrt{(\tfrac{-5}{3})^2 + (\tfrac{-10}{3})^2 + (\tfrac{-7}{3})^2 - (\tfrac{17}{9})^2}|\sigma_0| = \sqrt{1277}/9|\sigma_0| \approx 3,97|\sigma_0|.$$

4. The principal stresses σ and the normals of the planes where the principal stresses act $\boldsymbol{n}$, are obtained from the eigenvalue problem

$$\begin{bmatrix} \sigma_0 - \sigma & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & 4\sigma_0 - \sigma & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & \sigma_0 - \sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

A homogeneous equation system has a nontrivial solution only if the coefficient matrix is singular, thus

$$
\begin{aligned}
&\det \begin{bmatrix} \sigma_0-\sigma & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & 4\sigma_0-\sigma & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & \sigma_0-\sigma \end{bmatrix} = \\
&= (\sigma_0-\sigma)\begin{vmatrix} 4\sigma_0-\sigma & 6\sigma_0 \\ 6\sigma & \sigma_0-\sigma \end{vmatrix} - 2\sigma_0 \begin{vmatrix} 2\sigma_0 & 6\sigma_0 \\ 3\sigma & \sigma_0-\sigma \end{vmatrix} + 3\sigma_0 \begin{vmatrix} 2\sigma_0 & 4\sigma_0-\sigma \\ 3\sigma & 6\sigma_0 \end{vmatrix} \\
&= 0,
\end{aligned}
$$

from which the characteristic equation

$$-\sigma^3 + 6\sigma_0\sigma + 40\sigma_0^2\sigma = 0$$

is obtained. Solution for the principal stresses is then $10\sigma_0, 0, -4\sigma_0$.
For $10\sigma_0$ the corresponding direction of the principal stress space is obtained from

$$\begin{bmatrix} -9\sigma_0 & 2\sigma_0 & 3\sigma_0 \\ 2\sigma_0 & -6\sigma_0 & 6\sigma_0 \\ 3\sigma_0 & 6\sigma_0 & -9\sigma_0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

from where the solution $n_1 : n_2 : n_3 = 3 : 6 : 5$ is obtained. Directions corresponding to the other principal stresses can be obtained in a similar fashion, and they are $-2 : 1 : 0$ and $1 : 2 : -3$. Notice that the directions are mutually orthogonal.

Example 2.2. *A stress state of a continuum body is given by the stress matrix*

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau_0 & \tau_0 \\ \tau_0 & 0 & \tau_0 \\ \tau_0 & \tau_0 & 0 \end{bmatrix}$$

Determine the principal stresses and the corresponding principal directions.

Solution. The principal stresses σ and the principal directions $\boldsymbol{n}$ can be solved from the eigenvalue problem

$$\begin{bmatrix} -\sigma & \tau_0 & \tau_0 \\ \tau_0 & -\sigma & \tau_0 \\ \tau_0 & \tau_0 & -\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

To have a non-trivial solution for $\boldsymbol{n}$, the determinant of the coefficient matrix has to vanish

$$\det \begin{bmatrix} -\sigma & \tau_0 & \tau_0 \\ \tau_0 & -\sigma & \tau_0 \\ \tau_0 & \tau_0 & -\sigma \end{bmatrix} = -\sigma \begin{vmatrix} -\sigma & \tau_0 \\ \tau_0 & -\sigma \end{vmatrix} - \tau_0 \begin{vmatrix} \tau_0 & \tau_0 \\ \tau_0 & -\sigma \end{vmatrix} + \tau_0 \begin{vmatrix} \tau_0 & -\sigma \\ \tau_0 & \tau_0 \end{vmatrix} = 0,$$

from which the characteristic equation is obtained

$$-\sigma^3 + 3\tau_0^2\sigma + 2\tau_0^3 = 0.$$

Since $\operatorname{tr}\boldsymbol{\sigma} = 0$ the stress matrix is purely deviatoric. The position of the stress state on the π-plane, which is the spesific deviatoric plane going through the origin of the principal stress space can be determined if the radius $\rho = \sqrt{\boldsymbol{s} : \boldsymbol{s}} = \sqrt{2J_2}$ and the Lode angle θ is known. The deviatoric invariants J_2 and J_3 have the values

$$J_2 = \tfrac{1}{2}s_{ij}s_{ji} = 3\tau_0^2, \quad J_3 = \det\boldsymbol{s} = \det\boldsymbol{\sigma} = 2\tau_0^3,$$

thus $\rho = \sqrt{2J_2} = \sqrt{6}|\tau_0|$ and the Lode angle θ can be solved from equation

$$\cos 3\theta = \frac{3\sqrt{3}J_3}{2J_2^{3/2}} = 1,$$

resulting in $\theta = 0°$. Thus the current point in the stress space is located on the deviatoric plane at distance $\sqrt{6}\tau_0$ from the origo on a line parallel to the projection of the largest principal stress axis onto the deviatoric plane, see Fig. 2.5.

The principal stresse can be obtained by using (2.38) and substituting $\theta = 0°$, resulting in $\sigma_1 = s_1 = (2/\sqrt{3})\sqrt{J_2} = 2\tau_0$. The other two principal stresses are obtained after substituting $\theta_2 = 120°$ and $\theta = -120°$, giving

$$\sigma_2 = s_2 = -\tau_0, \quad \text{and} \quad \sigma_3 = s_3 = -\tau_0.$$

It is always recommendable to check the results, since the deviator is traceless $s_1 + s_2 + s_3 = 0$, and $J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) = 3\tau_0^2$ and furhermore $J_3 = s_1 s_2 s_3 = 2\tau_0^3$.

The principal directions can be obtained when substituting the principal stresses back to the eigenvalue problem. For the case $\sigma_1 = 2\tau_0$:

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

from where $n_1 = \frac{1}{2}(n_2 + n_3)$ and $n_2 = n_3$. The direction of the normal where the principal stress $2\tau_0$ acts is 1:1:1.

Directions corresponding to the double eigenvalue $-\tau_0$ can be obtained from

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

resulting in a single equation $n_1 + n_2 + n_3 = 0$. This condition shows that the principal stress $-\tau_0$ is acting on an arbitrary plane, the normal of which is perpendicular to the direction 1:1:1.

Example 2.3. *Consider a stress state expressed by the matrix*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 + \alpha\sigma_0 \end{bmatrix}$$

where α is a dimensionless constant. Draw the stress state both in the (σ_m, ρ)-coordinate system and in the deviatoric plane as a function of the parameter $\alpha \in [-2, 2]$.

Solution: The mean stress is $\sigma_{\rm m} = \frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = (1 + \frac{1}{3}\alpha)\sigma_0$ and the deviatoric stress matrix

$$\boldsymbol{s} = \begin{bmatrix} -\frac{1}{3}\alpha\sigma_0 & 0 & 0 \\ 0 & -\frac{1}{3}\alpha\sigma_0 & 0 \\ 0 & 0 & \frac{2}{3}\alpha\sigma_0 \end{bmatrix},$$

from where the radius ρ on the deviatoric plane can be determined as $\rho = \sqrt{2/3}|\alpha\sigma_0|$. Solving σ_0 as a function of the mean stress $\sigma_{\rm m}$ and substituting the result in the expression of ρ, gives

$$\rho = \left| \sqrt{\tfrac{2}{3}} \frac{\alpha}{1 + \frac{1}{3}\alpha} \sigma_{\rm m} \right|,$$

which present lines on the $(\sigma_{\rm m}, \rho)$-plane. The slope of these lines depends on the parameter α. However, when drawing these lines in the $(\sigma_{\rm m}/\sigma_0, \rho/\sigma_0)$-coordinate system, the expressions

$$\sigma_{\rm m}/\sigma_0 = 1 + \tfrac{1}{3}\alpha, \quad \text{and} \quad \rho/\sigma_0 = \sqrt{2/3}|\alpha|,$$

is used. Fixing two points, one on the $\sigma_{\rm m}$-axis and the second on ρ-axis, gives an easy interpretation.

The Lode angle θ on the deviatoric plane is determined from

$$\cos 3\theta = \frac{3\sqrt{3}J_3}{2J_2^{3/2}}.$$

Calculating the deviatoric invariants: $J_2 = \frac{1}{3}\alpha^2\sigma_0^2$ and $J_3 = s_1 s_2 s_3 = \dfrac{2}{27}\alpha^3\sigma_0^3$, gives

$$\cos 3\theta = \frac{\alpha\sigma_0}{|\alpha\sigma_0|}.$$

Notice that J_3 has sign, but J_2 as a quadratic quantity is always positive or zero. If σ_0 is positive, then $\cos 3\theta = \pm 1$ depends on the sign of $\alpha\sigma_0$. If α and σ_0 have same sign, the Lode angle $\theta = 0$ and if α and σ_0 have different signs, the Lode angle has the value $\theta = \pi/3$.

Chapter 3
Balance equations

3.1 Balance of momentum

The Newton's second law postulate for a set of particles that the time rate of change of the total momentum equals to the sum of all the external forces acting of the set. For a continuum the mass of a body occupying a volume V is given as $\int \rho \, dV$, and the rate of change of change of the total momentum of the mass is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \boldsymbol{v} \, \mathrm{d}V,$$

where $\mathrm{d}/\mathrm{d}t$ denotes the material time derivative. The postulate of the momentum balance can be stated as[1]

$$\frac{d}{dt} \int_V \rho \boldsymbol{v} \, dV = \int_S \boldsymbol{t} \, dS + \int_V \rho \boldsymbol{b} \, dV, \tag{3.1}$$

where $\boldsymbol{t}$ is the surface traction vector and $\boldsymbol{b}$ is the body force density per unit mass. By using the Cauchy's stress theorem stating that $\boldsymbol{t} = \boldsymbol{\sigma}^T \boldsymbol{n}$ and using the Gauss divergence theorem the surface integral can be transformed to a volume integral resulting in equation[2]

$$\int_V \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \, \mathrm{d}V = \int_V \left(\nabla \cdot \boldsymbol{\sigma}^T + \rho \boldsymbol{b} \right) \, \mathrm{d}V, \tag{3.2}$$

which can be rearranged in the form

$$\int_V \left(\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \nabla \cdot \boldsymbol{\sigma}^T - \rho \boldsymbol{b} \right) \mathrm{d}V = 0. \tag{3.3}$$

[1] Also known as the balance of linear momentum.

[2] In the literature the transpose of the stress is often missing. Either (i) the meaning of the indexes of the stress tensor is defined differently (e.g. in [16]), or (ii) the divergence operator is defined in another way (e.g. in [28]).

In the index notation it has the form[3]

$$\int_V \left(\rho \frac{dv_i}{dt} - \frac{\partial \sigma_{ji}}{\partial x_j} - \rho b_i \right) \mathrm{d}V = 0. \tag{3.4}$$

Since the balance has to be satisfied in every volume of the material body, the integrand of (3.4) has to be zero and the local form of the momentum balance can be written as

$$\rho \frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i, \tag{3.5}$$

or in the coordinate free notation

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla \cdot \boldsymbol{\sigma}^T + \rho \boldsymbol{b}. \tag{3.6}$$

It should be noted that the form (3.6) of the equations of motion is valid in any coordinate system while the index form in eq. (3.5) is expressed in rectangular cartesian coordinate system.

In the case of static equilibrium the acceleration $d\boldsymbol{v}/dt$ is zero, the equations of motion simplifies to the form

$$-\frac{\partial \sigma_{ji}}{\partial x_j} = \rho b_i, \quad \text{or in coordinate free notation} \quad -\nabla \cdot \boldsymbol{\sigma}^T = \rho \boldsymbol{b}. \tag{3.7}$$

These three equations do not contain any kinematical variables, however, they do not in general suffice to determine the stress distribution; it is a statically indeterminate problem except some special cases.

3.2 Balance of moment of momentum

In the absense of distributed couples the postulate of the balance of moment of momentum is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V (\boldsymbol{r} \times \rho \boldsymbol{v}) \,\mathrm{d}V = \int_S \boldsymbol{r} \times \boldsymbol{t} \,\mathrm{d}S + \int_V (\boldsymbol{r} \times \rho \boldsymbol{b}) \,\mathrm{d}V, \tag{3.8}$$

or in indicial notation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \epsilon_{ijk} x_j \rho v_k \,\mathrm{d}V = \int_S \epsilon_{ijk} x_j t_k \,\mathrm{d}S + \int_V \epsilon_{ijk} x_j \rho b_k \,\mathrm{d}V. \tag{3.9}$$

As in the case of the momentum balance, transforming the surface integral to a a volume integral results in equations

$$\int_V \epsilon_{ijk} \frac{\mathrm{d}}{\mathrm{d}t} (x_j v_k) \rho \,\mathrm{d}V = \int_V \epsilon_{ijk} \left[\frac{\partial (x_j \sigma_{nk})}{\partial x_n} + x_j \rho b_k \right] \mathrm{d}V. \tag{3.10}$$

[3] Equations (3.3) and (3.4) are also called as Cauchy's (1827) or Euler's (∼1740) first law of motion.

Since $\mathrm{d}x_j/\mathrm{d}t = v_j$, this becomes

$$\int_V \epsilon_{ijk}\left(v_j v_k + x_j \frac{\mathrm{d}v_k}{\mathrm{d}t}\right)\rho\,\mathrm{d}V = \int_V \epsilon_{ijk}\left[x_j\left(\frac{\partial \sigma_{nk}}{\partial x_n} + \rho b_k\right) + \delta_{jm}\sigma_{mk}\right]\mathrm{d}V. \quad (3.11)$$

Due to the symmetry of $v_j v_k$ the product $\epsilon_{ijk} v_j v_k = 0$, and after rearrangements the following form is obtained

$$\int_V \epsilon_{ijk}\left[x_j\left(\rho\frac{\mathrm{d}v_k}{\mathrm{d}t} - \frac{\partial \sigma_{nk}}{\partial x_n} - \rho b_k\right) + \sigma_{jk}\right]\mathrm{d}V = 0. \quad (3.12)$$

Since the term in the parenthesis vanishes, resulting in equations

$$\int_V \epsilon_{ijk}\sigma_{jk}\,\mathrm{d}V = 0, \quad (3.13)$$

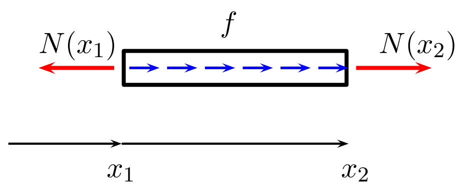
which have to be valid for every volume

$$\epsilon_{ijk}\sigma_{jk} = 0, \quad \text{i.e.} \quad \begin{cases} \sigma_{23} - \sigma_{32} = 0, & \text{for } i = 1, \\ \sigma_{31} - \sigma_{13} = 0, & \text{for } i = 2, \\ \sigma_{12} - \sigma_{21} = 0, & \text{for } i = 3, \end{cases} \quad (3.14)$$

showing the symmetry of the stress matrix $\sigma_{ij} = \sigma_{ji}$.

3.3 Solved example problems

Example 3.1. *Derive the equilibrium equations of an axially loaded bar.*



Solution. The force equilibrium in the horizontal direction is

$$N(x_2) - N(x_1) + \int_{x_1}^{x_2} f(x)\mathrm{d}x = 0,$$

which can be written as

$$\Big|_{x_1}^{x_2} N(x) + \int_{x_1}^{x_2} f\,\mathrm{d}x = 0,$$

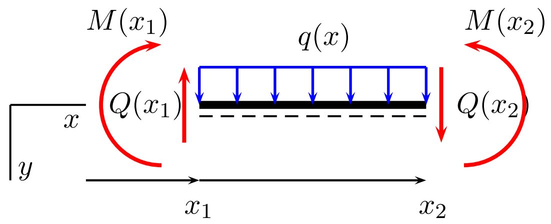
and furthermore

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}N}{\mathrm{d}x} + f \right) \mathrm{d}x = 0.$$

Since the values x_1 and x_2 are arbitrary it can be deduced that

$$-\frac{\mathrm{d}N}{\mathrm{d}x} = f, \quad x \in (0, L).$$

Example 3.2. *Derive the equilibrium equations of a beam model, loaded by a vertical force intensity* $q(x)$.



Solution. The force equilibrium in the vertical direction is

$$Q(x_2) - Q(x_1) + \int_{x_1}^{x_2} q(x)\mathrm{d}x = 0,$$

which can be written as

$$\Big|_{x_1}^{x_2} Q(x) + \int_{x_1}^{x_2} q(x)\,\mathrm{d}x = 0,$$

and furthermore

$$\int_{x_1}^{x_2} \left(\frac{\mathrm{d}Q}{\mathrm{d}x} + q \right) \mathrm{d}x = 0.$$

Since the values x_1 and x_2 are arbitrary it can be deduced that

$$-\frac{\mathrm{d}Q}{\mathrm{d}x} = q, \quad x \in (0, L). \tag{3.15}$$

The moment equilibrium equation with respect to an arbitrary point x_0 is

$$M(x_1) - M(x_2) + Q(x_2)(x_2 - x_0) - Q(x_1)(x_1 - x_0) + \int_{x_1}^{x_2} q(x)(x - x_1)\,\mathrm{d}x = 0,$$

which can be written as

$$-\Big|_{x_1}^{x_2} M(x) + \Big|_{x_1}^{x_2} Q(x)(x - x_0) + \int_{x_1}^{x_2} q(x)(x - x_0)\,\mathrm{d}x.$$

Proceeding in a similar way as in the previous example gives

$$-\int_{x_1}^{x_2} \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left[Q(x)(x-x_0)\right]\,\mathrm{d}x + \int_{x_1}^{x_2} q(x)(x-x_0)\,\mathrm{d}x = 0,$$

$$-\int_{x_1}^{x_2} \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_1}^{x_2} \left(Q + (x-x_0)\frac{\mathrm{d}Q}{\mathrm{d}x}\right)\,\mathrm{d}x + \int_{x_1}^{x_2} q(x)(x-x_0)\,\mathrm{d}x = 0$$

$$\int_{x_1}^{x_2} \left(Q - \frac{\mathrm{d}M}{\mathrm{d}x}\right)\,\mathrm{d}x + \int_{x_1}^{x_2} (x-x_0)\left(\frac{\mathrm{d}Q}{\mathrm{d}x} + q\right)\,\mathrm{d}x = 0.$$

Due to the vertical force equilibrium equation (3.15) the last integral vanishes and the moment equilibrium equations results in

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x}, \tag{3.16}$$

from which

$$-\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = q.$$

Example 3.3. *Determine the shear stress distribution in a cross-section for a beam with solid rectangular cross-section.*

Solution. In the Euler-Bernoulli beam model, the shear force cannot be obtained through the kinematical and constitutive equations, due to the kinematical constraint. However, the distribution of the shear stress in the cross-section can be obtained from the general equilibrium equations, which in the plane case are

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 & \text{horizontal equilibrium} \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 & \text{vertical equilibrium} \end{cases}.$$

In the Euler-Bernoulli beam model the axial strain has a linear variation aloong the cross-section height and assuming linear elastic material the normal stress σ_x also has a linear variation

$$\sigma_x = \frac{M}{I} y.$$

Assuming that the beam's cross-section is uniform in the axial direction, I = constant, it is obtained

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{M'}{I} y = \frac{Q}{I} y,$$

where the symmetry property of the stress tensor is taken into account. After integration it is obtained

$$\tau_{xy} = -\frac{Q(x)}{2I} y^2 + C,$$

where C is the integration constant. From the stress-free boundary conditions

$$\tau_{xy}(x, \pm h/2) = 0,$$

the value for C is obtained as

$$C = \frac{Qh^2}{8I}.$$

Thus

$$\tau_{xy} = \frac{Qh^2}{8I}\left[1 - 4\left(\frac{y}{h}\right)^2\right] = \frac{3Q}{2bh}\left[1 - 4\left(\frac{y}{h}\right)^2\right] = \frac{3Q}{2A}\left[1 - 4\left(\frac{y}{h}\right)^2\right].$$

The maximun shear stress is located on the neutral axis and it is 50 % higher than the average shear stress Q/A.

Chapter 4
Kinematical relations

4.1 Motion of a continuum body

Motion of a continuum body $\mathcal{B}$ embedded in a three-dimensional Euclidean space and occupying a domain Ω will be studied. Consider a point P which has an initial position $\boldsymbol{X}$ at time $t = 0$. At time $t > 0$ the body occcupies another configuration and the motion of the particle P is described by mapping

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t), \quad \text{or in index notation} \quad x_i = \chi_i(X_k, t). \tag{4.1}$$

The motion $\boldsymbol{\chi}$ is assumed to be invertible and sufficiently many times differentiable. The displacement vector is defined as

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}. \tag{4.2}$$

4.2 Deformation gradient

The most important measure of deformation in non-linear continuum mechanics is the deformation gradient, which will be introduced next. Consider a material curve Γ at the initial configuration, a position of a point on this curve is given as $\boldsymbol{X} = \Gamma(\xi)$, where ξ denotes a parametrization, see figure 4.1. Notice that the material curve does not depend on time. During the motion, the material curve deforms into curve

$$\boldsymbol{x} = \boldsymbol{\gamma}(\xi, t) = \boldsymbol{\chi}(\Gamma(\xi), t). \tag{4.3}$$

The tangent vectors of the material and deformed curves are denoted as $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$, respectively, and defined as

$$\mathrm{d}\boldsymbol{X} = \Gamma'(\xi)\mathrm{d}\xi, \tag{4.4}$$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{\gamma}'(\xi, t)\mathrm{d}\xi = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}\Gamma'(\xi)\mathrm{d}\xi = \boldsymbol{F}\mathrm{d}\boldsymbol{X}, \tag{4.5}$$

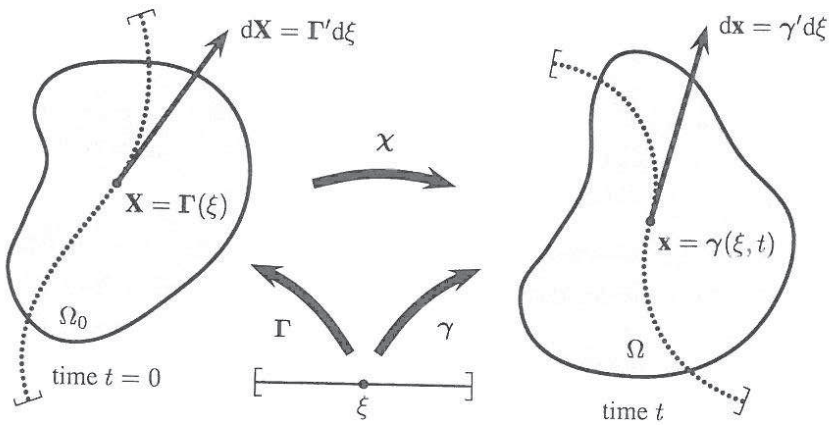


Figure 4.1: Deformation of a material curve, figure from [16, page 70].

since on the deformed curve $\boldsymbol{x} = \gamma(\xi, t) = \boldsymbol{\chi}(\Gamma(\xi), t)$. The quantity $\boldsymbol{F}$ is called the *deformation gradient* and it describes the motion in the neighbourhod of a point. It is defined as

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \quad \text{or in indicial notation} \quad F_{ij} = \frac{\partial \chi_i}{\partial X_j}. \tag{4.6}$$

The deformation gradient reduces into identity tensor $\boldsymbol{I}$ if there is no motion, or the motion is a rigid translation. However, rigid rotation will give a deformation gradient not equal to the identity.

4.3 Definition of strain tensors

Let us investigate the change of length of a line element. Denoting the length of a line element in the deformed configuration as $\mathrm{d}s$ and as $\mathrm{d}S$ in the initial configuration, thus

$$\begin{aligned} \frac{1}{2}[(\mathrm{d}s)^2 - (\mathrm{d}S)^2] &= \frac{1}{2}(\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}) = \frac{1}{2}(\boldsymbol{F}\mathrm{d}\boldsymbol{X}\cdot\boldsymbol{F}\mathrm{d}\boldsymbol{X} - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}) \\ &= \frac{1}{2}\mathrm{d}\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})\cdot\mathrm{d}\boldsymbol{X} = \mathrm{d}\boldsymbol{X}\cdot\boldsymbol{E}\mathrm{d}\boldsymbol{X}, \end{aligned} \tag{4.7}$$

where the tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) \tag{4.8}$$

is called the *Green-Lagrange strain tensor*.

Let us express the Green-Lagrange strain in terms of displacement vector $\boldsymbol{u}$. It is first observed that the deformation gradient takes the form

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}, \tag{4.9}$$

where the tensor $\partial \boldsymbol{u}/\partial \boldsymbol{X}$ is called the *displacement gradient*. Thus, the Green-Lagrange strain tensor takes the form

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}\left[\left(\boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T \left(\boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) - \boldsymbol{I}\right] \\ &= \frac{1}{2}\left[\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)\right], \end{aligned} \tag{4.10}$$

or in index notation

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right). \tag{4.11}$$

If the elements of the displacement gradient are small in comparison to unity, i.e.

$$\frac{\partial u_i}{\partial X_j} \ll 1, \tag{4.12}$$

then the quadratic terms can be neglected and the infinitesimal strain tensor can be defined as the symmetric part of the displacement gradient

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \approx E_{ij}. \tag{4.13}$$

Let us define a stretch vector $\boldsymbol{\lambda}$ in the direction of a unit vector $\boldsymbol{n}_0$ as

$$\boldsymbol{\lambda} = \boldsymbol{F}\boldsymbol{n}_0, \tag{4.14}$$

and the length of the stretch vector $\lambda = |\boldsymbol{\lambda}|$ is called the stretch ratio or simply the stretch. The square of the stretch ratio is

$$\lambda^2 = \boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = \boldsymbol{n}_0\cdot\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{n}_0 = \boldsymbol{n}_0\cdot\boldsymbol{C}\boldsymbol{n}_0, \tag{4.15}$$

where the tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ is called the *right Cauchy-Green* strain tensor. The attribute *right* comes from the fact that the deformation gradient operates on the right hand side. The right Cauchy-Green strain tensor is symmetric and positive definite tensor, i.e. $\boldsymbol{C} = \boldsymbol{C}^T$ and $\boldsymbol{n} \cdot \boldsymbol{C}\boldsymbol{n} > 0,\ \forall \boldsymbol{n} \neq 0$.

For values $0 < \lambda < 1$, a line element is compressed and elongated for values $\lambda > 1$.

The deformation gradient can also be decomposed multiplicatively as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}, \tag{4.16}$$

where $\boldsymbol{R}$ is an othogonal tensor ($\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$) describing the rotation of a material element and $\boldsymbol{U}$ and $\boldsymbol{V}$ are symmetric positive definite tensors describing the deformation. The decomposition (4.16) is also called the *polar decomposition*. The tensor $\boldsymbol{U}$ is called as the *right stretch tensor* and $\boldsymbol{V}$ the *left stretch tensor*.

The square of the stretch can be expressed as

$$\lambda^2 = \boldsymbol{\lambda}\cdot\boldsymbol{\lambda} = \boldsymbol{n}_0\cdot\boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U}\boldsymbol{n}_0 = \boldsymbol{n}_0\cdot\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{n}_0 = \boldsymbol{n}_0\cdot\boldsymbol{U}^2\boldsymbol{n}_0. \tag{4.17}$$

Other strain measures can be defined as

$$\boldsymbol{E}^{(m)} = \frac{1}{m}(\boldsymbol{U}^m - \boldsymbol{I}). \tag{4.18}$$

For $m = 2$, we obtain the Green-Lagrange strain tensor which have already been discussed. With $m = 0$ we obtain the Hencky or logarithmic strain tensor

$$\boldsymbol{E}^{(0)} = \ln \boldsymbol{U}. \tag{4.19}$$

The logarithmic strain[1] has a special position in non-linear continuum mechanics, especially in formulating constitutive equations, since it can be additively decomposed into volumetric and isochoric parts similarly as the small strain tensor $\boldsymbol{\varepsilon}$.

For $m = 1$, we obtain

$$\boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}, \tag{4.20}$$

which is called the *Biot strain tensor*. If the deformation is rotation free, i.e. $\boldsymbol{R} = \boldsymbol{I}$, the Biot strain tensor coincides with the small strain tensor $\boldsymbol{\varepsilon}$. It is much used in dimensionally reduced continuum models, such as beams, plates and shells.

4.4 Geometric intepretation of the strain components

Let us investigate the extension $\varepsilon = \lambda - 1$ of a line element, for instance in a direction $\boldsymbol{n}_0 = (1, 0, 0)^T$, thus

$$\lambda_{(1)} = \sqrt{C_{11}}, \quad E_{11} = \tfrac{1}{2}(C_{11} - 1) \quad \Rightarrow C_{11} = 1 + 2E_{11}$$
$$\Rightarrow \lambda = \sqrt{1 + 2E_{11}} \quad \Rightarrow \varepsilon = \sqrt{1 + 2E_{11}} - 1 \tag{4.21}$$

[1]The logarithmic strain is sometimes called also as the true strain. Such naming is not used in this text, all properly defined strain measures are applicable, since the definition of strain is a geometrical construction. Naturally, the choice of strain measure dictates the choise of the stress. However, deeper discussion on this topic is beyond the present lecture notes.

Secondly, let us compute the angle change of two unit vectors $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$. In the deformed configuration they are $\boldsymbol{n}_1 = \boldsymbol{F}\boldsymbol{N}_1$ and $\boldsymbol{n}_1 = \boldsymbol{F}\boldsymbol{N}_2$ and the angle between them can be determined from

$$\cos\theta_{12} = \frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|} = \frac{\boldsymbol{N}_1\cdot\boldsymbol{C}\boldsymbol{N}_2}{\sqrt{\boldsymbol{N}_1\cdot\boldsymbol{C}\boldsymbol{N}_1}\sqrt{\boldsymbol{N}_2\cdot\boldsymbol{C}\boldsymbol{N}_2}}. \tag{4.22}$$

If we choose the directions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ as

$$\boldsymbol{N}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{N}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{4.23}$$

then

$$\cos\theta_{12} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}} = \frac{C_{12}}{\lambda_{(1)}\lambda_{(2)}} = \frac{2E_{12}}{\sqrt{(1+2E_{11})(1+2E_{22})}}. \tag{4.24}$$

Using the trigonometric identity

$$\sin(\tfrac{1}{2}\pi - \theta_{12}) = \cos\theta_{12} \tag{4.25}$$

and if $E_{11}, E_{22} \ll 1$ then

$$\tfrac{1}{2}\pi - \theta_{12} \approx 2E_{12}. \tag{4.26}$$

Thus, the component E_{12} is approximately one half of the angle change of the two direction vectors.

4.5 Definition of the infinitesimal strain

Let us investigate the motion of two neighbouring points, which are denoted as $P\,Q$ in the undeformed configuration. After deformation these points occupy the positions marked by p and q. Displacement of the point Q relative to P is defined as, see fig. 4.2,

$$\mathrm{d}\boldsymbol{u} = \boldsymbol{u}_Q - \boldsymbol{u}_P. \tag{4.27}$$

Length of the vector $\vec{PQ}$ is denoted as $\mathrm{d}S$, thus

$$\frac{\mathrm{d}u_i}{\mathrm{d}S} = \frac{\partial u_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}S}, \tag{4.28}$$

where the Jacobian matrix $\boldsymbol{J} = \partial\boldsymbol{u}/\partial\boldsymbol{x}$ can be divided additively into a symmetric and an antisymmetric part as

$$\boldsymbol{J} = \boldsymbol{\varepsilon} + \boldsymbol{\Omega}, \tag{4.29}$$

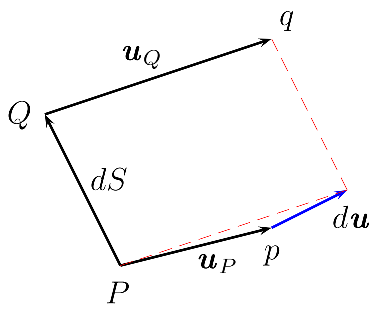


Figure 4.2: Relative displacement $d\boldsymbol{u}$ of Q relative to P.

where the symmetric part $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{z} \end{bmatrix}, \tag{4.30}$$

and the antisymmetric part $\boldsymbol{\Omega}$ is the infinitesimal rotation tensor

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix}. \tag{4.31}$$

Written in the displacement components, these tensor have the expressions

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{and} \quad \Omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{4.32}$$

The infinitesimal rotation matrix is a skew matrix and when operating with a vector the following relation holds

$$\boldsymbol{\Omega}\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{a}, \tag{4.33}$$

where $\boldsymbol{a}$ is an arbitrary vector and $\boldsymbol{\omega}$ is the vector

$$\boldsymbol{\omega} = -\Omega_{23}\boldsymbol{e}_1 - \Omega_{31}\boldsymbol{e}_2 - \Omega_{12}\boldsymbol{e}_3, \quad \text{or} \quad \omega_i = -\tfrac{1}{2}\epsilon_{ijk}\Omega_{jk}. \tag{4.34}$$

Expressed in terms of the displacement vector $\boldsymbol{u}$ the infinitesimal rotation vector $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} = \tfrac{1}{2}\nabla \times \boldsymbol{u}. \tag{4.35}$$

It should be emphasised that the rotation matrix Ω near the point P describes the rigid body rotation only if the elements Ω_{ij} are small.

4.5.1 Principal strains

The principal strains ε are obtained from the linear eigenvalue problem

$$(\varepsilon_{ij} - \varepsilon\delta_{ij})n_j = 0, \tag{4.36}$$

where the vector n_i defines the normal direction of the principal strain plane. Thus, the characteristic polynomial has the form

$$-\varepsilon^3 + I_1\varepsilon^2 + I_2\varepsilon + I_3 = 0, \tag{4.37}$$

where the strain invariants $I_i, i = 1, \dots, 3$ are

$$I_1 = \mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}, \tag{4.38}$$

$$I_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\varepsilon}^2) - (\mathrm{tr}\boldsymbol{\varepsilon})^2] = \tfrac{1}{2}(\varepsilon_{ij}\varepsilon_{ji} - I_1^2), \tag{4.39}$$

$$I_3 = \det(\boldsymbol{\varepsilon}). \tag{4.40}$$

If the coordinate axes are chosen to coincide with the axes of principal strains, the strain matrix will be a diagonal matrix

$$\boldsymbol{\varepsilon} = [\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{4.41}$$

The invariants $I_1, \dots, I_3$ expressed in terms of the principal strains $\epsilon_1, \dots, \epsilon_3$ have the forms

$$I_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \tag{4.42}$$

$$I_2 = -\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 - \varepsilon_3\varepsilon_1, \tag{4.43}$$

$$I_3 = \varepsilon_1\varepsilon_2\varepsilon_3. \tag{4.44}$$

4.5.2 Deviatoric strain

As in the case of the stress tensor, the infinitesimal strain tensor can be additively decomposed into a deviatoric part and an isotropic part as

$$\varepsilon_{ij} = e_{ij} + \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}, \tag{4.45}$$

where the deviatoric strain tensor is denoted as $\boldsymbol{e}$. In the literature the notation $\boldsymbol{\epsilon}'$ is also used. By definition the, deviatoric strain tensor is traceless

$$\mathrm{tr}\boldsymbol{e} = 0. \tag{4.46}$$

The eigenvalues of the deviatoric strain e_i can be solved from the equation

$$|e_{ij} - e\delta_{ij}| = 0, \tag{4.47}$$

and the characteristic equation is

$$-e^3 + J_1 e^2 + J_2 e + J_3 = 0, \tag{4.48}$$

where the invariants $J_1, \dots, J_3$ have expressions

$$J_1 = \mathrm{tr}\,\boldsymbol{e} = e_{ii} = e_{11} + e_{22} + e_{33} = 0, \tag{4.49}$$
$$J_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{e}^2) - (\mathrm{tr}\,\boldsymbol{e})^2] = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{e}^2) = \tfrac{1}{2}e_{ij}e_{ji} \tag{4.50}$$
$$= \tfrac{1}{6}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2], \tag{4.51}$$
$$J_3 = \det \boldsymbol{e} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{e}^3)\tfrac{1}{3}(e_1^3 + e_2^3 + e_3^3) = e_1 e_2 e_3. \tag{4.52}$$

For small strains the first invariant $I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \equiv \varepsilon_{\mathrm{v}}$ describes the relative volume change.

The octahedral strains are defined similarly as for the stress

$$\varepsilon_{\mathrm{oct}} = \tfrac{1}{3}I_1 = \tfrac{1}{3}\varepsilon_{\mathrm{v}}, \tag{4.53}$$
$$\gamma_{\mathrm{oct}}^2 = \tfrac{8}{3}J_2. \tag{4.54}$$

For the first sight, the equation (4.54) might look strange as compared to the expression of the octahedral stress, but we have to remember that $\gamma_{xy} = 2\epsilon_{xy}$, etc.

4.6 Solved example problems

Example 4.1. *The following equations define the deformation state of the body:*

1. $x_1 = X_1, \quad x_2 = X_2 + \alpha X_1, \quad x_3 = X_3,$
2. $x_1 = \sqrt{2\alpha X_1 + \beta}, \quad x_2 = \gamma X_2, \quad x_3 = \delta X_3,$
3. $x_1 = X_1\cos(\alpha X_3) + X_2\sin(\alpha X_3), \quad x_2 = -X_1\sin(\alpha X_3) + X_2\cos(\alpha X_3),$ $x_3 = (1 + \alpha\beta)X_3.$

Determine the deformation gradient $\boldsymbol{F}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$. In addition determine also the small strain and rotation tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\Omega}$, respectively.

Solution. The deformation gradient expressed in terms of the displacement gradient is

$$\boldsymbol{F} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}},$$

or using the index notation

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}.$$

Case a. Let's determine first the displacements and the displacement gradient

$$\begin{aligned} u_1 &= x_1 - X_1 = 0, \\ u_2 &= x_2 - X_2 = \alpha X_1, \\ u_3 &= x_3 - X_3 = 0. \end{aligned}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The deformation gradient is

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Green-Lagrange strain tensor is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\begin{bmatrix} \alpha^2 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The small strain tensor is

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \frac{1}{2}\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the infinitesimal rotation tensor is

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} - \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \frac{1}{2}\begin{bmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Draw the deformation state in (X_1, X_2)-plane. What kind of deformation it is?

Case b. The dislacement vector has components

$$\begin{aligned} u_1 &= x_1 - X_1 = \sqrt{2\alpha X_1 + \beta} - X_1, \\ u_2 &= x_2 - X_2 = (\gamma - 1)X_2, \\ u_3 &= x_3 - X_3 = (\delta - 1)X_3. \end{aligned}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} = \begin{bmatrix} \alpha/\sqrt{2\alpha X_1 + \beta} - 1 & 0 & 0 \\ 0 & \gamma - 1 & 0 \\ 0 & 0 & \delta - 1 \end{bmatrix}$$

The deformation gradient is

$$\boldsymbol{F} = \begin{bmatrix} \alpha/\sqrt{2\alpha X_1 + \beta} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \delta \end{bmatrix}.$$

The Green-Lagrange strain tensor is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\begin{bmatrix} \alpha^2/(2\alpha X_1 + \beta) - 1 & 0 & 0 \\ 0 & \gamma^2 - 1 & 0 \\ 0 & 0 & \delta^2 - 1 \end{bmatrix}.$$

The small strain tensor is

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \begin{bmatrix} \alpha/\sqrt{2\alpha X_1 + \beta} - 1 & 0 & 0 \\ 0 & \gamma - 1 & 0 \\ 0 & 0 & \delta - 1 \end{bmatrix},$$

and the infinitesimal rotation tensor is

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} - \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

What kind of deformation it is? For the small strain assumption to be valid what are the restrictions should be imposed to the constants α, β, γ and δ?

Case c. The displacement vector is

$$\begin{aligned} u_1 &= X_1(\cos(\alpha X_3) - 1) + X_2 \sin(\alpha X_3), \\ u_2 &= -X_1 \sin(\alpha X_3) + X_2(\cos(\alpha X_3) - 1), \\ u_3 &= \alpha\beta X_3. \end{aligned}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha X_3) - 1 & \sin(\alpha X_3) & -\alpha X_1 \sin(\alpha X_3) + \alpha X_2 \cos(\alpha X_3) \\ -\sin(\alpha X_3) & \cos(\alpha X_3) - 1 & -\alpha X_1 \cos(\alpha X_3) - \alpha X_2 \sin(\alpha X_3) \\ 0 & 0 & \alpha\beta \end{bmatrix}$$

The deformation gradient is

$$\boldsymbol{F} = \begin{bmatrix} \cos(\alpha X_3) & \sin(\alpha X_3) & -\alpha X_1 \sin(\alpha X_3) + \alpha X_2 \cos(\alpha X_3) \\ -\sin(\alpha X_3) & \cos(\alpha X_3) & -\alpha X_1 \cos(\alpha X_3) - \alpha X_2 \sin(\alpha X_3) \\ 0 & 0 & 1 + \alpha\beta \end{bmatrix}.$$

The Green-Lagrange strain tensor is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & \alpha X_2 \\ 0 & 0 & -\alpha X_1 \\ \alpha X_2 & -\alpha X_1 & \alpha^2(X_1^2 + X_2 + \beta^2) + 2\alpha\beta \end{bmatrix}.$$

What kind of deformation state it is?

If we assume small displacements and strains then we have to assume that the angle α is small as well as the parameter β. Therefore $\sin(\alpha X_3) \approx \alpha X_3$ and $\cos(\alpha X_3) \approx 1$. Neglecting the quadratic terms, the displacement gradient is thus

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} 0 & \alpha X_3 & \alpha X_2 \\ -\alpha X_3 & 0 & -\alpha X_1 \\ 0 & 0 & \alpha\beta \end{bmatrix}.$$

and the infinitesimal strain tensor is

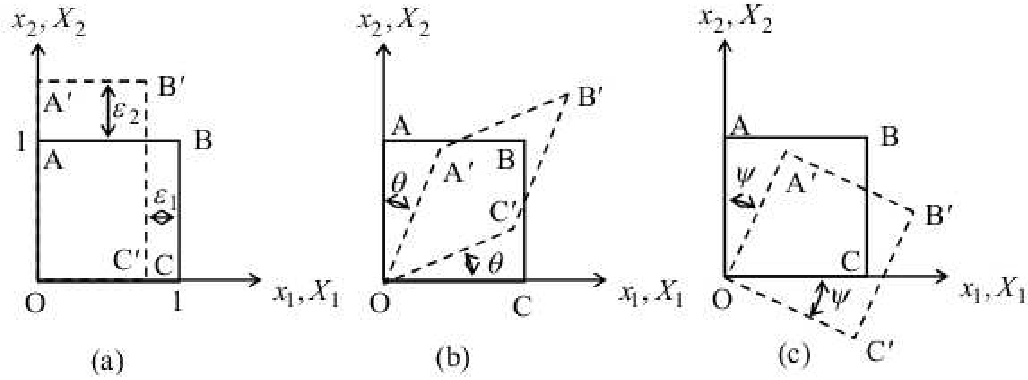
$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & \alpha X_2 \\ 0 & 0 & -\alpha X_1 \\ \alpha X_2 & -\alpha X_1 & 2\alpha\beta \end{bmatrix}.,$$

and the infinitesimal rotation tensor is

$$\boldsymbol{\Omega} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} - \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^{\mathrm{T}}\right) = \frac{1}{2}\begin{bmatrix} 0 & 2\alpha X_3 & \alpha X_2 \\ -2\alpha X_3 & 0 & -\alpha X_1 \\ -\alpha X_2 & \alpha X_1 & 0 \end{bmatrix}.$$

Draw the deformation state in (X_2, X_3)-plane. What kind of deformation it is?

Example 4.2. *A unit square OABC deforms to a quadrilateral shape OA'B'C' with the three forms shown below. Write down in each case the displacement fields* u_1, u_2 *as a function of material coordinates, i.e. the coordinates describing the material point in the undeformed configuration* (X_1, X_2)*. Further determine the deformation gradient* $\mathbf{F}$ *and the Green-Lagrange strain tensor* $\mathbf{E}$*. Determine also the infinitesimal strain tensor used in linear theory* $\boldsymbol{\varepsilon}$ *and the rotation tensor* $\boldsymbol{\Omega}$*.*



Solution. The deformation state is homogeneous, thus the displacement field can be determined as

$$u_i(X_1, X_2) = a_i + b_i X_1 + c_i X_2,$$

where a_i, b_i and c_i are constants. We can determine the coefficients using three points.

In the (a) case:

$$\begin{aligned}
x_1(0,0) &= a_1 = 0,\\
x_1(1,0) &= a_1 + b_1 = 1-\varepsilon_1 \quad \Rightarrow \quad b_1 = 1-\varepsilon_1\\
x_1(0,1) &= c_1 = 0,\\
x_2(0,0) &= a_2 = 0,\\
x_2(1,0) &= b_2 = 0,\\
x_2(0,1) &= c_2 = 1+\varepsilon_2,
\end{aligned}$$

thus $x_1 = (1-\varepsilon_1)X_1$, and $x_2 = (1+\varepsilon_2)X_2$ and the displacement field is

$$\begin{aligned}
u_1 &= x_1 - X_1 = -\varepsilon_1 X_1,\\
u_2 &= x_2 - X_2 = \varepsilon_2 X_2.
\end{aligned}$$

The deformation gradient is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = \begin{bmatrix} 1-\varepsilon_1 & 0 \\ 0 & 1+\varepsilon_2 \end{bmatrix},$$

and the Green-Lagrange strain tensor is

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) = \begin{bmatrix} -\varepsilon_1 + \frac{1}{2}\varepsilon_1^2 & 0 \\ 0 & \varepsilon_2 + \frac{1}{2}\varepsilon_2^2 \end{bmatrix},$$

The small strain matrix is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix},$$

and the infinitesimal rotation matrix is a zero matrix.

In the (b) case:

$$\begin{aligned}
x_1(0,0) &= a_1 = 0,\\
x_1(1,0) &= a_1 + b_1 = \cos\theta \quad \Rightarrow \quad b_1 = \cos\theta\\
x_1(0,1) &= c_1 = \sin\theta,\\
x_2(0,0) &= a_2 = 0,\\
x_2(1,0) &= b_2 = \sin\theta,\\
x_2(0,1) &= c_2 = \cos\theta.
\end{aligned}$$

and so on. The result for the Green-Larrange strain tensor is

$$\boldsymbol{E} = \begin{bmatrix} 0 & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & 0 \end{bmatrix}.$$

In order to be consistent with the small displacements and strain hypothesis, the angle θ should be small, thus $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Then the small strain, small displacement strain and rotation matrices follow.

In the (c) case:

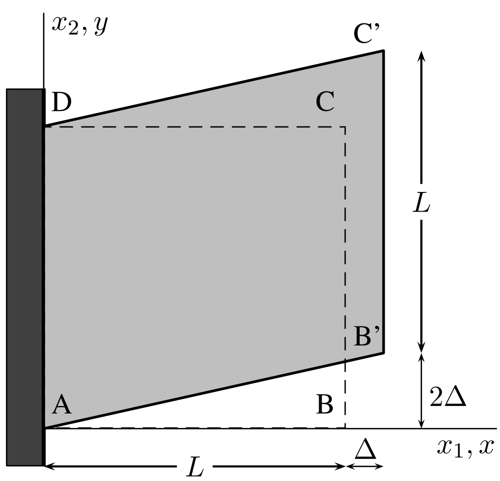
$$\begin{aligned}
&x_1(0,0) = a_1 = 0,\\
&x_1(1,0) = a_1 + b_1 = \cos\psi \quad \Rightarrow \quad b_1 = \cos\psi\\
&x_1(0,1) = c_1 = \sin\psi,\\
&x_2(0,0) = a_2 = 0,\\
&x_2(1,0) = b_2 = -\sin\psi,\\
&x_2(0,1) = c_2 = \cos\psi.
\end{aligned}$$

and so on. The result for the Green-Lagrange strain tensor is

$$\boldsymbol{E} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the motion is pure rigid body motion.

Example 4.3. *A square plate ABCD with a side length L as shown below deform to the state AB'C'D. Determine the deformation gradient $\boldsymbol{F}$, the Green-Lagrange strain tensor $\boldsymbol{E}$ and the infinitesimal strain tensor ε. Determine also the deformed length AC' of the diagonal by using these three deformation measures.*



Solution. The displacement field is then $u_1(X_1, X_2) = \Delta(X_1/L)$ and $u_2(X_1, X_2) = 2\Delta(X_1/L)$. Deformation gradient is $F_{ij} = \delta_{ij} + \partial u_i/\partial X_j$, thus

$$\boldsymbol{F} = \begin{pmatrix} 1+\Delta/L & 0 \\ 2\Delta/L & 1 \end{pmatrix}.$$

The Green-Lagrange strain tensor is $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\cdot\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$:

$$\boldsymbol{C} = \begin{pmatrix} (1+\Delta/L)^2 + 4(\Delta/L)^2 & 2\Delta/L \\ 2\Delta/L & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{E} = \begin{pmatrix} \Delta/L + \frac{5}{2}(\Delta/L)^2 & \Delta/L \\ \Delta/L & 0 \end{pmatrix}$$

The infinitesimal strain tensor, i.e. the engineering strain tensor is $\varepsilon_{ij} = \frac{1}{2}(\partial u_i/\partial X_j + \partial u_j/\partial X_i)$

$$\boldsymbol{\varepsilon} = \frac{\Delta}{L}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is noticed that the engineering strain $\boldsymbol{\varepsilon}$ is a good approximation of the Green-Lagrange strain $\boldsymbol{E}$ if the displacements are small, i.e. $\Delta/L \ll 1$.

Denoting the vector defining the undeformed diagonal AC as $\boldsymbol{a}$ and the deformed diagonal as $\boldsymbol{a}'$, thus $\boldsymbol{a} = L(\boldsymbol{e}_1 + \boldsymbol{e}_2)$ and

$$\boldsymbol{a}' = \boldsymbol{F}\cdot\boldsymbol{a} = \begin{pmatrix} 1+\Delta/L \\ 1+2\Delta/L \end{pmatrix} L = \begin{pmatrix} L+\Delta \\ L+2\Delta \end{pmatrix}.$$

The length of $\boldsymbol{a}'$ is

$$|\boldsymbol{a}'| = \sqrt{(L+\Delta)^2 + (L+2\Delta)^2} = \sqrt{1 + 3\Delta/L + \tfrac{5}{2}(\Delta/L)^2}\sqrt{2}L$$

Since the deformation is homogeneous and the diagonal is straight, the deformed length of the diagonal can be computed directly using the definition of the Green-Lagrange strain (4.7):

$$|\boldsymbol{a}'|^2 - |\boldsymbol{a}|^2 = 2\boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{a},$$

thus

$$|\boldsymbol{a}'|^2 = |\boldsymbol{a}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{a} = 3L^2 + 2(3\Delta L + \tfrac{5}{2}\Delta^2) = 2L^2(1 + 3(\Delta/L) + \tfrac{5}{2}(\Delta/L)^2)$$

and

$$|\boldsymbol{a}'| = \sqrt{1 + 3(\Delta/L) + \tfrac{5}{2}(\Delta/L)^2}\sqrt{2}L \qquad (4.55)$$

Naturally the same result is obtained as with the deformation gradient.

The deformed length computed from the linear strain measure $\boldsymbol{\varepsilon}$ is

$$|\boldsymbol{a}'| = (1 + \varepsilon_a)|\boldsymbol{a}|,$$

where ε_a is the strain in the direction of $\boldsymbol{a}$

$$\varepsilon_a = \boldsymbol{n}_a\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{n}_a$$

and $\boldsymbol{n}_a$ is the unit vector in the direction of $\boldsymbol{a}$. Thus the deformed length of the diagonal is

$$|\boldsymbol{a}'| = (1 + \tfrac{3}{2}\Delta/L)\sqrt{2}L. \tag{4.56}$$

Remembering the series expansion of $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ and applying it in (4.55) gives

$$|\boldsymbol{a}'| = (1 + \tfrac{3}{2}\Delta/L + \tfrac{1}{8}(\Delta/L)^2 + \cdots)\sqrt{2}L.$$

If $\Delta/L \ll 1$ then the engineering strain is a good approximation of the Green-Lagrange strain.

Chapter 5
Elastic constitutive models

5.1 Introduction

Elastic behaviour is due to reversible movements of atoms, molecules or cells and thus elastic response of a material is independent of the load history. For many materials like metals, composites, concrete and wood the elastic strains are limited up to strains $0.2 - 0.5\%$ [26].

The most general form of elasticity is called as *Cauchy-elasticity* and it essentially means that there exists one-to-one relation between stress and strain

$$\sigma_{ij} = f_{ij}(\varepsilon_{kl}), \quad \text{or} \quad \varepsilon_{ij} = g_{ij}(\sigma_{kl}). \tag{5.1}$$

The tensor valued tensor functions f_{ij} and g_{ij} are called as response functions. For non-linear Cauchy-elastic models, the loading-unloading process may yield hysteresis, which is incompatible with the notion of elasticity, where the response should be reversible. For more detailed discussion of Cauchy elasticity, see [31]. In this lecture notes Cauchy-elasticity is not treated.

Another form of elasticity, where the constitutive equations are expressed in rate-form

$$\dot{\sigma}_{ij} = f_{ij}(\sigma_{kl}, \dot{\varepsilon}_{mn}) \tag{5.2}$$

is called *hypo-elastic*. If the material is incrementally linear, it can be written in the form

$$\dot{\sigma}_{ij} = C_{ijkl}(\sigma_{mn})\dot{\varepsilon}_{kl}. \tag{5.3}$$

The most rigorous form of elasticity is called as *hyper-elasticity*, and the constitutive equations of a hyper-elastic model can be derived from a potential, i.e. the strain energy function $W = W(\varepsilon_{ij})$ as

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}. \tag{5.4}$$

Alternatively, the hyperelastic constitutive models can be derived from a complementary function, depending on stress, such that

$$\varepsilon_{ij} = \frac{\partial W^{\mathrm{c}}}{\partial \sigma_{ij}}. \tag{5.5}$$

These two potentials W and W^{c} are related with each other by the *Legendre-Fenchel transformation*

$$W^{\mathrm{c}} = \sigma_{ij}\varepsilon_{ij} - W. \tag{5.6}$$

Symmetry is an important aspect when modelling material behavior. As shown by Chadwick et al. [7] there are eight possible linear elastic symmetries: isotropic symmetry and seven anisotropic symmetries, triclinic, monoclinic, trigonal, orthotropic, hexagonal (transversely isotropic), tetragonal, and cubic [9]. In this lecture notes only isotropic and the most common anisotropic i.e. the transversely isotropic and orthotropic symmetries are described.

5.2 Isotropic elasticity

A material which behaviour is independent of the direction in which the response is measured is called *isotropic*. Therefore also the strain energy density should be an isotropic tensor valued scalar function

$$W = W(\boldsymbol{\varepsilon}) = W(\boldsymbol{\varepsilon}') = W(\boldsymbol{\beta}\boldsymbol{\varepsilon}\boldsymbol{\beta}^T) = W(I_1, I_2, I_3), \tag{5.7}$$

where I_1, I_2 and I_3 are the principal invariants of the strain tensor and $\boldsymbol{\beta}$ is the transformation tensor from the $\boldsymbol{x}$-coordinate system to the $\boldsymbol{x}'$-system, i.e. $\boldsymbol{x}' = \boldsymbol{\beta}\boldsymbol{x}$. Alternatively the strain energy density function W can be written as

$$W = W(I_1, J_2, J_3), \quad \text{or} \quad W = W(I_1, \tilde{I}_2, \tilde{I}_3), \tag{5.8}$$

where J_2 and J_3 are the invariants of the deviatoric strain tensor and $\tilde{I}_2, \tilde{I}_3$ the generic invariants defined as

$$\tilde{I}_2 = \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\varepsilon}^2), \quad \tilde{I}_3 = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon}^3). \tag{5.9}$$

Equations (5.7) and (5.8) are special forms of *representation theorems*, for which an alternative form can be written as: the most general form of an isotropic elastic material model can be written as

$$\boldsymbol{\sigma} = a_0\boldsymbol{I} + a_1\boldsymbol{\varepsilon} + a_2\boldsymbol{\varepsilon}^2, \tag{5.10}$$

where the coefficients a_0, a_1 and a_2 can be non-linear functions of the strain invariants. Proof for the representation theorem (5.10) can be found e.g. in ref. [40, Appendix].

Al alternative form to (5.10) can be formulated using the complementary potential resulting in

$$\boldsymbol{\varepsilon} = b_0\boldsymbol{I} + b_1\boldsymbol{\sigma} + b_2\boldsymbol{\sigma}^2, \tag{5.11}$$

where b_0, b_1 and b_2 can be non-linear functions of *stress* invariants. In many cases this form gives more illustrative description of physically relevant constitutive parameters.

From (5.10) and (5.11) it can be easily seen that the principal directions of the strain- and stress tensors coincide for an isotropic elastic material.

For a linear isortropic elastic material the constitutive equation (5.10) reduces to

$$\boldsymbol{\sigma} = a_0 \boldsymbol{I} + a_1 \boldsymbol{\varepsilon}, \tag{5.12}$$

where a_1 has to be a constant and the scalar a_0 can depend only linearly on strain, i.e. $a_0 = \lambda I_1 = \lambda \operatorname{tr}(\boldsymbol{\varepsilon})$, thus

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}) \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}, \quad \text{or} \quad \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{5.13}$$

where λ, μ are the Lamé constants, and μ equals to the shear modulus, i.e. $\mu = G$. To relate the Lamé's constants to the modulus of elasticity E and the Poisson's ratio ν, it is useful to invert equation (5.13) as follows. First, solve the volume change ε_{kk}

$$\sigma_{ii} = 3\lambda \varepsilon_{kk} + 2\mu \varepsilon_{ii} = (3\lambda + 2\mu)\varepsilon_{jj} \quad \Rightarrow \quad \varepsilon_{kk} = \frac{1}{3\lambda + 2\mu} \sigma_{kk}, \tag{5.14}$$

and substituting it back to (5.13) gives

$$\varepsilon_{ij} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{kk} + \frac{1}{2\mu} \sigma_{ij}. \tag{5.15}$$

Writing equations (5.15) componentwise

$$\varepsilon_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \sigma_{11} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} (\sigma_{22} + \sigma_{33}) = \frac{1}{E} \sigma_{11} - \frac{\nu}{E} (\sigma_{22} + \sigma_{22}), \tag{5.16}$$

$$\varepsilon_{22} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \sigma_{22} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} (\sigma_{11} + \sigma_{33}) = \frac{1}{E} \sigma_{22} - \frac{\nu}{E} (\sigma_{11} + \sigma_{33}), \tag{5.17}$$

$$\varepsilon_{33} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} \sigma_{33} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} (\sigma_{11} + \sigma_{22}) = \frac{1}{E} \sigma_{33} - \frac{\nu}{E} (\sigma_{11} + \sigma_{22}), \tag{5.18}$$

$$\varepsilon_{12} = \frac{1}{2\mu} \sigma_{12} = \frac{1}{2G} \sigma_{12}, \tag{5.19}$$

$$\varepsilon_{23} = \frac{1}{2\mu} \sigma_{23} = \frac{1}{2G} \sigma_{23}, \tag{5.20}$$

$$\varepsilon_{31} = \frac{1}{2\mu} \sigma_{31} = \frac{1}{2G} \sigma_{31}. \tag{5.21}$$

From (5.16)-(5.21) it can be seen that $\mu = G$ and $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$. Also the physical meaning of the Posson's ratio is clear from eqs. (5.16)-(5.18). If, for example, the body is under uniaxial stress in the x_1-direction, the Poisson's ratio is expressed as

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}. \tag{5.22}$$

If the decomposition of strain into volumetric and deviatoric parts is susbtituted into eq. (5.13)

$$\begin{aligned}\sigma_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu(e_{ij} + \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}) \\ &= (\lambda + \tfrac{2}{3}\mu)\varepsilon_{kk}\delta_{ij} + 2\mu e_{ij} \\ &= K\varepsilon_{\mathrm{v}}\delta_{ij} + 2Ge_{ij},\end{aligned} \tag{5.23}$$

where $\varepsilon_{\mathrm{v}} = \varepsilon_{kk} = I_1$ is the volumetric strain and K is the bulk modulus. It can be seen that the constitutive equation (5.23) can be slit into volymetric-pressure and deviatoric strain-stress relations as

$$p = -K\varepsilon_{\mathrm{v}}, \quad \text{and} \quad s_{ij} = 2Ge_{ij}, \tag{5.24}$$

where the pressure p is defined as $p = -\sigma_{\mathrm{m}} = -\sigma_{kk}/3$.

Linearly elastic constitutive equations can be written either in the form

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}, \quad \text{or} \quad \boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{\sigma}, \tag{5.25}$$

where $\boldsymbol{C}$ is the *material stiffness tensor* or matrix, and $\boldsymbol{D}$ is the *compliance tensor/matrix* or the *material flexibility tensor/matrix*. They are obviously related as $\boldsymbol{C} = \boldsymbol{D}^{-1}$ and they are symmetric positive definite operators, i.e. all their eigenvalues are positive.

The strain energy function for a linearly elastic isotropic material can be given e.g. in the following forms

$$W = \tfrac{1}{2}KI_1^2 + 2GJ_2, \tag{5.26}$$

$$= \tfrac{1}{2}\lambda I_1^2 + 2\mu\tilde{I}_2. \tag{5.27}$$

Since the bulk and shear modulus have to be positive, the Young's modulus and the Poisson's ratio ν have to satisfy the following inequalities

$$E > 0, \qquad -1 < \nu < \tfrac{1}{2}. \tag{5.28}$$

For natural materials, the Poisson's ratio is usually positive. Incompressibility is approached when the Posson's ratio is near 1/2. For metals it is usually in the range 0.25-0.35 and for concrete it is near 0.2. Cork has an almost zero Poisson's ratio which make it a good material for sealing wine bottles. Materials with negative Poisson's ratio are called auxetics.

Relations between the different elasticity coefficients are given in the following equations [28, pages 293-294],[30, table 3.1.1 on page 71]:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = K - \frac{2}{3}G = \frac{G(E-2G)}{(3G-E)}, \tag{5.29}$$

$$\mu \equiv G = \frac{E}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda), \tag{5.30}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{3K-2G}{2(3K+G)}, \tag{5.31}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda}, \tag{5.32}$$

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} = \frac{\lambda(1+\nu)}{3\nu} = \frac{GE}{3(3G-E)}. \tag{5.33}$$

5.2.1 Material parameter determination

5.3 Transversely isotropic elasticity

A material is called transversely isotropic if the behaviour of it is isotropic in a plane and different in the direction of the normal of that isotropy plane. The strain energy density function, or the complementary strain energy function can be written as

$$W = W(\boldsymbol{\varepsilon}, \boldsymbol{M}) = W(\boldsymbol{Q}\boldsymbol{\varepsilon}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^T), \tag{5.34}$$

or

$$W^{\mathrm{c}} = W^{\mathrm{c}}(\boldsymbol{\sigma}, \boldsymbol{M}) = W(\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^T), \tag{5.35}$$

where $\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^T$ is called the structural tensor and the unit vector $\boldsymbol{m}$ defines the normal of the isotropy plane. The ortogonal tensor $\boldsymbol{Q}$ belongs to te transverse isotropy group

$$\mathcal{G}_1 = \{\boldsymbol{Q} \in \mathcal{O}(V),\ \boldsymbol{Q}\boldsymbol{m} = \boldsymbol{m}\}, \tag{5.36}$$

where $\mathcal{O}(V)$ is the ortogonal group in three dimensional Euclidean space V. Thus the transverse isotropy group $\mathcal{G}_1$ contains all rotations $\boldsymbol{Q}$, which are preserved in the direction $\boldsymbol{m}$.

Examples of transversely isotropic materials are those having unidirectional reinforcment , stratified soils and rocks, crystalline materials with hexagonal close packed structure.

As in the isotropic case, the complementary approach gives an easier way to interpret the material constants. Therefore the representation theorem is given first using the complementary strain energy function.

Figure 5.1: Stratified rock at Grand Canyon shows clearly transversely isotropic structure. Courtesy by Luca Galuzzi.

The representation theorem of a transversely isotropic solid says that the complementary strain energy density function can depend on five invariants

$$W^{\mathrm{c}} = W^{\mathrm{c}}(I_1, I_2, I_3, I_4, I_5), \tag{5.37}$$

where the invarinats I_i are

$$I_1 = \mathrm{tr}\,\boldsymbol{\sigma}, \quad I_2 = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}^2), \quad I_3 = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}^3), \quad I_4 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}), \quad I_5 = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}). \tag{5.38}$$

The invariants I_4 and I_5 can also be written as

$$I_4 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}) = \boldsymbol{m}^T\boldsymbol{\sigma}\boldsymbol{m}, \qquad I_5 = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}) = \boldsymbol{m}^T\boldsymbol{\sigma}^2\boldsymbol{m}. \tag{5.39}$$

The constitutive equation is thus

$$\boldsymbol{\varepsilon} = \frac{\partial W^{\mathrm{c}}}{\partial \boldsymbol{\sigma}} = \frac{\partial W^{\mathrm{c}}}{\partial I_1}\boldsymbol{I} + \frac{\partial W^{\mathrm{c}}}{\partial I_2}\boldsymbol{\sigma} + \frac{\partial W^{\mathrm{c}}}{\partial I_3}\boldsymbol{\sigma}^2 + \frac{\partial W^{\mathrm{c}}}{\partial I_4}\boldsymbol{M} + \frac{\partial W^{\mathrm{c}}}{\partial I_5}(\boldsymbol{\sigma}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\sigma}). \tag{5.40}$$

If we restrict to linear elasticity, the coefficients $\partial W^{\mathrm{c}}/\partial I_i$ has to satisfy

$$\frac{\partial W^{\mathrm{c}}}{\partial I_1} = b_1 I_1 + b I_4, \tag{5.41}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_2} = b_2, \tag{5.42}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_3} = 0, \tag{5.43}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_4} = b_3 I_1 + b_4 I_4, \tag{5.44}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_1} = b_5, \tag{5.45}$$

since all the terms in (5.40) have to be linear in $\boldsymbol{\varepsilon}$. Due to the identity

$$\frac{\partial^2 W^c}{\partial I_i \partial I_j} = \frac{\partial^2 W^c}{\partial I_j \partial I_i}, \tag{5.46}$$

we have now

$$\frac{\partial}{\partial I_4}\left(\frac{\partial W^c}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W^c}{\partial I_4}\right) \qquad \text{thus} \qquad b = b_3. \tag{5.47}$$

Transversely isotropic linear solid has thus five material coefficients, and the constitutive equation can be written as

$$\boldsymbol{\varepsilon} = (b_1 \operatorname{tr}\boldsymbol{\sigma} + b_3 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}))\boldsymbol{I} + b_2\boldsymbol{\sigma} + (b_3 \operatorname{tr}\boldsymbol{\sigma} + b_4 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}))\boldsymbol{M} + b_5(\boldsymbol{\sigma}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\sigma}). \tag{5.48}$$

If the isotropy plane coincides with the x_2, x_3-plane, i.e. $\boldsymbol{m}$ is in the direction of the x_1-axis, physically comprehensible material parameters are the Young's modulus $E_2 = E_3 = E_T$ and the Poisson's ratio $\nu_{23} = \nu_{32} = \nu_T$ in the isotropy plane x_2, x_3. The three remaining elastic coefficients are the Young's modulus E_L in the longitudinal x_1-direction, the Poisson's ratio associated with the x_1-direction and a direction in the x_2, x_3-plane, $\nu_{12} = \nu_{13} \equiv \nu_L$ and the shear modulus $G_{12} = G_{13} = G_L$. Notice that the coefficients E_L, G_L and ν_L are independent of each other.

Example 5.1. *Express (5.48) in Voigt's notation and find out the relationship between the parameters $b_1, \ldots, b_5$ and the physically meaningfull elasticity coefficients E_L, G_L, ν_L, E_T and ν_T. Assume that the longitudinal direction coincides to the x_1 axis, i.e. the transverse isotropy plane is (x_2, x_3)-plane. In the Voigt notation use the following ordering of the stress and strain components: $\hat{\boldsymbol{\sigma}} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{23}, \tau_{13}, \tau_{12}]^T$ and $\hat{\boldsymbol{\varepsilon}} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{13}, \gamma_{12}]^T$. Find the form of the flexibility matrix D: $\hat{\boldsymbol{\varepsilon}} = \mathsf{D}\hat{\boldsymbol{\sigma}}$.*

Solution. Now the longitudinal direction is $\boldsymbol{m} = (1, 0, 0)^{\mathrm{T}}$, thus

$$\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\sigma} = \begin{pmatrix} 2\sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & 0 \\ \tau_{13} & 0 & 0 \end{pmatrix},$$

and

$$I_1 = \operatorname{tr}\boldsymbol{\sigma} = \sigma_{11} + \sigma_{22} + \sigma_{33}, \quad I_4 = \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}) = \sigma_{11}.$$

Hence

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} = [b_1(\sigma_{11}+\sigma_{22}+\sigma_{33}) + b_3\sigma_{11}] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b_2 \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \varepsilon_{23} & \sigma_{33} \end{pmatrix}$$

$$+ [b_3(\sigma_{11} + \sigma_{22} + \sigma_{33}) + b_4\sigma_{11}] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b_5 \begin{pmatrix} 2\sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & 0 & 0 \\ \tau_{13} & 0 & 0 \end{pmatrix}.$$

Collecting the results gives

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix} = \begin{bmatrix} b_1+b_2+b_4+2(b_3+b_5) & b_1+b_3 & b_1+b_3 & 0 & 0 & 0 \\ b_1+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 \\ b_1+b_3 & b_1 & b_1+b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2+b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2+b_5 \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{pmatrix}. \tag{5.49}$$

Putting the above expression into the Voigt notation with $\gamma_{ij} = 2\varepsilon_{ij}$, we get

$$\hat{\boldsymbol{\varepsilon}} = \mathsf{D}\hat{\boldsymbol{\sigma}}$$

where

$$\mathsf{D} = \begin{bmatrix} b_1+b_2+b_4+2(b_3+b_5) & b_1+b_3 & b_1+b_3 & 0 & 0 & 0 \\ b_1+b_3 & b_1+b_2 & b_1 & 0 & 0 & 0 \\ b_1+b_3 & b_1 & b_1+b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(b_2+b_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(b_2+b_5) \end{bmatrix}. \tag{5.50}$$

From the above expression we can immedately notice that the shear modulus in the isotropy plane $G_{23} = G_T$ can be expressed by b_2 as

$$\tau_{23} = G_T\,\gamma_{23}, \quad \text{hence} \quad b_2 = \frac{1}{2G_T} = \frac{1+\nu_T}{E_T}.$$

The Poisson's ratio in the isotropy plane ν_T is the opposite value of the ratio between the normal strain in the transverse and longitudinal directions caused by a normal stress in the longitudinal direction. Now the isotropy plane is the x_2, x_3-plane and the normal stress acts in the x_3-directions, then

$$\varepsilon_{22} = -\nu_T \varepsilon_{33}.$$

This results in

$$\varepsilon_{22} = b_1\sigma_{33} = b_1 E_T \varepsilon_{33} = -b_1 E_T \varepsilon_{22}/\nu_T,$$

therefore

$$b_1 = -\frac{\nu_T}{E_T}.$$

As a check, we can observe that

$$\varepsilon_{22} = (b_1+b_2)\sigma_{22} = \frac{1}{E_T}\sigma_{22}.$$

The term β_5 can be solved from the shear components in the plane (x_1, x_2) or (x_1, x_3):

$$b_2 + b_5 = \frac{1}{2G_L}, \quad \text{from which we get} \quad b_5 = \frac{1}{2}\left(\frac{1}{G_L} - \frac{1}{G_T}\right).$$

The coefficient b_3 can be solved by considering normal strain in the x_2-direction when the stress in acting in the longitudinal direction. The Poisson's ratio ν_L is defined as (when $\sigma_{11} \neq 0$)

$$\varepsilon_{22} = -\nu_T \varepsilon_{11}, \quad \text{or} \quad \varepsilon_{33} = -\nu_T \varepsilon_{11}.$$

Since $\sigma_{11} = E_L \varepsilon_{11}$ we get

$$\varepsilon_{22} = (b_1 + b_3)\sigma_{11} = (b_1 + b_3)E_L \varepsilon_{11}$$

from which we obtain

$$\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\nu_L = (b_1 + b_3)E_L,$$

and finaly we get b_3 as

$$b_3 = -\frac{\nu_L}{E_L} - b_1 = \frac{\nu_T}{E_T} - \frac{\nu_L}{E_L}.$$

The last coefficient b_4 can be solved from

$$\varepsilon_{11} = (b_1 + b_2 + b_4 + 2(b_3 + b_5))\sigma_{11} = \frac{1}{E_L}\sigma_{11}$$

which gives

$$b_4 = \frac{1}{E_L} - b_1 - b_2 - 2(b_3 + b_5) = \frac{1 + 2\nu_L}{E_L} + \frac{1}{E_T} - \frac{1}{G_L}.$$

As a result the coefficients can be collected as

$$b_1 = -\frac{\nu_T}{E_T}, \tag{5.51}$$

$$b_2 = \frac{1 + \nu_T}{E_T}, \tag{5.52}$$

$$b_3 = \frac{\nu_T}{E_T} - \frac{\nu_L}{E_L}, \tag{5.53}$$

$$b_4 = \frac{1 + 2\nu_L}{E_L} + \frac{1}{E_T} - \frac{1}{G_L}, \tag{5.54}$$

$$b_5 = \frac{1}{2}\left(\frac{1}{G_L} - \frac{1}{G_T}\right), \tag{5.55}$$

and the flexibility matrix has the form

$$\mathsf{D} = \begin{pmatrix} 1/E_L & -\nu_L/E_L & -\nu_L/E_L & 0 & 0 & 0 \\ -\nu_L/E_L & 1/E_T & -\nu_T/E_T & 0 & 0 & 0 \\ -\nu_L/E_L & -\nu_T/E_T & 1/E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_L & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_L \end{pmatrix}. \tag{5.56}$$

We can immediately notice that the flexibility matrix of linearly elastic transversely isotropic solid reduces that of isotropic one when $\nu = \nu_T = \nu_L, E = E_T = E_L$ and $G = G_L = G_T = E/2(1+\nu)$.

It can be also seen that the constitutive equation with the flexibility matrix (**??**) can be written in the form

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{12} & D_{23} & D_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(D_{22}-D_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44} & 0 \\ 0 & 0 & & 0 & & D_{44} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{pmatrix}, \tag{5.57}$$

where

$$D_{11} = \frac{1}{E_L}, \quad D_{22} = \frac{1}{E_T}, \quad D_{12} = -\frac{\nu_L}{E_L}, \quad D_{23} = -\frac{\nu_T}{E_T}, \quad D_{44} = \frac{1}{G_L}, \tag{5.58}$$

and

$$2(D_{22}-D_{23}) = \frac{2(1+\nu_T)}{E_T} = \frac{1}{G_T}. \tag{5.59}$$

5.3.1 Thermodynamic restrictions to the material parameters

As in the case of linear isotropic elasticity, the compliance and stiffness matrices of the material have to be positive definite, cf. (5.28). The matrix is positive definite if all its principal minors are positive, thus

$$\begin{aligned} D_{11} > 0 \Rightarrow &\quad E_L > 0, \quad D_{22} > 0 \quad \Rightarrow \quad E_T > 0, \\ 2(D_{22}-D_{23}) > 0 \Rightarrow &\quad G_T > 0, \quad D_{44} > 0 \quad \Rightarrow G_L > 0, \end{aligned} \tag{5.60}$$

$$\begin{aligned} \begin{vmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{vmatrix} > 0 \Rightarrow &\quad D_{11}D_{22} - D_{12}^2 > 0 \quad \Rightarrow 1 - \frac{E_T}{E_L}\nu_L^2 > 0 \\ \Rightarrow &-\sqrt{E_L/E_T} < \nu_L < \sqrt{E_L/E_T}, \end{aligned} \tag{5.61}$$

$$\begin{aligned} \begin{vmatrix} D_{22} & D_{23} \\ D_{23} & D_{22} \end{vmatrix} > 0 \Rightarrow &\quad D_{22}^2 - D_{23}^2 > 0 \quad \Rightarrow \quad \frac{1}{E_T^2} - \frac{\nu_T^2}{E_T^2} > 0 \\ \Rightarrow &\quad 1 - \nu_T^2 > 0 \quad \Rightarrow \quad -1 < \nu_T < 1, \end{aligned} \tag{5.62}$$

$$\begin{aligned} \begin{vmatrix} D_{11} & D_{12} & D_{12} \\ D_{12} & D_{22} & D_{23} \\ D_{12} & D_{23} & D_{22} \end{vmatrix} > 0 \Rightarrow &\quad (1-\nu_T^2)E_L - 2E_T\nu_L^2(1+\nu_T) > 0 \\ \Rightarrow &-\sqrt{\frac{E_L(1-\nu_T)}{2E_T}} < \nu_L < \sqrt{\frac{E_L(1-\nu_T)}{2E_T}}. \end{aligned} \tag{5.63}$$

It is seen that due to restriction (5.62) the inequality (5.63) is more restrictive than (5.61). As a summary the thermodynamic restrictions to the material parameters for a linear transversely isotropic elastic material are

$$E_L > 0, \quad E_T > 0, \quad G_L > 0, \tag{5.64}$$

$$-1 < \nu_T < 1, \tag{5.65}$$

$$-\sqrt{\frac{E_L(1-\nu_T)}{2E_T}} < \nu_L < \sqrt{\frac{E_L(1-\nu_T)}{2E_T}}. \tag{5.66}$$

5.3.2 Monotonicity conditions

The thermodynamic restrictions have necessarily to be fulfilled. However, an additional restrictions emerge if the longitudinal and transverse moduli are considered as extreme values for the Young's modulus E in an arbitrary direction. To obtain conditions for monotonous dependence, it is equivalent to consider the applied uniaxial stress in the x_1-axis direction and the longitudinal direction $\boldsymbol{m}$ forms an angle α w.r.t. the x_1-direction. Therefore $\boldsymbol{m} = (\cos\alpha, \sin\alpha, 0)^{\mathrm{T}}$ and using the following notations for brevity $c = \cos\alpha$ and $s = \sin\alpha$, it is obtained

$$\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^T = \begin{pmatrix} c^2 & sc & 0 \\ sc & s^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\sigma}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\sigma} = \begin{pmatrix} 2c^2\sigma_x & sc\sigma_x & 0 \\ sc\sigma_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_x \quad I_4 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}) = c^2\sigma_x.$$

The strain tensor has now the form

$$\boldsymbol{\varepsilon} = (b_1\sigma_x + b_3c^2\sigma_x)\boldsymbol{I} + b_2\boldsymbol{\sigma} + (b_3\sigma_x + b_4c^2\sigma_x)\boldsymbol{M} + b_5(\boldsymbol{\sigma}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\sigma}),$$

from which we can obtain the strain in the x_1-axis direction

$$\varepsilon_x = (b_1 + b_2 + 2(b_3 + b_5)c^2 + b_4c^4)\sigma_x,$$

which has the following expression written in the stiffness form

$$\sigma_x = \frac{1}{b_1 + b_2 + 2(b_3 + b_5)c^2 + b_4c^4}\varepsilon_x.$$

The Young's modulus in the α-direction is thus

$$E(\alpha) = \frac{1}{b_1 + b_2 + 2(b_3 + b_5)\cos^2\alpha + b_4\cos^4\alpha}.$$

Now we can investigate if the denominator $f(x) = b_1 + b_2 + 2(b_3 + b_5)x + b_4 x^2$ have extreme values when $0 < x < 1$. The function f has zero derivative at $x = c^2 = -(b_3 + b_5)/b_4$. For f to be monotoneous in the interval $0 \le x \le 1$, the expressions $b_3 + b_5$ and b_4 have to have same sign and the function f do not have extreme values in the interval.

In the example 5.1 the coefficients $b_1, \dots, b_5$ are given in terms of E_L, E_T, G_L, ν_L and ν_T in equations (5.51)-(5.55). It is now assumed that $E_L > E_T$. Considering the equation

$$c^2 = -\frac{b_3 + b_5}{b_4},$$

in order to have a real solution it is required that

$$-\frac{b_3 + b_5}{b_4} > 1, \quad \text{or} \quad -\frac{b_3 + b_5}{b_4} < 0.$$

Considering first the condition $-(b_3 + b_5)/b_4 > 1$, from which the following condition is obtained provided that $b_4 > 0$:

$$-(b_3 + b_5) > b_4 \quad \Rightarrow \quad -\frac{\nu_T}{E_T} + \frac{\nu_L}{E_L} + \frac{1}{2G_T} - \frac{1}{2G_L} > \frac{1 + 2\nu_L}{E_L} + \frac{1}{E_T} - \frac{1}{G_L},$$

and after some intermediate steps the inequality

$$G_L < \frac{E_L}{2(1 + \nu_L)}$$

is obtained. The condition $b_3 + b_5 < 0$ results in

$$G_L > \frac{E_L}{2(\nu_L + E_L/E_T)}.$$

As an example consider the case $E_T/E_L = 2/5 = 0.4$ and $\nu_L = \nu_T = 1/4 = 0.25$. These values provide the following limits for G_L:

$$\frac{G_L}{E_L} < \frac{1}{2(1 + \nu_L)} = \frac{2}{5} = 0.4 \quad \text{and} \quad \frac{G_L}{E_L} > \frac{1}{2(\nu_L + E_L/E_T)} = \frac{2}{11} \approx 0.182.$$

In Fig. 5.2 the cases $G_L/E_L = 0.5$ (the uppermost curve), $G_L/E_L = 0.3$ (the middle curve) and $G_L/E_L = 0.15$ (the lowest curve).

5.3.3 Material parameter determination

The linear elasticity constants for transversely isotropic solid can be determined from the following tests, where it is assumed that the longitudinal direction coincides with the x_1-axis direction.

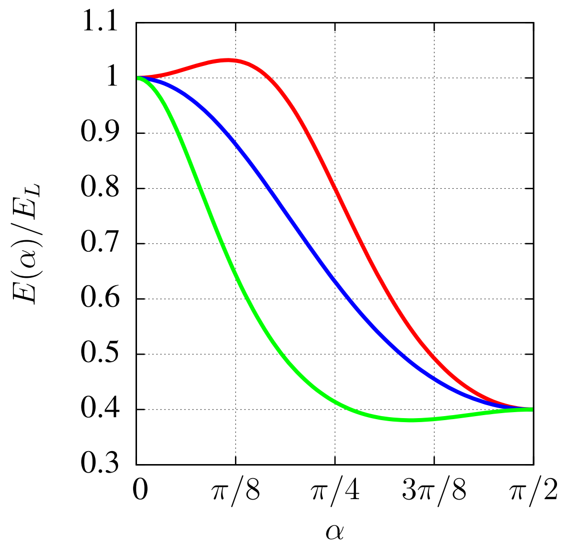


Figure 5.2: Young's modulus in different orientation with respect to the longitudinal direction, red curve G_L/E_L = 0.5, blue curve G_L/E_L = 0.3, green curve G_L/E_L= 0.15, $\nu_L = \nu_T$ = 0.25, E_T/E_L = 0.4.

1. Apply a stress in the longitudinal direction 1, i.e. σ_{11}, and measure $\varepsilon_{11}, \varepsilon_{22} = \varepsilon_{33}$, then $E_1 = E_L = \sigma_{11}/\varepsilon_{11}$ and $\nu_L = \nu_{12} = \nu_{13} = -\varepsilon_{22}/\varepsilon_{11}$.

2. Apply a stress in the transverse direction, i.e. σ_{22}, and measure strain in the three perpendicular direction $\varepsilon_{11}, \varepsilon_{22}$ and ε_{33}, then $E_2 = E_T = \sigma_{22}/\varepsilon_{22}$, $\nu_{23} = -\varepsilon_{33}/\varepsilon_{22} = \nu_T$

3. Apply a shear stress in the 1-2 plane, then $G_{12} = G_L = \tau_{12}/\gamma_{12}$. Note $G_{12} = G_{13}$.

4. This test is not necessary. Shear in the isotropy plane, i.e. in the 2-3 plane. $G_{23} = \tau_{23}/\gamma_{23}$. Could also be obtained from $G_{23} = E_2/(1+\nu_{23}) = G_T = E_T/(1+\nu_T)$.

5.3.4 Stiffness form of the transversly isotropic linear elastic model

Using similar arguments which resulted the equation (5.48), we get the stiffness form of the constitutive equation as

$$\boldsymbol{\sigma} = (a_1 \operatorname{tr} \boldsymbol{\varepsilon} + a_3 \operatorname{tr}(\boldsymbol{\varepsilon M}))\boldsymbol{I} + a_2\boldsymbol{\varepsilon} + (a_3 \operatorname{tr} \boldsymbol{\varepsilon} + a_4 \operatorname{tr}(\boldsymbol{\varepsilon M}))\boldsymbol{M} + a_5(\boldsymbol{\varepsilon M} + \boldsymbol{M\varepsilon}). \quad (5.67)$$

As in the case of the flexibility approach, assume that the longitudinal direction equals to the x_1-axis direction, the stiffness matrix C in the Voigt notation has the form

$$\mathsf{C} = \begin{pmatrix} a_1+a_2+a_4+2(a_3+a_5) & a_1+a_3 & a_1+a_3 & 0 & 0 & 0 \\ a_1+a_3 & a_1+a_2 & a_1 & 0 & 0 & 0 \\ a_1+a_3 & a_1 & a_1+a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(a_2+a_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(a_2+a_5) \end{pmatrix}, \tag{5.68}$$

the the order of stress and strain components are as in (5.50). It is now easy to start with the term a_i from shear in the isotropy plane, resulting in

$$\tau_{23} = G_T\gamma_{23} \quad \Rightarrow \quad a_2 = 2G_T. \tag{5.69}$$

Next, considering shear in the plane containing the longitudinal axis, i.e. the plane x_1, x_2 or x_1, x_3:

$$\tau_{12} = G_L\gamma_{12}, \quad \tau_{31} = G_L\gamma_{31} \quad \Rightarrow \quad \frac{1}{2}(a_2+a_5) = G_L, \tag{5.70}$$

hence

$$G_T + \frac{1}{2}a_5 = G_L \quad \Rightarrow \quad a_5 = 2(G_L - G_T). \tag{5.71}$$

Notice that in the isotropy plane the following relations hold

$$a_1 = \lambda_T = \frac{G_T(E_T - 2G_T)}{3G_T - E_T} = \frac{\nu_T E_T}{(1+\nu_T)(1-2\nu_T)}, \quad G_T = \frac{E_T}{2(1+\nu_T)}. \tag{5.72}$$

Stress in the x_1-axis direction produces strain state $\varepsilon_{22} = -\nu_L\varepsilon_{11}, \quad \varepsilon_{33} = -\nu_L\varepsilon_{11}$, hence

$$\begin{aligned} \sigma_{11} = E_L\varepsilon_{11} &= (a_1+a_2+a_4+2(a_3+a_5))\varepsilon_{11} + (a_1+a_3)\varepsilon_{22} + (a_1+a_3)\varepsilon_{33} \\ &= (a_1+a_2+a_4+2(a_3+a_5))\varepsilon_{11} - 2\nu_L(a_1+a_3)\varepsilon_{11} \\ &= \left[(1-2\nu_L)\lambda_T + 2G_T + 2(1-\nu_L)a_3 + a_4 + 4(G_L - G_T)\right]\varepsilon_{11}, \end{aligned} \tag{5.73}$$

from which follows the relations

$$E_L = 2(1-\nu_L)a_3 + a_4 + (1-2\nu_L)\lambda_T + 4G_L - 2G_T. \tag{5.74}$$

Conditions $\sigma_{22} = \sigma_{33} = 0$ result in the same relationship

$$\sigma_{22} = (\lambda_T + a_3)\varepsilon_{11} + (\lambda_T + 2G_T)\varepsilon_{22} + \lambda_T\varepsilon_{33} = [\lambda_T + a_3 - 2\nu_L(\lambda_T + G_T)]\varepsilon_{11} = 0, \tag{5.75}$$

from which the constant a_3 can be solved as

$$a_3 = 2\nu_L(\lambda_T + G_T) - \lambda_T = (2\nu_L - 1)\lambda_T + 2\nu_L G_T. \tag{5.76}$$

At last the value for the constant a_4 is obtained

$$\begin{aligned}\alpha_4 &= E_L + 2G_T - 4G_L + (2\nu_L - 1)\lambda_T - 2(1-\nu_L)\left[(2\nu_L - 1)\lambda_T + 2\nu_L G_T\right] \\ &= E_L - 4G_L + G_T + (1-2\nu_L)^2(G_T + \lambda_T). \end{aligned} \tag{5.77}$$

The transversally isotropic linear elastic stiffness matrix in the special case where the longitudinal direction coincides with the x_1-axis direction has the form

$$\mathsf{C} = \begin{pmatrix} E_L + 4\nu_L^2(\lambda_T + G_T) & 2\nu_L(\lambda_T + G_T) & 2\nu_L(\lambda_T + G_T) & 0 & 0 & 0 \\ 2\nu_L(\lambda_T + G_T) & \lambda_T + 2G_T & \lambda_T & 0 & 0 & 0 \\ 2\nu_L(\lambda_T + G_T) & \lambda_T & \lambda_T + 2G_T & 0 & 0 & 0 \\ 0 & 0 & 0 & G_T & 0 & 0 \\ 0 & 0 & 0 & 0 & G_L & 0 \\ 0 & 0 & 0 & 0 & 0 & G_L \end{pmatrix}, \tag{5.78}$$

where

$$\lambda_T = \frac{\nu_T E_T}{(1+\nu_T)(1-2\nu_T)}, \quad G_T = \frac{E_T}{2(1+\nu_T)}. \tag{5.79}$$

As in the case of the flexibility approach, the constants (5.69)-(5.77) reduce to the known Lamé constants $a_1 = \lambda = \nu E/[(1+\nu)(1-2\nu)], a_2 = 2\mu = 2G$, and $a_3 = a_4 = a_5 = 0$.

As a summary the stiffness coefficients a_i in (5.67) are

$$\begin{aligned} a_1 &= \lambda_T = \frac{\nu_T E_T}{(1+\nu_T)(1-2\nu_T)} = \frac{G_T(E_T - 2G_T)}{3G_T - E_T}, \\ a_2 &= \frac{E_T}{1+\nu_T} = 2G_T, \\ a_3 &= (2\nu_L - 1)\lambda_T + 2\nu_L G_T, \\ a_4 &= E_L - 4G_L + G_T + (1-2\nu_L)^2(G_T + \lambda_T), \\ a_5 &= 2(G_L - G_T). \end{aligned}$$

5.4 Orthotropic material

A material is called orthotropic if it has three perpendicular symmetry planes. Let's denote the unit vectors normal to the symmetry planes as $\boldsymbol{m}_1, \boldsymbol{m}_2$ and $\boldsymbol{m}_3$. Due to the orthogonality $\boldsymbol{m}_i \cdot \boldsymbol{m}_j = \delta_{ij}$. The structural tensors associated with these direction vectors are $\boldsymbol{M}_i = \boldsymbol{m}_i \boldsymbol{m}_i^T$, and they satisfy

$$\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3 = \boldsymbol{I}, \tag{5.80}$$

due to the orthogonality. As it was noticed in the case of transverse isotropy, it is easier to obtain the representation of the material parameters in the compliance form (5.5). Thus, the complementary strain energy is now written and due to the constraint (5.80) only two structural tensors are necessary to describe the behaviour of an orthotropic material

$$W^{\rm c} = W^{\rm c}(\boldsymbol{\sigma}, \boldsymbol{M}_1, \boldsymbol{M}_2) = W^{\rm c}(\boldsymbol{\beta}\boldsymbol{\sigma}\boldsymbol{\beta}^T, \boldsymbol{\beta}\boldsymbol{M}_1\boldsymbol{\beta}^T, \boldsymbol{\beta}\boldsymbol{M}_2\boldsymbol{\beta}^T). \tag{5.81}$$

The representation theorem of an orthotropic solid says that the complementary strain energy density function can depend on seven invariants

$$W^{\rm c} = W^{\rm c}(\mathrm{tr}\,\boldsymbol{\sigma},\ \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}^2),\ \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}^3),\ \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1),\ \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2),\ \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_1),\ \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_2)). \tag{5.82}$$

It can be written in a form, where all the structural tensors $\boldsymbol{M}_i$ are symmetrically present. Notice that

$$\boldsymbol{\sigma}\boldsymbol{M}_1 + \boldsymbol{\sigma}\boldsymbol{M}_2 + \boldsymbol{\sigma}\boldsymbol{M}_3 = \boldsymbol{\sigma}(\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3) = \boldsymbol{\sigma}, \tag{5.83}$$

$$\boldsymbol{M}_1\boldsymbol{\sigma} + \boldsymbol{M}_2\boldsymbol{\sigma} + \boldsymbol{M}_3\boldsymbol{\sigma} = (\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3)\boldsymbol{\sigma} = \boldsymbol{\sigma}, \tag{5.84}$$

thus summing by parts gives

$$\boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\sigma}) + \tfrac{1}{2}(\boldsymbol{\sigma}\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{\sigma}) + \tfrac{1}{2}(\boldsymbol{\sigma}\boldsymbol{M}_3 + \boldsymbol{M}_3\boldsymbol{\sigma}), \tag{5.85}$$

and

$$\mathrm{tr}\ \boldsymbol{\sigma} = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1) + \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2) + \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3). \tag{5.86}$$

In a similar way it can be deduced

$$\mathrm{tr}\,(\boldsymbol{\sigma}^2) = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_1) + \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_2) + \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_3). \tag{5.87}$$

In other words, the invariants $\mathrm{tr}\ \boldsymbol{\sigma}, \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1)$ and $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2)$ can be replaced by the invariants $I_1 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1), I_2 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2)$ and $I_3 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3)$. In a similar way the invariants $\mathrm{tr}(\boldsymbol{\sigma}^2), \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_1)$ and $\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_2)$ can be replaced by the invariants $I_4 = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_1), I_5 = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_2)$ and $I_6 = \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_3)$. If we now denote the cubic invariant $I_7 = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}^3)$, the complementary strain energy density function for an orthotropic material can be written as a function of these seven invariants as

$$W^{\rm c} = W^{\rm c}(I_1, \ldots, I_7), \tag{5.88}$$

and the constitutive equation has the form

$$\begin{aligned}\boldsymbol{\varepsilon} =& \frac{\partial W^{\rm c}}{\partial \boldsymbol{\sigma}} = \sum_{i=1}^{7} \frac{\partial W^{\rm c}}{\partial I_i}\frac{\partial I_i}{\partial \boldsymbol{\sigma}} \\ =& \frac{\partial W^{\rm c}}{\partial I_1}\boldsymbol{M}_1 + \frac{\partial W^{\rm c}}{\partial I_2}\boldsymbol{M}_2 + \frac{\partial W^{\rm c}}{\partial I_3}\boldsymbol{M}_3 + \frac{\partial W^{\rm c}}{\partial I_4}(\boldsymbol{\sigma}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\sigma}) \\ &+ \frac{\partial W^{\rm c}}{\partial I_5}(\boldsymbol{\sigma}\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{\sigma}) + \frac{\partial W^{\rm c}}{\partial I_6}(\boldsymbol{\sigma}\boldsymbol{M}_3 + \boldsymbol{M}_3\boldsymbol{\sigma}) + \frac{\partial W^{\rm c}}{\partial I_7}\boldsymbol{\sigma}^2.\end{aligned} \tag{5.89}$$

If we now restrict to a linear model, the coefficients $\partial W^{\mathrm{c}}/\partial I_i$ has to satisfy the following conditions

$$\frac{\partial W^{\mathrm{c}}}{\partial I_1} = b_1 I_1 + c_1 I_2 + c_2 I_3, \tag{5.90}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_2} = b_2 I_1 + b_3 I_2 + c_3 I_3, \tag{5.91}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_3} = b_4 I_1 + b_5 I_2 + b_6 I_3, \tag{5.92}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_4} = b_7, \tag{5.93}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_5} = b_8, \tag{5.94}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_6} = b_9, \tag{5.95}$$

$$\frac{\partial W^{\mathrm{c}}}{\partial I_7} = 0. \tag{5.96}$$

Due to the identity of the second derivatives (5.46), we have

$$\frac{\partial}{\partial I_2}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_2}\right) \qquad \Rightarrow \quad c_1 = b_2, \tag{5.97}$$

$$\frac{\partial}{\partial I_3}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_1}\right) = \frac{\partial}{\partial I_1}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_3}\right) \qquad \Rightarrow \quad c_2 = b_4, \tag{5.98}$$

$$\frac{\partial}{\partial I_3}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_2}\right) = \frac{\partial}{\partial I_2}\left(\frac{\partial W^{\mathrm{c}}}{\partial I_3}\right) \qquad \Rightarrow \quad c_3 = b_5. \tag{5.99}$$

The constitutive equation is thus

$$\begin{aligned}\boldsymbol{\varepsilon} = \; & [b_1 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1) + b_2 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2) + b_4 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3)]\,\boldsymbol{M}_1 \\ & + [b_2 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1) + b_3 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2) + b_5 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3)]\,\boldsymbol{M}_2 \\ & + [b_4 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1) + b_5 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2) + b_6 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3)]\,\boldsymbol{M}_3 \\ & + b_7(\boldsymbol{\sigma}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\sigma}) + b_8(\boldsymbol{\sigma}\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{\sigma}) + b_9(\boldsymbol{\sigma}\boldsymbol{M}_3 + \boldsymbol{M}_3\boldsymbol{\sigma}).\end{aligned} \tag{5.100}$$

If the directions of the unit vectors $\boldsymbol{m}_i$ coincide with the coordinate axis, the material coefficients a_i and b_i can be expressed in terms of physically comprehensible material constants, which for orthotropic material are the Young's moduli in the 1,2 and 3 material directions E_1, E_2 and E_3, the Poisson's ratios ν_{ij}, defined as a ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the i-direction, i.e.

$$\varepsilon_j = -\nu_{ij}\varepsilon_i = -\nu_{ij}\frac{\sigma_i}{E_i}, \quad \text{no sum in } \; i, \tag{5.101}$$

and the shear moduli in the 1-2, 2-3 and 1-3 planes G_{12}, G_{23} and G_{13}.

If the axes of orthotropy coincide to the coordinate axes, the compliance matrix has the form

$$\boldsymbol{D} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix}, \tag{5.102}$$

which can be easily deduced by using the superposition principle. Due to the symmetry requirement of the compliance matrix $\boldsymbol{D}$ the following relations have to hold

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \qquad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}, \qquad \frac{\nu_{13}}{E_1} = \frac{\nu_{21}}{E_3}, \tag{5.103}$$

or written in a more easily memorized form

$$\nu_{ij} E_j = \nu_{ji} E_i. \tag{5.104}$$

The material parameters b_i in (5.100) can be easily obtained when comparing (5.100) and (5.102), resulting in

$$b_1 = \frac{1}{E_1} + \frac{1}{2G_{23}} - \frac{1}{2G_{12}} - \frac{1}{2G_{13}}, \tag{5.105}$$

$$b_3 = \frac{1}{E_2} + \frac{1}{2G_{13}} - \frac{1}{2G_{12}} - \frac{1}{2G_{23}}, \tag{5.106}$$

$$b_6 = \frac{1}{E_3} + \frac{1}{2G_{12}} - \frac{1}{2G_{23}} - \frac{1}{2G_{13}}, \tag{5.107}$$

$$b_2 = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}, \tag{5.108}$$

$$b_4 = -\frac{\nu_{31}}{E_3} = -\frac{\nu_{13}}{E_1}, \tag{5.109}$$

$$b_5 = -\frac{\nu_{23}}{E_2} = -\frac{\nu_{32}}{E_3}, \tag{5.110}$$

$$b_7 = \frac{1}{4}\left(\frac{1}{G_{12}} + \frac{1}{G_{13}} - \frac{1}{G_{23}}\right), \tag{5.111}$$

$$b_8 = \frac{1}{4}\left(\frac{1}{G_{12}} + \frac{1}{G_{23}} - \frac{1}{G_{13}}\right), \tag{5.112}$$

$$b_9 = \frac{1}{4}\left(\frac{1}{G_{23}} + \frac{1}{G_{13}} - \frac{1}{G_{12}}\right). \tag{5.113}$$

Starting from the energy density $W(I_1, \dots, I_6)$ a similar expression can be obtained

$$\begin{aligned}
\boldsymbol{\sigma} &= (a_1 I_1 + a_2 I_2 + a_4 I_3)\boldsymbol{M}_1 + (a_2 I_1 + a_3 I_2 + a_5 I_3)\boldsymbol{M}_2 + (a_4 I_1 + a_5 I_2 + a_6 I_3)\boldsymbol{M}_3 \\
&\quad + a_7(\boldsymbol{\varepsilon}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\varepsilon}) + a_8(\boldsymbol{\varepsilon}\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{\varepsilon}) + a_9(\boldsymbol{\varepsilon}\boldsymbol{M}_3 + \boldsymbol{M}_3\boldsymbol{\varepsilon}) \\
&= \quad [a_1 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_1) + a_2 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_2) + a_4 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_3)]\, \boldsymbol{M}_1 \\
&\quad + [a_2 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_1) + a_3 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_2) + a_5 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_3)]\, \boldsymbol{M}_2 \\
&\quad + [a_4 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_1) + a_5 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_2) + a_6 \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_3)]\, \boldsymbol{M}_3 \\
&\quad + a_7(\boldsymbol{\varepsilon}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\varepsilon}) + a_8(\boldsymbol{\varepsilon}\boldsymbol{M}_2 + \boldsymbol{M}_2\boldsymbol{\varepsilon}) + a_9(\boldsymbol{\varepsilon}\boldsymbol{M}_3 + \boldsymbol{M}_3\boldsymbol{\varepsilon}),
\end{aligned} \tag{5.114}$$

where the invariants $I_1, \dots I_6$ are now

$$\begin{aligned}
&I_1 = \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_1), \quad I_2 = \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_2), \quad I_3 = \operatorname{tr}(\boldsymbol{\varepsilon}\boldsymbol{M}_3) \\
&\qquad\qquad I_4 = \operatorname{tr}(\boldsymbol{\varepsilon}^2\boldsymbol{M}_1), \quad I_5 = \operatorname{tr}(\boldsymbol{\varepsilon}^2\boldsymbol{M}_2), \quad I_6 = \operatorname{tr}(\boldsymbol{\varepsilon}^2\boldsymbol{M}_3).
\end{aligned} \tag{5.115}$$

Constitutive equation (5.114) can be expressed in the Voigt notation in the special case where the principal material directions coincide to the coordinate axes in the following form

$$\hat{\boldsymbol{\sigma}} = \mathsf{C}\hat{\boldsymbol{\varepsilon}}, \tag{5.116}$$

where the material stiffness matrix C is

$$\mathsf{C} = \begin{pmatrix}
a_1 + 2a_7 & a_2 & a_4 & 0 & 0 & 0 \\
a_2 & a_3 + 2a_8 & a_5 & 0 & 0 & 0 \\
a_4 & a_5 & a_6 + 2a_9 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(a_8 + a_9) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}(a_7 + a_9) & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(a_7 + a_8)
\end{pmatrix} \tag{5.117}$$

where the order of the stress and strain components is as in (**??**). Parameters a_7, a_8 ja a_9 can be easily expressed in terms of shear modulae as

$$a_7 = G_{13} + G_{12} - G_{23}, \quad a_8 = G_{12} + G_{23} - G_{13}, \quad a_9 = G_{13} + G_{23} - G_{12}. \tag{5.118}$$

Other constants $a_1, \dots, a_6$ can be solved by considerin uniaxial stress state in direction

1,2 and 3, which results in

$$a_1 = \frac{1-\nu_{23}\nu_{32}}{\Delta}E_1 + 2(G_{23} - G_{13} - G_{12}), \tag{5.119}$$

$$a_3 = \frac{1-\nu_{13}\nu_{31}}{\Delta}E_2 + 2(G_{13} - G_{12} - G_{23}), \tag{5.120}$$

$$a_6 = \frac{1-\nu_{12}\nu_{21}}{\Delta}E_3 + 2(G_{12} - G_{23} - G_{13}), \tag{5.121}$$

$$a_2 = \frac{\nu_{21}+\nu_{23}\nu_{31}}{\Delta}E_1 = \frac{\nu_{12}+\nu_{13}\nu_{32}}{\Delta}E_2, \tag{5.122}$$

$$a_4 = \frac{\nu_{31}+\nu_{32}\nu_{21}}{\Delta}E_1 = \frac{\nu_{13}+\nu_{12}\nu_{23}}{\Delta}E_3, \tag{5.123}$$

$$a_5 = \frac{\nu_{32}+\nu_{31}\nu_{12}}{\Delta}E_2 = \frac{\nu_{23}+\nu_{21}\nu_{13}}{\Delta}E_3, \tag{5.124}$$

where $\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{32}\nu_{21}\nu_{13}$.

The linear elastic material stiffness matrix for an orthotropic solid in the special case that the principal material directions coincide to the coordinate axis direction is

$$\mathsf{C} = \begin{pmatrix} (1-\nu_{23}\nu_{32})E_1/\Delta & (\nu_{12}+\nu_{13}\nu_{32})E_2/\Delta & (\nu_{13}+\nu_{12}\nu_{23})E_3/\Delta & 0 & 0 & 0 \\ & (1-\nu_{13}\nu_{31})E_2/\Delta & (\nu_{23}+\nu_{21}\nu_{13})E_3/\Delta & 0 & 0 & 0 \\ & & (1-\nu_{12}\nu_{21})E_3/\Delta & 0 & 0 & 0 \\ & & & G_{23} & 0 & 0 \\ & & & & G_{13} & 0 \\ & \text{symm.} & & & & G_{12} \end{pmatrix}. \tag{5.125}$$

5.4.1 Thermodynamic restriction to the material parameters

Since the compliance matrix has to be positive definite, an immediate consequence is that the elasticity- and shear moduli has to be positive

$$E_1 > 0, \quad E_2 > 0, \quad E_3 > 0, \quad G_{12} > 0, \quad G_{23} > 0, \quad \text{and} \quad G_{13} > 0. \tag{5.126}$$

In addition the following minors have to be positive

$$\begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 \\ -\nu_{12}/E_1 & 1/E_2 \end{vmatrix} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2} > 0, \tag{5.127}$$

$$\begin{vmatrix} 1/E_2 & -\nu_{32}/E_3 \\ -\nu_{23}/E_2 & 1/E_3 \end{vmatrix} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3} > 0, \tag{5.128}$$

$$\begin{vmatrix} 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & 1/E_3 \end{vmatrix} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3} > 0, \tag{5.129}$$

and

$$\begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 \end{vmatrix} = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{32}\nu_{21}\nu_{13}}{E_1E_2E_3} > 0. \quad (5.130)$$

Since the Young's moduli are positive, the inequalities (5.127)-(5.129) can be written in the form

$$1-\nu_{ij}\nu_{ji} > 0, \quad (5.131)$$

which after taking the reciprocal relation (5.104) into account has the form

$$1-\nu_{ij}^2 E_j/E_i > 0, \quad \text{or} \quad |\nu_{ij}| < \sqrt{E_i/E_j}. \quad (5.132)$$

The positive definiteness is thus quaranteed if the inequalities for the moduli (5.126) together with the inqualities (5.132) and

$$1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{32}\nu_{21}\nu_{13} > 0 \quad (5.133)$$

for the Poisson's ratios are satisfied.

5.4.2 Monotonicity conditions

As in the case of transversely isotropic linear elasticity, see section 5.3.2, a reasonable assumption is that the Young's moduli have their extreme values in the principal directions of orthotropy and they varies monotonously in the intermediate directions. Due to orthotropy three different planes have to be investigated. Considering first a uniaxial stress state in the x_1-x_2 plane, the stress tensor can be written as

$$\boldsymbol{\sigma} = \sigma \boldsymbol{n}_{12} \otimes \boldsymbol{n}_{12}, \quad \text{where} \quad \boldsymbol{n}_{12} = (\cos\psi_{12}, \sin\psi_{12}, 0)^{\mathrm{T}}. \quad (5.134)$$

Using notation $c_{12} = \cos\psi_{12}$ and $s_{12} = \sin\psi_{12}$, the strain tensor has the following form

$$\boldsymbol{\varepsilon} = \big[(b_1c_{12}^2 + b_2s_{12}^2)\boldsymbol{M}_1 + (b_2c_{12}^2 + b_3s_{12}^2)\boldsymbol{M}_2 + (b_4c_{12}^2 + b_5s_{12}^2)\boldsymbol{M}_3 + \\ + b_7(\boldsymbol{n}_{12}\cdot\boldsymbol{m}_1)(\boldsymbol{n}_{12}\otimes\boldsymbol{m}_1 + \boldsymbol{m}_1\otimes\boldsymbol{n}_{12}) + b_8(\boldsymbol{n}_{12}\cdot\boldsymbol{m}_2)(\boldsymbol{n}_{12}\otimes\boldsymbol{m}_2 + \boldsymbol{m}_2\otimes\boldsymbol{n}_{12})\big]\sigma. \quad (5.135)$$

The only non-zero strain components are thus $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ and ε_{33}, having expressions

$$\varepsilon_{11} = [(b_1+2b_7)c_{12}^2 + b_2s_{12}^2]\sigma, \quad (5.136)$$
$$\varepsilon_{22} = [(b_3+2b_8)s_{12}^2 + b_2c_{12}^2]\sigma, \quad (5.137)$$
$$\varepsilon_{12} = (b_7+b_8)s_{12}c_{12}\sigma, \quad (5.138)$$
$$\varepsilon_{33} = (b_4c_{12}^2 + b_5s_{12}^2)\sigma. \quad (5.139)$$

It can be noticed that stress- and strain tensors are not coaxial, i.e. they have different pricipal directions. This is typical for ansiotropic materials. Strain in the direction $\boldsymbol{n}_{12}$ is

$$\varepsilon(\boldsymbol{n}_{12}) = \boldsymbol{n}_{12}\cdot\boldsymbol{\varepsilon}\boldsymbol{n}_{12} = c_{12}^2\varepsilon_{11} + 2s_{12}c_{12}\varepsilon_{12} + s_{12}\varepsilon_{22}, \tag{5.140}$$

and the Young's modulus in the direction $\boldsymbol{n}_{12}$ is

$$E(\boldsymbol{n}_{12}) = \frac{E_1}{\left(1 + 2\nu_{12} + \dfrac{E_1}{E_2} - \dfrac{E_1}{G_{12}}\right)c_{12}^4 + 2\left(\dfrac{E_1}{2G_{12}} - \dfrac{E_1}{E_2} - \nu_{12}\right)c_{12}^2 + \dfrac{E_1}{E_2}}, \tag{5.141}$$

Denoting $x = c_{12}^2$ and investigating monotonicity of the denominator

$$f(x) = \left(1 + 2\nu_{12} + \frac{E_1}{E_2} - \frac{E_1}{G_{12}}\right)x^2 + 2\left(\frac{E_1}{2G_{12}} - \frac{E_1}{E_2} - \nu_{12}\right)x + \frac{E_1}{E_2}. \tag{5.142}$$

Function f is monotonic in the interval $x \in [0, 1]$ if its derivative wrt x does not have any zero values in that interval. This condition is satisfied if

$$\frac{g_1}{g_2} = \frac{\nu_{12} + \dfrac{E_1}{E_2} - \dfrac{E_1}{2G_{12}}}{1 + 2\nu_{12} + \dfrac{E_1}{E_2} - \dfrac{E_1}{G_{12}}} > 1, \quad \text{or} \quad \frac{\nu_{12} + \dfrac{E_1}{E_2} - \dfrac{E_1}{2G_{12}}}{1 + 2\nu_{12} + \dfrac{E_1}{E_2} - \dfrac{E_1}{G_{12}}} < 0. \tag{5.143}$$

The first condition gives the inequality

$$\frac{E_1}{G_{12}} > 2(1 + \nu_{12}), \tag{5.144}$$

if the denominator of expression (5.143) is positive, thus

$$\frac{E_1}{G_{12}} < 1 + 2\nu_{12} + \frac{E_1}{E_2} \quad \text{or} \quad \frac{E_1}{E_2} > \frac{E_1}{G_{12}} - 1 - 2\nu_{12}. \tag{5.145}$$

If the denominator is negative, the direction of the inequalities (5.144) and (5.145) change.

The latter of these conditions can only be true if the nominator and the denominator have different signs. Thus, the monotonicity condition is restriced to a domain defined by inequalities

$$\frac{E_1}{G_{12}} > 2(1 + \nu_{12}) \quad \text{and} \quad \frac{E_1}{G_{12}} < 2\left(\nu_{12} + \frac{E_1}{E_2}\right), \tag{5.146}$$

which is illustrated in Fig. 5.3.

In a similar way the monotonicity conditions can be treated for x_2, x_3- and x_3, x_1-planes and the resulting inequalities are

$$\frac{E_2}{G_{23}} > 2(1 + \nu_{23}) \quad \text{ja} \quad \frac{E_2}{G_{23}} < 2\left(\nu_{23} + \frac{E_2}{E_3}\right), \tag{5.147}$$

$$\frac{E_3}{G_{13}} > 2(1 + \nu_{13}) \quad \text{ja} \quad \frac{E_1}{G_{13}} < 2\left(\nu_{13} + \frac{E_1}{E_3}\right). \tag{5.148}$$

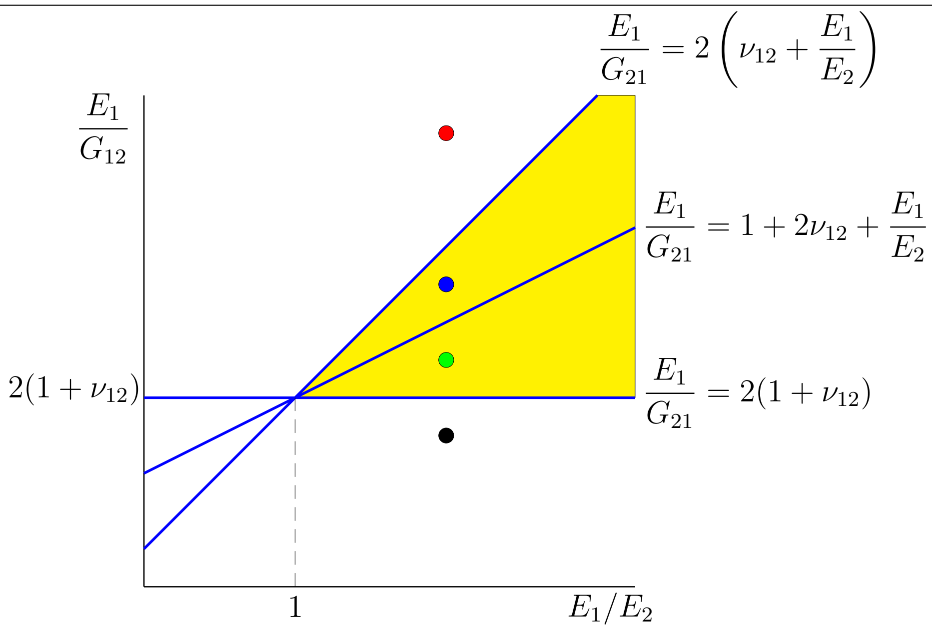


Figure 5.3: The monotonicity (5.146) domain is shown by yellow colour. Directional dependency of the Young's modulus for the cases marked by coloured dots is shown in Fig. 5.4. The equations of lines are written in the 1-2 plane.

In Fig. 5.4 the directional dependency of Young's modulus $\boldsymbol{n}_{12}(\psi_{12})$ for four cases shown in Fg. 5.3 where $E_1/E_2 = 2$ and $\nu_{12} = 0.25$, but the ratio E_1/G_{12} changes: $E_1/G_{12} = 6$ both nominator g_1 and denominator g_2 are negative, $E_1/G_{12} = 4$ nominator and denominator have different signs, $E_1/G_{12} = 3$ both nominator and denominator are positive and the inequality (5.144) is satisfied, $E_1/G_{12} = 2$ both nominator and denominator are positive, but the inequality (5.144) is violated.

In Table 5.1 the elastic constants of Balsa and Douglas fir are shown [23, Taulukko 52, sivu 165]. Humidity of the samples has been 9%. In Fig. 5.5 directional dependency of the Young's modulus on planes 1-2, 2-3 and 1-3 are shown. Direction 1 corresponts to the axial direction, i.e. the growth direction, 2 is radial- and 3 tangential direction. It can be noticed that the monotonicity consitions are only fulfilled in the 1-2 plane. In plane 2-3 the inequality $E_2/G_{23} > 2(1+\nu_{23})$ is violated, i.e with the Balsa values $1.48 \not> 3.32$, which corresponds to the black point in Fig. 5.3. Correspondngly in plane 1-3 the condition $E_1/G_{13} < 2(\nu_{13} + E_1/E_3)$ is violated, which for the Balsa values give $207.6 \not< 37.2$ 5.3. It is possible that there are measuring errors for the shear modulae G_{23} and G_{13}.

In Table 5.2 drained elasticity constants for human cortical bones of tibia and femur are shown [10, Table 11.2, page 357]. Direction 3 is the longitudinal direction of bone, 1 radial- and 2 tangential direction. Directional dependency of Young's modulae are shown in Fig. 5.6, and it can be noticed that the monotoncity conditions are satisfied in all planes.

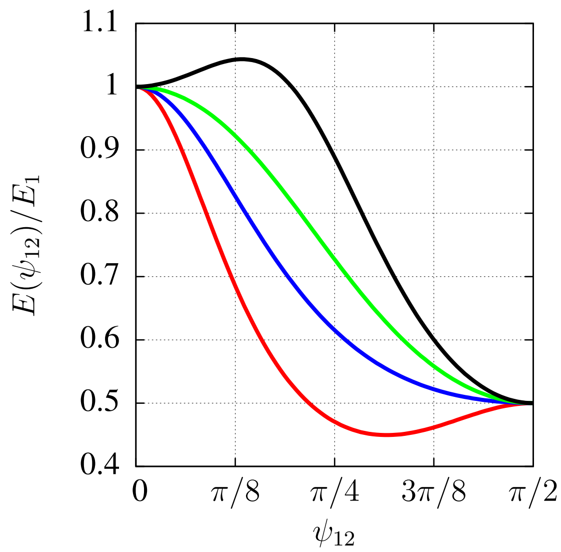


Figure 5.4: Directional dependency of Young's modulus $\boldsymbol{n}_{12}(\psi_{12})$ when $E_1/E_2 = 2$ and $\nu_{12} = 0.25$. From bottom to top the curves correspond to points $E_1/G_{12} = 6, 4, 3$, and 2, see Fig. 5.3.

Table 5.1: Elasticity constants for Balsa and Douglas fir [23, Table 52, page 165].

species	E_1 MPa	E_2 MPa	E_3 MPa	ν_{23} -	ν_{13} -	ν_{12} -	G_{23} MPa	G_{13} MPa	G_{12} MPa
Balsa	6274	296	103	0.66	0.49	0.23	200	33	310
Douglas fir	16400	1300	900	0.62	0.37	0.43	910	79	1180

Table 5.2: Drained elasticity constants for human cortical bone [10, Table 11.2, page 357].

bone	. test method	E_1 MPa	E_2 MPa	E_3 MPa	ν_{23} -	ν_{13} -	ν_{12} -	G_{23} MPa	G_{13} MPa	G_{12} MPa
tibia	mechanical	6910	8510	18400	0.14	0.12	0.49	4910	3560	2410
femur	ultrasound	12000	13400	20000	0.235	0.222	0.376	6230	5610	4530

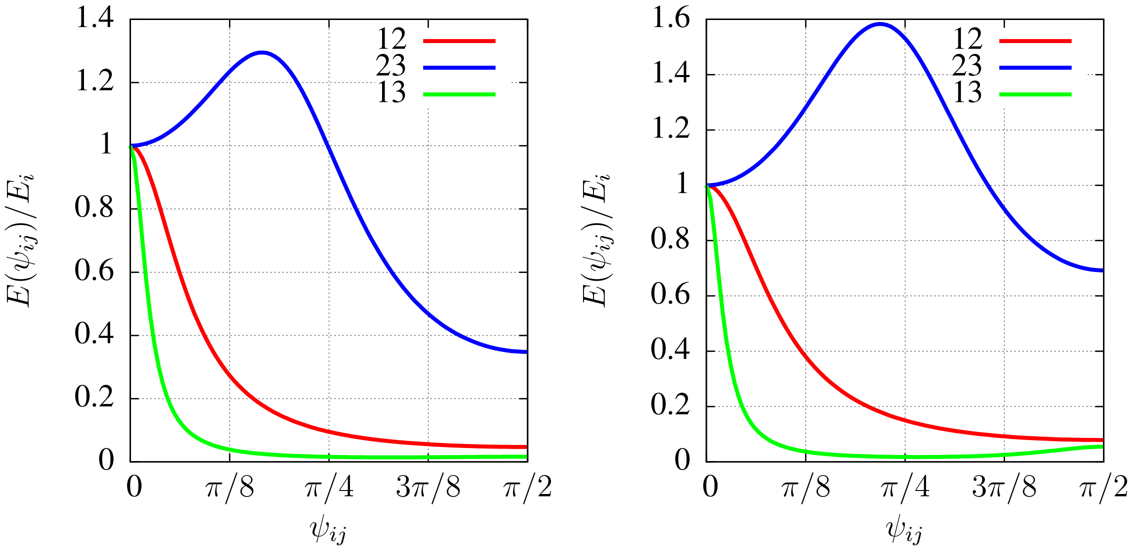


Figure 5.5: Effect of direction $\boldsymbol{n}_{ij}(\psi_{ij})$ on the Young's modulus for Balsa (LHS) and Douglas fir (RHS).

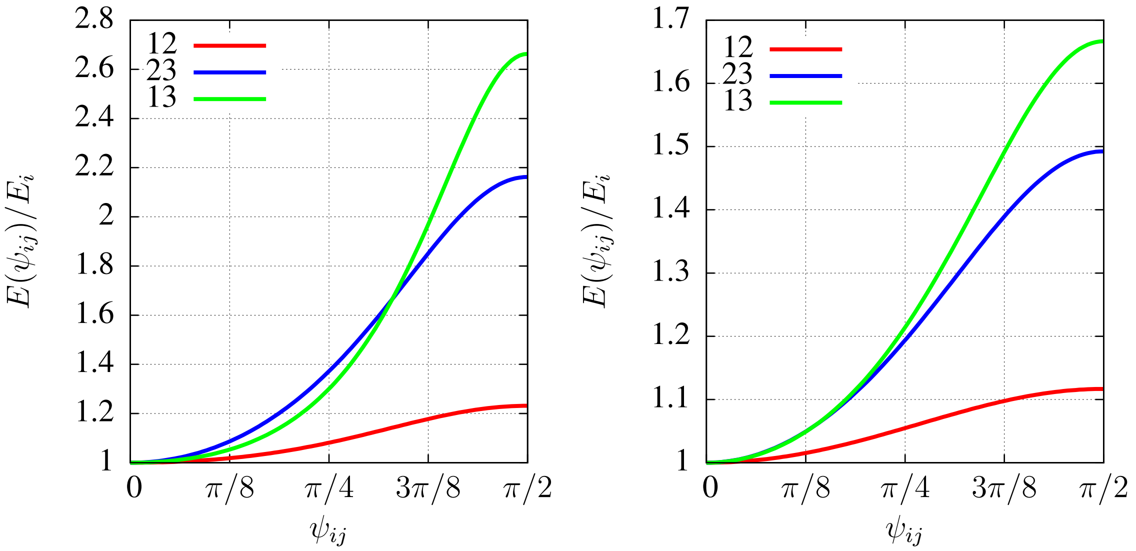


Figure 5.6: Effect of orientation $\boldsymbol{n}_{ij}(\psi_{ij})$ on Young's modulus for tibia- (LHS) and femur (RHS).

5.5 Thermoelasticity

If linear kinematics is assumed, the strain tensor is the symmetric part of the displacement gradient, see section 4.5

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left(\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^T\right). \tag{5.149}$$

If temperature changes occur they produce thermal strains $\boldsymbol{\varepsilon}^{\mathrm{te}}$ which have to be substracted from the total strains to obtain stress producing elastic strains

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{th}}). \tag{5.150}$$

For isotropic solids the thermal strains are obtained as

$$\boldsymbol{\varepsilon}^{\mathrm{th}} = \alpha(T - T_{\mathrm{ref}})\boldsymbol{I} = \alpha\Delta T\boldsymbol{I}, \tag{5.151}$$

where α is the linear coefficient of thermal expansion CTE, which usually also depends on temperature. At temperature T_{ref} the body is at its stress free state. In view of (5.23), inclusion of thermal strains yields

$$\begin{aligned}\boldsymbol{\sigma} &= K\operatorname{tr}(\boldsymbol{\varepsilon}^{\mathrm{e}})\boldsymbol{I} + 2G\boldsymbol{e} = K(\operatorname{tr}(\boldsymbol{\varepsilon}) - 3\alpha\Delta T)\boldsymbol{I} + 2G\boldsymbol{e}\\ &= \lambda\operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2G\boldsymbol{\varepsilon} - 3K\alpha\Delta T\boldsymbol{I}.\end{aligned} \tag{5.152}$$

In the case of transverse isotopy, there are two independent CTE's, one in the longitudinal direction α_L and one in the transverse isotropy plane α_T. The thermal strain in a general cordinate system can be expressed as

$$\boldsymbol{\varepsilon}^{\mathrm{th}} = (\alpha_T\boldsymbol{I} + (\alpha_L - \alpha_T)\boldsymbol{M})\,\Delta T, \tag{5.153}$$

where $\boldsymbol{M}$ is the structural tensor of transverse isotropy, see section 5.3.

For orthotropic material the thermal strain is simply in a general coordinate system

$$\varepsilon^{\mathrm{th}} = (\alpha_1\boldsymbol{M}_1 + \alpha_2\boldsymbol{M}_2 + \alpha_3\boldsymbol{M}_3)\,\Delta T, \tag{5.154}$$

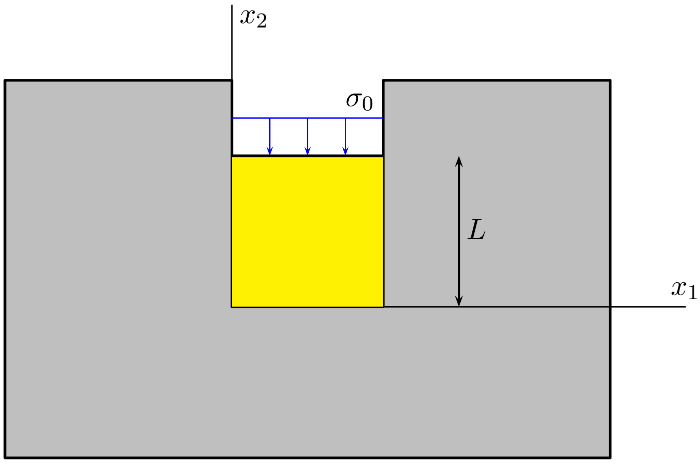
where α_i and $\boldsymbol{M}_i, i = 1, \ldots, 3$ are the CTE's in the principal axes of orthotropy and the structural tensors, respectively.

5.6 Solved example problems

Example 5.2. *The yellow cube has frictionless supports at its lower face and with two vertical faces. Two faces having normals in the x_3-axis are free as well as the upper face. Determine the stress and strain state in the cube when the temparature changes from a stress free state by the value $\Delta T = 80°C$, and a vertical pressure of $\sigma_0 = 50$ MPa is imposed to the upper face of the cube. What are the particuar values if the cube is made of*

1. *ordinary construction steel, or*
2. *aluminium 7075-T6.*

The edge length of the cube is $L = 20$ *cm.*



Solution. The constitutive equation for an isotropic linear thermoelastic solid (5.152) is now expressed in the index notation

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - 2K\alpha\Delta T\delta_{ij}.$$

Due to the boundary conditions $\varepsilon_{11} = 0$ and $\sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$. Due to the external pressure $\sigma_{22} = -\sigma_0$. There are thus three unknowns $\sigma_{11}, \varepsilon_{22} = \varepsilon_{33}$. The strains ε_{22} and ε_{33} can be solved from

$$\sigma_{22} = \lambda(\varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{22} - 3K\alpha\Delta T = -\sigma_0, \qquad (5.155)$$

$$\sigma_{33} = \lambda(\varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{33} - 3K\alpha\Delta T = 0. \qquad (5.156)$$

Substracting (5.156) from (5.155), it is obtained

$$2G(\varepsilon_{33} - \varepsilon_{22}) = \sigma_0 \quad \Rightarrow \quad \varepsilon_{33} = \frac{\sigma_0}{2G} + \varepsilon_{22}.$$

Substituting it to (5.156) gives

$$\varepsilon_{22} = -\frac{1 + \frac{\lambda}{2G}}{2(\lambda + G)}\sigma_0 + \frac{3K}{2(\lambda + G)}\alpha\Delta T = -\frac{1 - \nu^2}{E}\sigma_0 + (1 + \nu)\alpha\Delta T,$$

which yields

$$\varepsilon_{33} = \frac{\lambda}{4G(\lambda + G)}\sigma_0 + \frac{3K}{2(\lambda + G)}\alpha\Delta T = \frac{\nu(1 + \nu)}{E}\sigma_0 + (1 + \nu)\alpha\Delta T.$$

The unknown stress can now be determined from

$$\sigma_{11} = \lambda(\varepsilon_{22} + \varepsilon_{11}) - 3K\alpha\Delta T = \frac{\nu(2\nu^2 + \nu - 1)}{(1+\nu)(1-2\nu)}\sigma_0 - E\alpha\Delta T.$$

For a typical construction steel $E = 210$ GPa $\nu = 0.3$ and $\alpha = 12 \cdot 10^{-6}$ 1/°C. It results in $\sigma_{11} = -219$ MPa, $\varepsilon_{22} = 1.03 \cdot 10^{-3}, \varepsilon_{33} = 1.34 \cdot 10^{-3}$.

For alminium alloy 7075-T6: $E = 72$ GPa $\nu = 0.33$ and $\alpha = 24 \cdot 10^{-6}$ 1/°C. It results in $\sigma_{11} = -159$ MPa, $\varepsilon_{22} = 1.93 \cdot 10^{-3}, \varepsilon_{33} = 2.86 \cdot 10^{-3}$.

Example 5.3. *The complementary strain energy density is given as*

$$W^{\mathrm{c}} = a_1 J_2 + a_2 I_1 J_2,$$

where a_1 and a_2 are material parameters and $I_1 = \operatorname{tr}\boldsymbol{\sigma},\ J_2 = \frac{1}{2}\operatorname{tr}(\boldsymbol{s}^2)$.

1. *Determine the constitutive equation.*
2. *Determine the strain state under uniaxial normal stress σ_{11}.*
3. *Determine the apparent "Poisson's" ratio $\tilde{\nu} = -\varepsilon_{22}/\varepsilon_{11}$ as a function of the stress σ_{11}.*
4. *Determine the strain state under shear stress σ_{12}.*
5. *Can the Young's modulus E and shear modulus G be determined independently? Determine at least one of the parameters a_1 or a_2 as a functio of the shear modulus G. How can the other parameter to be found?*
6. *What kind of response the model predicts in pure hydrostatic stress state?*
7. *Determine the relative volume change in cases (b) and (d).*

Solution. Strains can be obtained from the complementary strain energy density by differentiating it with respect to stress

$$\boldsymbol{\varepsilon} = \frac{\partial W^{\mathrm{c}}}{\partial \boldsymbol{\sigma}} \quad \text{or in index form} \quad \varepsilon_{ij} = \frac{\partial W^{\mathrm{c}}}{\partial \sigma_{ij}} = \frac{\partial W^{\mathrm{c}}}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial W^{\mathrm{c}}}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}}.$$

Derivatives of the invariants are

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \frac{\partial \sigma_{kk}}{\partial \sigma_{ij}} = \delta_{ki}\delta_{kj} = \delta_{ij},$$

and

$$\frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \frac{1}{2}\frac{\partial s_{kl}s_{lk}}{\partial \sigma_{ij}} = \frac{1}{2}\left(\frac{\partial s_{kl}}{\partial \sigma_{ij}} + s_{kl}\frac{\partial s_{lk}}{\partial \sigma_{ij}}\right). \tag{5.157}$$

Derivative of the deviatoric stress with respect to the stress tensor is

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} - \frac{1}{3}\frac{\partial \sigma_{pp}}{\partial \sigma_{ij}}\delta_{kl} = \delta_{ki}\delta_{lj} - \tfrac{1}{3}\delta_{pi}\delta_{pj}\delta_{kl} = \delta_{ki}\delta_{lj} - \tfrac{1}{3}\delta_{ij}\delta_{kl},$$

and substituting it into (5.157) gives

$$\begin{aligned}\frac{\partial J_2}{\partial \boldsymbol{\sigma}} &= \frac{1}{2}\left[\left(\delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)s_{lk} + s_{kl}\left(\delta_{li}\delta_{kj} - \frac{1}{3}\delta_{ij}\delta_{lk}\right)\right]\\ &= \frac{1}{2}\left(s_{ji} - \frac{1}{3}\delta_{ij}s_{kk} + s_{ji} - \frac{1}{3}\delta_{ij}s_{kk}\right) = s_{ji} = s_{ij}\end{aligned}$$

since the stress tensor and its deviator are symmetric.

1. The constitutive equation is now

$$\begin{aligned}\varepsilon_{ij} &= a_1\frac{\partial J_2}{\partial \sigma_{ij}} + a_2\frac{\partial I_1}{\partial \sigma_{ij}}J_2 + a_2 I_1\frac{\partial J_2}{\partial \sigma_{ij}}\\ &= a_1 s_{ij} + a_2 J_2\delta_{ij} + a_2 I_1 s_{ij} = a_2 J_2 \delta_{ij} + (a_1 + a_2 I_1)s_{ij}.\end{aligned} \tag{5.158}$$

2. Under uniaxial stress σ_{11} the stress invariants are $I_1 = \sigma_{11}$ and $J_2 = \frac{1}{3}\sigma_{11}^2$ and the strain response is

$$\begin{aligned}\varepsilon_{11} &= a_2\tfrac{1}{3}\sigma_{11}^2 + (a_1 + a_2\sigma_{11})\tfrac{2}{3}\sigma_{11} = \tfrac{2}{3}a_1\sigma_{11} + a_2\sigma_{11}^2,\\ \varepsilon_{22} &= \varepsilon_{33} = a_2\tfrac{1}{3}\sigma_{11}^2 + (a_1 + a_2\sigma_{11})(-\tfrac{1}{3}\sigma_{11}) = -\tfrac{1}{3}a_1\sigma_{11},\\ \varepsilon_{12} &= \varepsilon_{13} = \varepsilon_{23} = 0.\end{aligned}$$

3. The apparent Poisson's ratio is immediately determined from the previous equations as

$$\tilde{\nu} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{-\frac{1}{3}a_1\sigma_{11}}{\frac{2}{3}a_1\sigma_{11} + a_2\sigma_{11}^2} = \frac{1}{2 + 3(a_2/a_1)\sigma_{11}}.$$

4. Under the shear stress σ_{12} the invariants are $I_1 = 0$ and $J_2 = \sigma_{12}^2$, the strain response is

$$\begin{aligned}&\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = a_2\sigma_{12}^2,\\ &\varepsilon_{12} = a_1\sigma_{12}, \quad \varepsilon_{13} = \varepsilon_{23} = 0.\end{aligned}$$

   Since $\varepsilon_{12} = \frac{1}{2}\gamma_{12}$ and if the engineering notation for the shear stress is used we obtain

$$\tau_{12} = \sigma_{12} = \frac{1}{2a_1}\gamma_{12}.$$

5. From the previous result the shear modulus is immediately obtained as

$$G = \frac{1}{2a_1} \quad a_1 = \frac{1}{2G}.$$

   Definition for the Young's modulus is

$$E = \left.\frac{\mathrm{d}\sigma_{11}}{\mathrm{d}\varepsilon_{11}}\right|_{\varepsilon_{11}=0} \quad \text{or inversely} \quad \frac{1}{E} = \left.\frac{\mathrm{d}\varepsilon_{11}}{\mathrm{d}\sigma_{11}}\right|_{\sigma_{11}=0} \quad \Rightarrow \quad \frac{1}{E} = \tfrac{2}{3}a_1.$$

It is clear that the shear and Young's moduli cannot be independetly determined.

It is also of interest to investigate if there exist extremum points in the stress-strain behaviour under uniaxial normal stress, case b.

$$\frac{\mathrm{d}\varepsilon_{11}}{\mathrm{d}\sigma_{11}} = \tfrac{2}{3}a_1 + 2a_2\sigma_{11} = 0 \quad \Rightarrow \quad \sigma_{11} = -\tfrac{1}{3}a_1/a_2.$$

The model predicts locking behaviour in the compressive side if the stress approaches $-\frac{1}{3}a_1/a_2$. The parameter a_2 can be estimated from a uniaxial stress-strain data if $\sigma_{11} > -\frac{1}{3}a_1/a_2$. Assuming now that the a_1 parameter is determined from the shear test, and denoting $a_2 = \alpha_2/G^2$, where α_2 is a dimensionless parameter, it is obtained

$$\begin{aligned}\varepsilon_{11} &= \tfrac{2}{3}a_1\sigma_{11} + a_2\sigma_{11}^2 \\ &= \tfrac{1}{3}\frac{\sigma_{11}}{G} + \alpha_2\left(\frac{\sigma_{11}}{G}\right)^2.\end{aligned}$$

The apparent Poisson's ratio is

$$\tilde{\nu} = \frac{1}{2 + 6\alpha_2\sigma_{11}/G}.$$

6. In hydrostatic stress state $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_0$ and thus $I_1 = 3\sigma_0$ and $s_{ij} = 0, J_2 = 0$ resulting $\varepsilon_{ij} = 0$. The behaviour in pure hydrostatic loading is incompressible.
7. Relative volume change in the case 2, uniaxial normal stress σ_{11}, is

$$\varepsilon_{\text{vol}} = \varepsilon_{kk} = \tfrac{2}{3}a_1\sigma_{11} + a_2\sigma_{11}^2 - 2\cdot\tfrac{1}{3}a_1\sigma_{11} = a_2\sigma_{11}^2,$$

and in the case 4, shear, is $\varepsilon_{kk} = 3a_2\sigma_{12}^2$. The volumetric behaviour under shear stress produces either increase or decrease of volme depending on the sign of a_2. Such behaviour can be observed in soil mechanics for example for dense sands and overconsoldated clays. However, this model in general is not very appealing due to other deficiences observed in case 6.

5.7 Exercises

1. The strain energy density W of a linearly elastic isotropic solid cen be expressed in the following equivalent forms

$$\begin{aligned}W_1 &= \tfrac{1}{2}a_1I_1^2 + b_1I_2, \\ W_2 &= \tfrac{1}{2}a_2I_1^2 + b_2J_2, \\ W_3 &= \tfrac{1}{2}a_3I_1^2 + b_3\tilde{I}_2,\end{aligned}$$

where I_1, I_2 are the linear and quadratic invariants of the infinitesimal strain tensor, $I_1 = \operatorname{tr} \boldsymbol{\varepsilon} = \varepsilon_{kk}, I_2 = \frac{1}{2}(\operatorname{tr}(\boldsymbol{\varepsilon}^2) - I_1^2)$ and J_2 is the quadratic invariant of the deviatorc part of the infinitensimal strain tensor $\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{1}{3}I_1\mathbf{I}$, $J_2 = \frac{1}{2}\operatorname{tr}(\boldsymbol{e}^2)$ and $\tilde{I}_2$ is a generic quadratic invariant $\tilde{I}_2 = \frac{1}{2}\operatorname{tr}(\boldsymbol{\varepsilon}^2)$.

Determine the coefficients a_i, b_i as a function of the Lamé constants $\lambda, \mu = G$ and alternatively using E, G and K.

2. As it can be noticed from exercise 5.3, construction of a well behaving non-linear isotropic elastic constitutive equation is a demanding task. If it is observed that the shear modulus depends on the volumetric deformation as

$$G(I_1) = G_0(1 - \alpha I_1^2),$$

where α is a dimensionless constant and I_1 the linear invariant of the strain tensor $I_1 = \operatorname{tr} \boldsymbol{\varepsilon}$. Try to formulate a hyperelastic constitutive equation in the form

$$\boldsymbol{\sigma} = k(I_1, J_2)\boldsymbol{I} + 2G_0(1 - \alpha I_1^2)\boldsymbol{e},$$

where $\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}$ is the strain deviator, i.e. find $k(I_1, J_2)$ such that the model is hyperelastic.

Investiate the behaviour of such model in the following deformation states. Draw the response curves.

(a) Uniaxial compression/tension, e.g. $\sigma_{11} \neq 0$. Determine and draw also $\tilde{\nu} = -\varepsilon_{22}/\varepsilon_{11}$.
(b) Shearing, e.g. $\sigma_{12} \neq 0$. What is the relative volume change in shear?
(c) Hydrostatic stress state superimposed with a shear in one direction.

You can choose the ratio between the bulk modulus and the shear modulus be 8/3 at the initial state, corresponding to the Poisson's ratio 1/3 for a linear isotropic solid.

3. The most general isotropic elastic constitutive equation is of the form

$$\boldsymbol{\sigma} = a_0\mathbf{I} + a_1\boldsymbol{\varepsilon} + a_2\boldsymbol{\varepsilon}^2,$$

where the coefficients a_i can depend of the invariants of the strain tensor $\boldsymbol{\varepsilon}$. The strain energy function W with respect to unit volume can be written as a function of the invariants of the strain tensor $\boldsymbol{\varepsilon}$ and its deviator $\boldsymbol{e}$, i.e. I_1, J_2 $J_3 = \det \mathbf{e}$

$$W = W(I_1, J_2, J_3).$$

(a) Determine the coefficients a_i expressed in terms of the derivatives of W.

(b) In order to be able to write the constitutive equation in the form $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ it is mandatory to have $a_2 \equiv 0$. Formulate a non-linear isotropic elastic constitutive equation using the bulk modulus K and shear modulus G, which are functions of the invariants I_1 and J_2:

$$K = K(I_1, J_2) \qquad \text{and} \qquad G = G(I_1, J_2).$$

In order to have a nyperelastic model K and G are not independent functions of I_1 and J_2. Derive the conditions they have to satisfy.

(c) For metals a good approximation is to assume the volumetric behaviour linearly elastic and that the shear modulus depends only on J_2. Then the volumetric and deviatoric behaviours are independent of each other. Assume now the relation

$$G(J_2) = G_0(1 + \alpha J_2),$$

where α is a dimensionless material parameter. Determine the strain energy function W. Determine and draw the stress-strain curve in a uniaxial stress state (e.g. σ_{11} and σ_{12}) using different values of α.

Hint: If a symmetric second order tensor $\boldsymbol{A}$ is deviatoric i.e. $\mathrm{tr}\,\boldsymbol{A} = 0$, then $J_3 = \det\,\boldsymbol{A} = \frac{1}{3}\mathrm{tr}(\boldsymbol{A}^3)$.

4. The material parameters for a transversally isotropic graphite epoxy laminate plate are in the transverse isotropy plane $E_T = 9.65$ GPa, $\nu_T = 0.6$ and in the longitudinal direction $E_L = 148$ GPa, $G_L = 4.55$ GPa, and $\nu_L = 0.3$. Is the set thermodynamically admissible? Does it satisfy the monotonicity conditions?

5. If $\boldsymbol{A}$ is a symmetric second order tensor and $\boldsymbol{m}$ is a vector. Show that the invariant $I_4 = \boldsymbol{m}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{m} = m_i A_{ij} m_j$ can also be written as $I_4 = \mathrm{tr}(\boldsymbol{A}\boldsymbol{M})$, where $\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}$, or in index notation $M_{ij} = m_i m_j$.

6. As in the previous problem $\boldsymbol{A}$ is a symmetric second order tensor and $\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^{\mathrm{T}}$ is the structural tensor where $\boldsymbol{m}$ is a unit vector. The invariant basis formed by the two elements, the unit vector $\boldsymbol{m}$ and tensor $\boldsymbol{A}$ have the following invariant elements

$$I_1 = \mathrm{tr}\,\boldsymbol{A}, \quad I_2 = \frac{1}{2}\mathrm{tr}(\boldsymbol{A}^2), \quad I_2 = \frac{1}{3}\mathrm{tr}(\boldsymbol{A}^3), \quad I_4 = \mathrm{tr}(\boldsymbol{A}\boldsymbol{M}), \quad I_5 = \mathrm{tr}(\boldsymbol{A}^2\boldsymbol{M}). \tag{5.159}$$

Show that the invariant $\tilde{I} = \mathrm{tr}(\boldsymbol{A}^3\boldsymbol{M})$ is reducible i.e. it can be written in terms of elements in the base (5.159).

Hint. Use the Cayley-Hamilton theorem which says that the tensor itself stisfies its characteristic polynomial

$$-\lambda^3 + \hat{I}_1\lambda^2 + \hat{I}_2\lambda + \hat{I}_3 = 0,$$

where $\hat{I}_i$ are the principal invariants of a tensor $\boldsymbol{A}$:

$$\hat{I}_1 = \mathrm{tr}\,\boldsymbol{A}, \quad \hat{I}_2 = \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{A}^2) - (\mathrm{tr}\,\boldsymbol{A})^2\right], \quad \hat{I}_3 = \det\,\boldsymbol{A}.$$

7. Investigate a unidirectionally reinforced material. For simplicity assume that the fibre cross-sections are squares having a side length $2b$. The fibres are in a regular lattice and the distance between the fibre centres is $2a$. The cross-section area of one fible is thus $4b$ and a reprerentative area could be $A_{\mathrm{r}} = 4a^2$. Assuming the material of the fibre and matrix isotropic having Young's moduli $E_{\mathrm{f}}, E_{\mathrm{m}}$ and Poisson's ratios $\nu_{\mathrm{f}}, \nu_{\mathrm{m}}$, respectively. Denote the volume fraction of the fibres as $f = (b/a)^2$ which is the same as the area ratio. Try to estimate the homogenised transversely isotropic material parameters E_L, E_T, G_L, ν_L and

ν_T expressed in terms of the corresponding values of the fibres are matrix and the volume ratio.

Hint. Consider a representative volume element (RVE) as a cube. Load the RVE either by a constant strain or a constant stress in proper directions and solve the average values of stress and strain, respectively.

Chapter 6
Elasto-plastic constitutive models

6.1 Introduction

On the contrary to elastic behaviour, the characteristic feature of plastic behaviour is irreversibility. If an elastic-plastic solid is first stressed above the elastic treshold and then the stress is removed, permanent strains are generated.

In the analysis of elasto-plastic behaviour of solids, three set of equations will be required to complete the analysis.

1. *Yield criterion*, to define the *borderline between elastic and plastic* behaviour.
2. *Flow rule*, which describe *how* the plastic strains evolve,
3. *Hardening rule*, which models the *change of the yield criterion* with evolving plastic strains.

These lecture notes give only introduction to the modelling of plasticity. More thorough exposition can be found in [14, 8, 26, 27, 31, 17].

6.2 Yield criteria

For an initially isotropic solid the yield criterion can only depend of the invariants of the stress tensor and possibly some parameters. Since the principal stresses form a valid set of invariants, the yield criterion can be expressed

$$f(\sigma_1, \sigma_2, \sigma_3) = 0. \tag{6.1}$$

Alternatively, the principal invariants of the stress tensor can be used. However, the yield function is usually expressed by using the set I_1, J_2 and $\cos 3\theta$, since they give a clear physical intepretation of the stress state.

To have a picture on the shape of the yield surface, it is advisable to determine its trace on the deviatoric- and meridian planes. On the meridian plane, the deviatoric radius

$\rho = \sqrt{2J_2}$, or the *effective stress* $\sigma_\mathrm{e} = \sqrt{3J_2}$, is shown as a function of the mean stress σ_m, or I_1, at certain value of the Lode angle θ on the deviatoric plane. Three meridian planes are of special interest: (i) the tensile meridian, (ii) the compressive meridian and (iii) the shear meridian. To give a physical meaning of these meridian planes, let's order the principal stresses as $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Therefore the intermediate principal stress can be expressed as a linear combination of the extreme ones, i.e.

$$\sigma_2 = (1-\alpha)\sigma_1 + \alpha\sigma_3, \quad \text{where} \quad 0 \leq \alpha \leq 1. \tag{6.2}$$

All stress states can therefore be expressed with the α-values in the range $[0, 1]$. The mean stress and the principal deviatoric stresses are

$$\sigma_\mathrm{m} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3}[(2-\alpha)\sigma_1 + (1+\alpha)\sigma_3], \tag{6.3}$$
$$s_1 = \sigma_1 - \sigma_\mathrm{m} = \tfrac{1}{3}(1+\alpha)(\sigma_1 - \sigma_3), \tag{6.4}$$
$$s_2 = \sigma_2 - \sigma_\mathrm{m} = \tfrac{1}{3}(1+2\alpha)(\sigma_1 - \sigma_3), \tag{6.5}$$
$$s_3 = \sigma_3 - \sigma_\mathrm{m} = \tfrac{1}{3}(\alpha-2)(\sigma_1 - \sigma_3). \tag{6.6}$$

The Lode angle has the expression (2.60)

$$\cos\theta = \frac{\sqrt{3}}{2}\frac{s_1}{\sqrt{J_2}} = \frac{1}{2}\frac{1+\alpha}{\sqrt{1-\alpha+\alpha^2}}. \tag{6.7}$$

Tensile meridian corrsponds to a stress state where a uniaxial tensile stress is superimposed to a hydrostatic stress state, thus $\sigma_1 > \sigma_2 = \sigma_3$, giving the value $\alpha = 1$ and the Lode angle $\theta = 0°$.

Compressive meridian corresponds to a stress state where a uniaxial compressive stress is superimposed to a hydrostatic stress state, thus $\sigma_1 = \sigma_2 > \sigma_3$, resulting into the value $\alpha = 0$ and the Lode angle $60°$.

Shear meridian is obtained when $\alpha = \frac{1}{2}$, thus corresponding to a stress state where a shear stress in the $1-3$-plane is superimposed to a hydrostatic stress state. The Lode angle has the value $\theta = 30°$.

For initially isotropic elastic solids, the yield criteria can be classified in two groups: (i) pressure independent and (ii) pressure dependent criteria. In this lecture notes only the two most important pressure independent yield criterion of Tresca and von Mises are described. Also their generalizations to pressure dependent forms which are the Drucker-Prager and Mohr-Coulomb yield criterion, respectively, are dealt with.

If the yield condition do not depend on the Lode angle θ, the trace of the yield surface in the deviatoric plane is circular. In general, for isotropic material the yield locus on the deviatoric plane is completely described in the sector $0 \le \theta \le 60°$. If both σ_{ij} and $-\sigma_{ij}$ will cause initial yield of a given material, as it is characteristic for metals, the yield curve in the deviatoric plane have symmetry about $\theta = 30°$, which implies that the tensile and compressive meridians have the same distrance from the hydrostatic axis. For a more detailed discussion on the symmetry properties of the yield surface see [31, section 8.2].

6.2.1 Tresca's yield criterion

For metals yielding is primarily due to slip in the crystal lattice. Tresca's criterion states that plastic deformations occur when the maximun shear stress attains a critical value

$$\tau_{\max} - k = 0,$$

where k is the yield stress in shear. Since $\tau_{\max} = (\sigma_1 - \sigma_3)/2$,in uniaxial tension the criterion has the form

$$\sigma_1 - 2k = 0, \quad \text{i.e.} \quad \sigma_1 - \sigma_\text{y} = 0,$$

where σ_y is the yield stress in uniaxial stress state. Notice, that similar expression is also obtained in uniaxial compression. Tresca's criterion do not depend on the hydrostatic pressure, i.e. on the first invariant of the stress tensor I_1.

6.2.2 Von Mises yield criterion

For metals the most used yield criterion is von Mises criterion, which can be written as

$$\sqrt{J_2} - k = 0, \tag{6.8}$$

where k is the yield stress in shear. Often, the criterion is given in the form

$$\sqrt{3J_2} - \sigma_\text{y} = 0, \quad \text{in short} \quad \sigma_\text{e} - \sigma_\text{y} = 0, \tag{6.9}$$

where σ_y is the yield stress in uniaxial tension/compression. The notation $\sigma_\text{e} = \sqrt{3J_2}$ is known as the *effective stress*. It is easily seen that the ratio between the uniaxial and shear yield stresses is $\sqrt{3} \approx 1,732$.

Von Mises yield criterion can be viewed in the principal stress space as a circular cylinder around the hydrostatic axis, and its cut with the surface $\sigma_3 = 0$ (plane stress state) is ellipse

$$\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} - \sigma_\text{y} = 0. \tag{6.10}$$

If the only nonzero components of the stress tensor are $\sigma_x = \sigma$ and $\tau_{xy} = \tau$, the yield criterion has the form

$$\sqrt{\sigma^2 + 3\tau^2} - \sigma_\text{y} = 0. \tag{6.11}$$

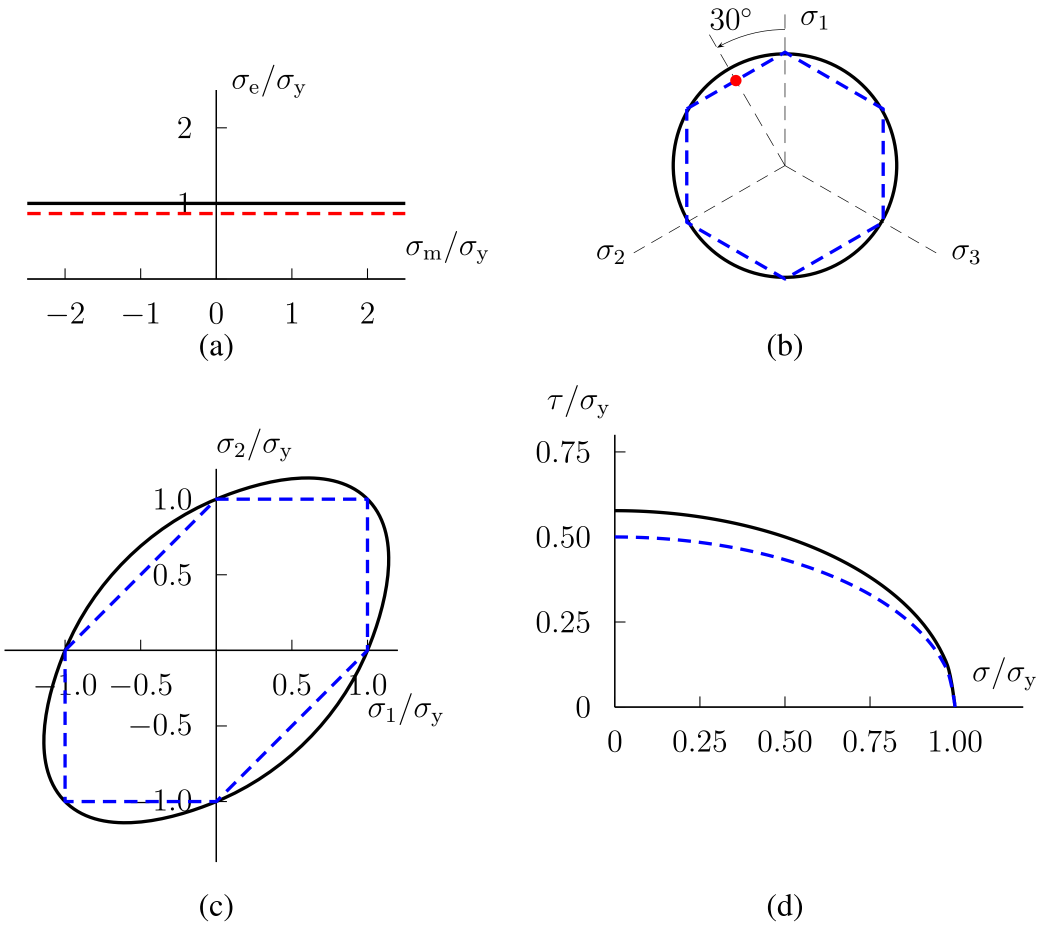


Figure 6.1: Von Mises (black) and Tresca (blue dashed lines) yield criteria. (a) in meridian plane (the shear meridian of Tresca criterion is drawn with a red line), (b) on the π-plane, (c) in plane stress state and (d) for (σ, τ)-stresses. The uniaxial tensile stress is matched, thus the tensile- and compressive meridians of Tresca and von Mises criteria coincide.

6.2.3 Drucker-Prager yield criterion

Drucker-Prager yield criterion, presented in 1952, is the most simple generalisation of the von-Mises criterion for pressure dependent plastic materials. In the deviatoric plane its shape is a circle with radius depending on the hydrostatic stress. Expressed by invariants I_1 and J_2, the criterion can be written in the form

$$f(I_1, J_2) = \sqrt{3J_2} + \alpha I_1 - \beta = \sigma_e + 3\alpha\sigma_m - \beta = 0, \tag{6.12}$$

or alternatively written in terms of I_1, ρ

$$f(I_1, \rho) = \rho + \sqrt{2/3}\alpha I_1 - \sqrt{2/3}\beta = 0. \tag{6.13}$$

The criterion is reduced to the von Mises criterion when $\alpha = 0$. Drucker-Prager (DP) yield criterion describes a linear dependency of yield on the hydrostatic stress and thus its ability to describe the plastic behaviour of pressure dependent real materials is very limited. The shape of DP-yield criterion on the meridian plane is a straight line, see fig. 6.2

The two material parameters α ja β can be determined e.g. by using two of the following four experiments: (i) uniaxial compression (f_c), (ii) uniaxial tension (f_t), (i) equibiaxial compression (f_{bc}), or (iv) equibiaxial tension (f_{bt}). Values of these material strengths can be expressed wit parameters α and β as

$$f_c = \frac{\beta}{1-\alpha}, \qquad f_t = \frac{\beta}{1+\alpha}, \tag{6.14}$$

$$f_{bc} = \frac{\beta}{1-2\alpha}, \qquad f_{bt} = \frac{\beta}{1+2\alpha}. \tag{6.15}$$

If the uniaxial and equibiaxial compressive strengths are known, the values for α and β are

$$\alpha = \frac{f_{bc} - f_c}{2f_{bc} - f_c} = \frac{(f_{bc}/f_c) - 1}{2(f_{bc}/f_c) - 1}, \quad \beta = (1-\alpha)f_c. \tag{6.16}$$

Alternatively, if the uniaxial strengths are known, the following expressions will be obtained

$$\alpha = \frac{f_c - f_t}{f_c + f_t}, \quad \beta = (1-\alpha)f_c. \tag{6.17}$$

If the ratio of uniaxial compressive strength with respect to the uniaxial tensile strength is denoted by m, $f_c = mf_t$, the expressions are

$$\alpha = \frac{m-1}{m+1}, \quad \beta = \frac{2}{m+1}f_t. \tag{6.18}$$

In the plane stress state ($\sigma_3 \equiv 0$) DP-criterion has the form

$$\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} + \alpha(\sigma_1 + \sigma_2) - \beta = 0, \tag{6.19}$$

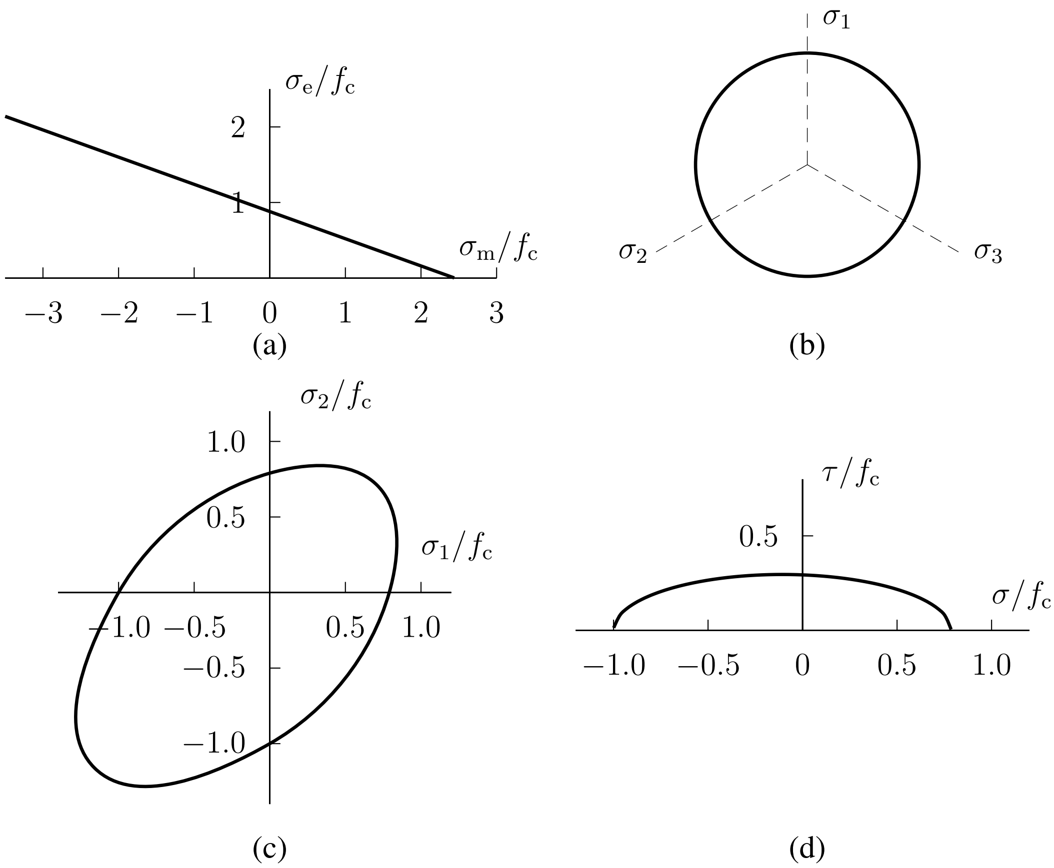


Figure 6.2: Drucker-Prager yield criterion: (a)on meridian plane, (b) on the π-plane, (c) in the plane stress state, (d) for (σ, τ)-stresses. In the figures the relation between the equibiaxial and compressive yield stressea is $f_{\rm bc} = 1.16 f_{\rm c}$, which implies $\alpha = 0.12$ and $\beta = 0.88 f_{\rm c}$.

which presents an ellipse in the (σ_1, σ_2)-plane, whose main axis makes 45°-angle with the σ_1-axis, see fig. 6.2c.

If the only nonzero components of the stress tensor are σ and τ, the DP-criterion expressed in terms of the uniaxial material strengths as follows

$$\sqrt{\sigma^2 + 3\tau^2} + \frac{m-1}{m+1}\sigma - \frac{2}{m+1} f_c = 0, \tag{6.20}$$

which is shown in fig. 6.2d.

6.2.4 Mohr-Coulomb yield criterion

Mohr-Coulomb yield criteria can be understood as a generalisation of Tresca's criterion to pressure dependent plastc materials.

Coulomb's criterion, dating back to the year 1773, is the oldest known yield or failure criterion. It assemes a linear relationship between the extreme principal stresses ($\sigma_1 \geq \sigma_2 \geq \sigma_3$)

$$m\sigma_1 - \sigma_3 - f_c = 0, \tag{6.21}$$

where $m = f_c/f_t$. Using the Mohr's circles, the criterion can be written also as

$$|\tau| + \mu\sigma - c = 0, \tag{6.22}$$

where the two material constans are μ and c. From the figure 6.3 it is obtained

$$\mu = \tan\phi, \tag{6.23}$$

where ϕ friction angle. For frictionless materials ($\phi = 0$) and the Mohr-Coulomb citerion (6.22) is reduced to the maximum shear criterion and the cohesion parameters c is equal to the yield stress in shear k.

Under pure hydrostatic stress $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and using equation (6.21), the following equation is obtained

$$\sigma = \frac{f_c}{m-1} = \frac{c}{\mu}. \tag{6.24}$$

The relation between the friction angle and the uniaxial strengths is

$$m = \frac{f_c}{f_t} = \frac{1+\sin\phi}{1-\sin\phi}. \tag{6.25}$$

Very usefull are also the relations

$$\mu = \tan\phi = \frac{m-1}{2\sqrt{m}}, \tag{6.26}$$

and

$$c = \frac{f_c}{2\sqrt{m}}. \tag{6.27}$$

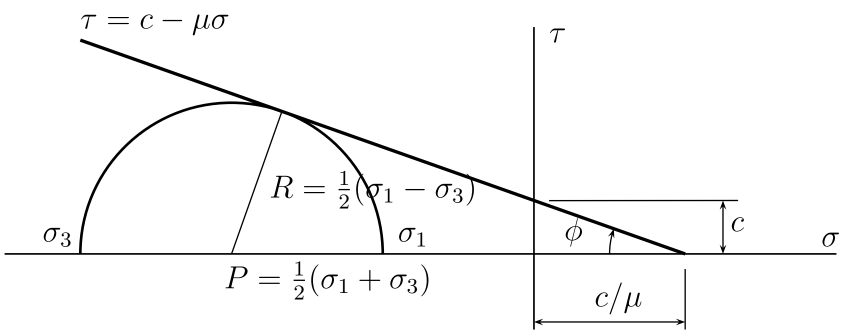


Figure 6.3: Mohr's circles and Coulomb's yield criterion.

Let's determine the equations for the straight meridian lines. Expressions of the invariants I_1 and ρ on the compressive meridian ($\sigma_1 = \sigma_2 > \sigma_3, \theta = 60°$) are

$$I_{1c} = 2\sigma_1 + \sigma_3, \quad \rho_c = \sqrt{2J_{2c}} = \sqrt{\tfrac{2}{3}}(\sigma_1 - \sigma_3). \tag{6.28}$$

Expressing the principal stresss σ_1 and σ_3 in terms of I_{1c} and ρ_c and substituting them into equation (6.21), the expression for the compressive meridian line is

$$\rho_c + \sqrt{\tfrac{2}{3}}\,\frac{m-1}{m+2}\,I_{1c} - \frac{\sqrt{6}}{m+2}\,f_c = 0, \qquad \text{or} \tag{6.29}$$

$$\sigma_e + 3\frac{m-1}{m+2}\sigma_m - \frac{3}{m+2}\,f_c = 0. \tag{6.30}$$

On the tensile meridian ($\sigma_1 > \sigma_2 = \sigma_3, \theta = 0°$) the expressions for the invariants are

$$I_{1t} = \sigma_1 + 2\sigma_3, \quad \rho_t = \sqrt{2J_{2t}} = \sqrt{\tfrac{2}{3}}(\sigma_1 - \sigma_3), \tag{6.31}$$

and the following equation for the tensile meridian is obtained

$$\rho_t + \sqrt{\tfrac{2}{3}}\,\frac{m-1}{2m+1}\,I_{1t} - \frac{\sqrt{6}}{2m+1}\,f_c = 0, \qquad \text{or} \tag{6.32}$$

$$\sigma_e + 3\frac{m-1}{2m+1}\sigma_m - \frac{3}{2m+1}\,f_c = 0. \tag{6.33}$$

Eliminating the invariant $I_1 = I_{1t} = I_{1c}$, the ratio between the radius of compressive and tensile meridians is obtained

$$\frac{\rho_c}{\rho_t} = \frac{2m+1}{m+2} = \frac{3+\sin\phi}{3-\sin\phi}. \tag{6.34}$$

The shape of the yield surface on the deviatoric plane is thus dependent on the ratio between the uniaxial strengths m.

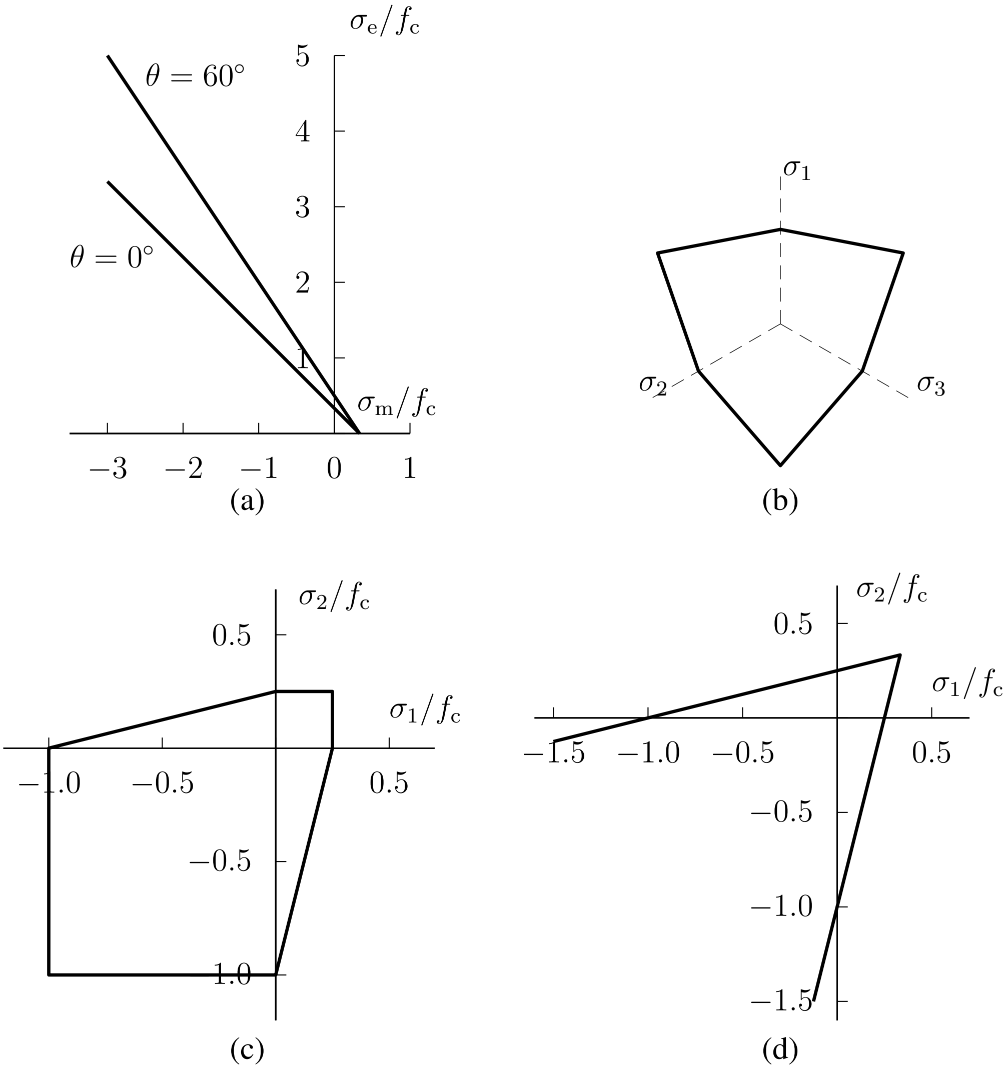


Figure 6.4: Illustrations of Mohr-Coulomb yield criterion when $m = 4$: (a) on meridian plane, (b) on the π-plane, (c) in the plane stress state and (d) in the plane strain state ($\nu = 1/3$).

6.3 Flow rule

Evolution equations for the plastic flow are assumed to be given in the following form

$$\dot{\varepsilon}^{\mathrm{p}}_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}} \qquad \text{and} \qquad \dot{\kappa}^{\alpha} = -\dot{\lambda}\frac{\partial g}{\partial K^{\alpha}}, \tag{6.35}$$

where g is the plastic potential, a function depending on the stress $\boldsymbol{\sigma}$ and the hardening parameters K^{α}. The factor λ is called the plastic multiplier. If a yield function is used for the plastic potential, the flow rule is called *associated*, otherwise it is called *non-associated*.

During the plastic deformation process the point of stress stays on the yield surface, thus $f(\sigma_{ij}, K^{\alpha}) = 0$ and also

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^{\alpha}}\dot{K}^{\alpha} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial K^{\alpha}}\frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}\dot{\kappa}^{\beta} = 0. \tag{6.36}$$

The equation above is called as the *consistency condition*. Inserting the evolution equation of the hardening variable κ in eq. (6.35) into the consistency condition (6.36), the result is

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} - \dot{\lambda}H, \tag{6.37}$$

where H is the plastic hardening modulus

$$H = \frac{\partial f}{\partial K^{\alpha}}\frac{\partial K^{\alpha}}{\partial \kappa^{\beta}}\frac{\partial g}{\partial K^{\beta}}. \tag{6.38}$$

Taking the time derivative of the constitutive equation

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^{\mathrm{p}}_{ij}) = C_{ijkl}(\dot{\varepsilon}_{ij} - \dot{\lambda}\frac{\partial g}{\partial \sigma_{kl}}). \tag{6.39}$$

Multiplying the above equation by parts from the left with the gradient of the yield surface, i.e. $\partial f/\partial \sigma_{ij}$, it is obtained

$$\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = \frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\dot{\varepsilon}_{kl} - \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\frac{\partial g}{\partial \sigma_{kl}}. \tag{6.40}$$

Taking the consistency condition (6.37) into account results in

$$\dot{\lambda} = \frac{1}{A}\frac{\partial f}{\partial \sigma_{ij}}C_{ijkl}\dot{\varepsilon}_{kl}. \tag{6.41}$$

Substituting the expression for the rate of the plastic multiplier back to the constitutive equation, gives

$$\dot{\sigma}_{ij} = C_{ijkl}\left(\dot{\varepsilon}_{kl} - \frac{1}{A}\frac{\partial f}{\partial \sigma_{mn}}C_{mnpq}\dot{\varepsilon}_{pq}\frac{\partial g}{\partial \sigma_{kl}}\right), \tag{6.42}$$

which after some rearrangements become

$$\dot{\sigma}_{ij} = \left(C_{ijkl} - \frac{1}{A} C_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} C_{pqkl} \right) \dot{\varepsilon}_{kl}, \tag{6.43}$$

defining the elastic-plastic constitutive operator as

$$C^{\text{ep}}_{ijkl} = C_{ijkl} - \frac{1}{A} C_{ijmn} \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{pq}} C_{pqkl}. \tag{6.44}$$

6.4 Hardening rule

In the previous example the evolution of the hardening variable K was not defined and the hardening expressed as increase of the yield stress in the form $\sigma_{\text{y}} = \sigma_{\text{y0}} + K(\kappa)$ results in isotropic expansion of the yield curve in the deviatoric plane, see fig. 6.5. Thus this type of hardening is called as *isotropic hardening*. Considering a material which is first loaded in the plastic region to a stress σ_{y}. In subsequent reversed loading the yield starts at the stress state $-\sigma_{\text{y}}$ if the material obeys the isotropic hardening rule. However, for metals lowering of the yield stress in reversed loading is observed. This phenomenon is known as *Bauschinger effect*, and *kinematic hardening* rules have been developed to model it. In ideal kinematic hardening, the size of the yield surface do not change, while the yield surface moves in the stress space, see fig. 6.5.

Some materials show change of the yield surface shape when plastically deformed. Such third type of hardening is called *distortional* or *anisotropic hardening*.

6.4.1 Isotropic hardening

Let us consider von Mises solid and assume that the yield stress σ_{y} is a function of a scalar internal variable κ as $\sigma_{\text{y}} = \sigma_{\text{y0}} + K(\kappa)$, where σ_{y0} is the yield stress of a virgin material. The yield condition is thus

$$f = \sqrt{3J_2} - (\sigma_{\text{y0}} + K(\kappa)) = 0. \tag{6.45}$$

Assuming associated flow, the evolution equations for the plastic variables are

$$\dot{\varepsilon}^{\text{p}}_{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda} \frac{3}{2} \frac{s_{ij}}{\sigma_{\text{y}}}, \tag{6.46}$$

$$\dot{\kappa} = -\dot{\lambda} \frac{\partial f}{\partial K} = \dot{\lambda}. \tag{6.47}$$

The plastic hardening modulus is

$$H = \frac{\partial f}{\partial K^\alpha} \frac{\partial K^\alpha}{\partial \kappa^\beta} \frac{\partial f}{\partial K^\beta} = \frac{\partial \sigma_{\text{y}}}{\partial \kappa}. \tag{6.48}$$

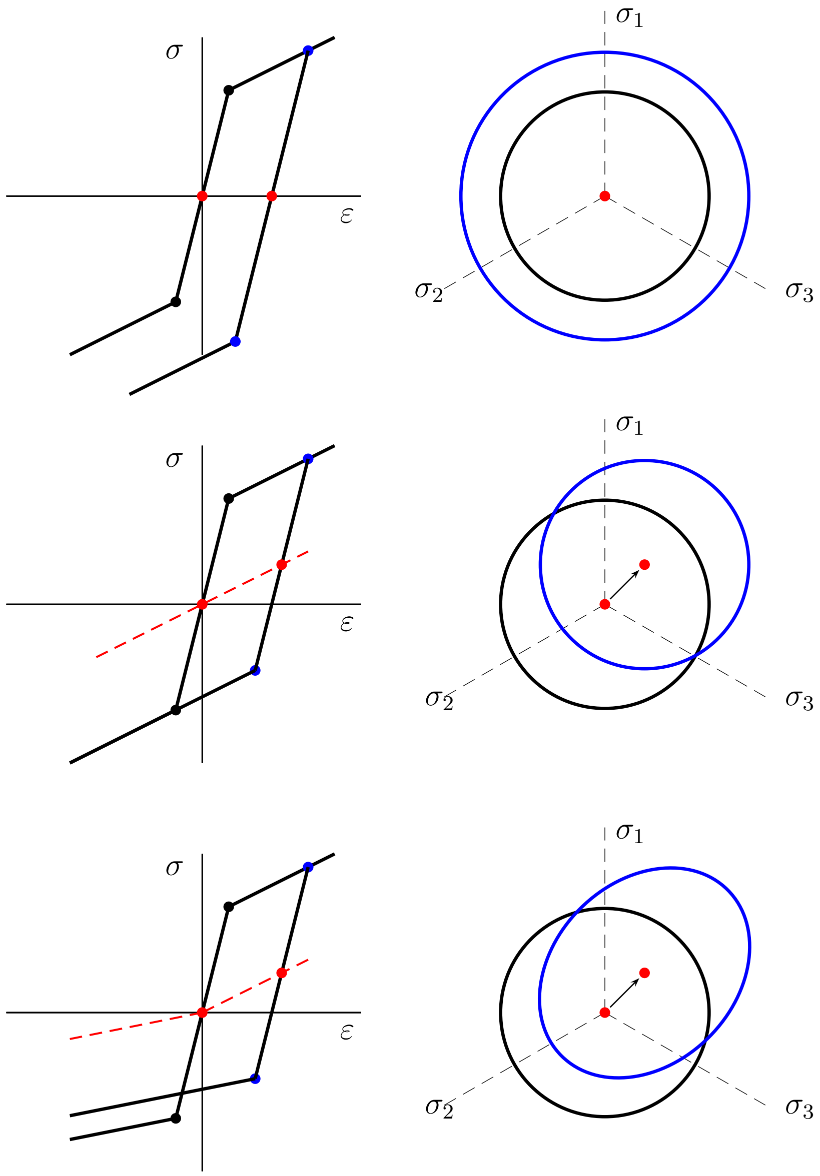


Figure 6.5: Linear isotropic (top), kinematic (middle) and kinematic + anisotropic hardening (bottom).

To make different strain evolutions in some sense comparable, let's define an equivalent plastic strain as

$$\bar{\varepsilon}^{\rm p} = \int \dot{\bar{\varepsilon}}^{\rm p}\,{\rm d}t, \qquad \text{where} \qquad \dot{\bar{\varepsilon}}^{\rm p} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}^{\rm p}_{ij}\dot{\varepsilon}^{\rm p}_{ij}}. \tag{6.49}$$

For von Mises model the plastic deformation is incompressible, i.e. $\operatorname{tr}\boldsymbol{\varepsilon}^{\rm p} = 0$, as can be seen from the flow rule (6.46), in uniaxial tension/compression test in the x_1-direction, the plastic part of the strain rate tensor has the following non-zero components

$$\varepsilon^{\rm p}_{11}, \quad \dot{\varepsilon}^{\rm p}_{22} = \dot{\varepsilon}^{\rm p}_{33} = -\tfrac{1}{2}\dot{\varepsilon}^{\rm p}_{11}. \tag{6.50}$$

The equivalent plastic strain rate is $\dot{\bar{\varepsilon}}^{\rm p} = \dot{\varepsilon}^{\rm p}_{11}$ and thus the equivalent plastic strain coincides to the uniaxial plastic strain.

Taking the flow rule (6.46) into account results in

$$\dot{\bar{\varepsilon}}^{\rm p} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}^{\rm p}_{ij}\dot{\varepsilon}^{\rm p}_{ij}} = \sqrt{\frac{3}{2}\frac{s_{ij}s_{ij}}{\sigma_{\rm y}^2}}\,\dot{\lambda} = \dot{\lambda}. \tag{6.51}$$

Thus we have obtained for associated flow of von Mises solid an important result that

$$\kappa = \lambda = \bar{\varepsilon}^{\rm p}. \tag{6.52}$$

Therefore the hardening modulus is

$$H = \frac{\partial \sigma_{\rm y}}{\partial \kappa} = \frac{\partial \sigma_{\rm y}}{\partial \bar{\varepsilon}^{\rm p}}. \tag{6.53}$$

Linear hardening is the most simple isotropic hardening rule

$$H = \frac{\partial \sigma_{\rm y}}{\partial \bar{\varepsilon}^{\rm p}} = \text{constant}, \tag{6.54}$$

thus $K = H\bar{\varepsilon}^{\rm p}$. In reality, the yield stress has an upper bound and

$$K = K_\infty(1 - \exp(-h\bar{\varepsilon}^{\rm p}/K_\infty)), \tag{6.55}$$

i.e.

$$\sigma_{\rm y} = \sigma_{\rm y0} + K_\infty(1 - \exp(-h\bar{\varepsilon}^{\rm p}/K_\infty)), \tag{6.56}$$

is videly used hardining equation. The plastic hardening modulus modulus is

$$H = \frac{\partial \sigma_{\rm y}}{\partial \bar{\varepsilon}^{\rm p}} = h\exp(-h\bar{\varepsilon}^{\rm p}/K_\infty). \tag{6.57}$$

This exponential hardening rule has two material parameters h and K_∞, which have a clear physical intepretation, see fig. 6.6.

The hardening rule (6.55) can be expressed in the rate form

$$\dot{K} = h\exp(h\bar{\varepsilon}^{\rm p}/K_\infty)\dot{\bar{\varepsilon}}^{\rm p}, \tag{6.58}$$

which can be written also in the form

$$\dot{K} = h(1 - K/K_\infty)\dot{\bar{\varepsilon}}^{\rm p}. \tag{6.59}$$

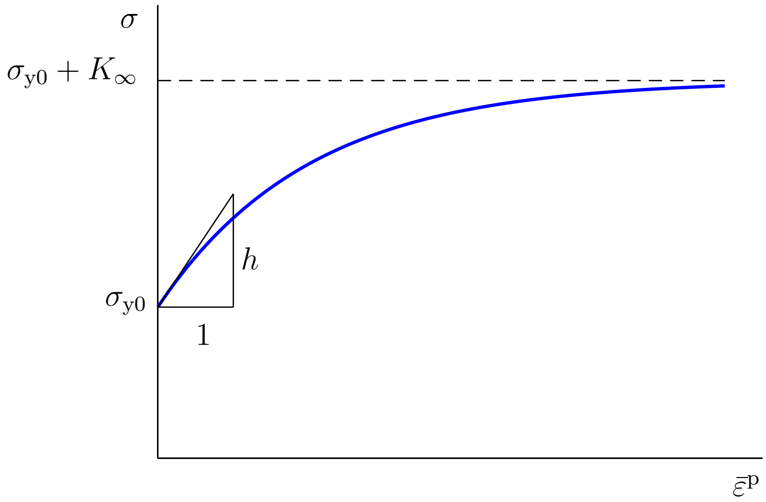


Figure 6.6: Hardening rule (6.56).

6.4.2 Kinematic hardening

Let's consider kinematically hardening von Mises model. Now the hardening parameter K is a second order tensor $\boldsymbol{\alpha}$, which defines the center of the yield curve in the deviatoric plane and it is called as the *back stress*. The yield surface is now defined as

$$f(\sigma_{ij}, \alpha_{ij}) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})} - \sigma_{\mathrm{y}0}. \tag{6.60}$$

Assuming associated flow rule, the plastic strain rate and the rate of the internal variable $\dot{\kappa}_{ij}$, dual to the back stress α_{ij} are

$$\dot{\varepsilon}^{\mathrm{p}}_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3}{2}\frac{s_{ij} - \alpha_{ij}}{\sigma_{\mathrm{y}0}}, \tag{6.61}$$

$$\dot{\kappa}_{ij} = -\dot{\lambda}\frac{\partial f}{\partial \alpha_{ij}} = \dot{\lambda}\frac{3}{2}\frac{s_{ij} - \alpha_{ij}}{\sigma_{\mathrm{y}0}} = \dot{\varepsilon}^{\mathrm{p}}_{ij}. \tag{6.62}$$

Thus, for kinematically hardening associated von Mises plasticity the ineternal variable equals to the plastic strain. Notice that the back stress tensor α has to be deviatoric to result in isochoric[1] plastic flow.

Two well know kinematic hardening rules are the Melan-Prager

$$\dot{\alpha}_{ij} = c\dot{\kappa}_{ij} = c\dot{\varepsilon}^{\mathrm{p}}_{ij}, \tag{6.63}$$

and the Ziegler's rule

$$\dot{\alpha}_{ij} = \dot{\lambda}\bar{c}(\sigma_{ij} - \alpha_{ij}), \tag{6.64}$$

[1]Isochoric = volume preserving.

where c and $\bar{c}$ are material parameters.

A prototype for non-linear kiematic hardening models is the Armstrong-Frederick model [1]

$$\dot{\alpha}_{ij} = h\left(\frac{2}{3}\dot{\varepsilon}^{\mathrm{p}}_{ij} - \frac{\alpha_{ij}}{\alpha_\infty}\dot{\bar{\varepsilon}}^{\mathrm{p}}\right), \tag{6.65}$$

where h and α_∞ are material parameters.

6.4.3 Distortional hardening

A seminal work on distortional, or anisotropic, hardening[2], is the work by Baltov and Sawczuk [4, 38], who expressed the yield function in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \boldsymbol{\varepsilon}^{\mathrm{p}}) = \sqrt{\frac{3}{2}N_{ijkl}(s_{ij} - \alpha_{ij})(s_{kl} - \alpha_{kl})} - \sigma_{\mathrm{y}} = 0, \tag{6.66}$$

where

$$N_{ijkl} = I_{ijkl} + A_0\varepsilon^{\mathrm{p}}_{ij}\varepsilon^{\mathrm{p}}_{kl}, \quad \text{and} \quad I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}). \tag{6.67}$$

The extra parameter A_0 controls the hardening, the yield surface can either expand if $A_0 < 0$ or contract if $A_0 > 0$. The Baltov-Sawczuk model seems to be one of the simplest generalisations to anisotropic hardening, which takes into account translation, rotation and distortion of the yield surface due to plastic straining.

Many other anisotropic hardening models have been introduced in the scientific literature, see e.g. [12, 36].

6.5 Anisotropic yield

6.5.1 Transverse isotropy

As in the case of elastic constitutive models, the material can posses different symmetry properties. The yield function can be formulated in terms of the proper integrity base. For transverse isotropy the most general yield function can be expressed as

$$f(I_1, I_2, I_3, I_4, I_5) = 0 \tag{6.68}$$

where the invariants are

$$I_1 = \operatorname{tr}\boldsymbol{\sigma},\ I_2 = \tfrac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}^2),\ I_3 = \tfrac{1}{3}\operatorname{tr}(\boldsymbol{\sigma}^3),\ I_4 = \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{M}),\ I_5 = \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}), \tag{6.69}$$

and $\boldsymbol{M} = \boldsymbol{m}\boldsymbol{m}^T$ is the structural tensor with the unit vector $\boldsymbol{m}$ defining the normal of the isotropy plane. In some cases it can be easier to operate with the deviatoric invariants

$$J_2 = \tfrac{1}{2}\operatorname{tr}(\boldsymbol{s}^2),\ J_3 = \tfrac{1}{3}\operatorname{tr}(\boldsymbol{s}^3),\ J_4 = \operatorname{tr}(\boldsymbol{s}\boldsymbol{M}),\ J_5 = \operatorname{tr}(\boldsymbol{s}^2\boldsymbol{M}). \tag{6.70}$$

[2] Chen and Han [8] call this kind of behaviour independent hardening.

Example 6.1. *Consider the following form of transversely isotropic yield function*

$$f(\boldsymbol{\sigma},\boldsymbol{M}) = \sqrt{k_1 J_2 + k_2 J_4^2 + k_3 J_5} - \sigma_{\text{yL}} = 0. \tag{6.71}$$

Determine the parameters k_1, k_2 and k_3 from the following tests results:

1. *uniaxial yield strength in the longitudinal direction σ_{yL},*
2. *uniaxial yield strength in the transverse isotropy plane σ_{yT},*
3. *and the shear strength in a plane containing the longitudinal axis τ_{yL}.*

Determine also the shear strength (τ_{yT}) which is predicted by the yield function. If $\sigma_{\text{yT}} = \sigma_{\text{yL}}$ and $\tau_{\text{yL}} = \sigma_{\text{yL}}/\sqrt{3}$, does the yield function (6.71) reduce to the von Mises yield function?

Solution. When the x_1-direction is chosen as the longitudinal direction, i.e. the normal direction of the isotropy plane, the structural tensor is

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us first investigate the yield in the longitudinal direction $\sigma_{11} = \sigma_{\text{yL}}$, then

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{\text{yL}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} \frac{2}{3}\sigma_{\text{yL}} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{\text{yL}} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{\text{yL}} \end{pmatrix},$$

$$\boldsymbol{s}^2 = \begin{pmatrix} \frac{4}{9}\sigma_{\text{yL}}^2 & 0 & 0 \\ 0 & \frac{1}{9}\sigma_{\text{yL}}^2 & 0 \\ 0 & 0 & \frac{1}{9}\sigma_{\text{yL}}^2 \end{pmatrix},$$

thus

$$J_2 = \tfrac{1}{3}\sigma_{\text{yL}}^2, \quad J_4 = \tfrac{2}{3}\sigma_{\text{yL}}, \quad J_5 = \tfrac{4}{9}\sigma_{\text{yL}}^2.$$

Substituting the above expressions in to the yield condition (6.71), we get

$$\tfrac{1}{3}k_1\sigma_{\text{yL}}^2 + \tfrac{4}{9}k_2\sigma_{\text{yL}}^2 + \tfrac{4}{9}k_3\sigma_{\text{yL}}^2 = \sigma_{\text{yL}}^2,$$

or

$$\tfrac{1}{3}k_1 + \tfrac{4}{9}k_2 + \tfrac{4}{9}k_3 = 1. \tag{6.72}$$

Investigating the yield in the transverse isotropy plane, and choosing $\sigma_{22} = \sigma_{\text{yT}}$ (equally we could choose $\sigma_{33} = \sigma_{\text{yT}}$), then

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\text{yT}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} -\frac{1}{3}\sigma_{\text{yT}} & 0 & 0 \\ 0 & \frac{2}{3}\sigma_{\text{yT}} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{\text{yT}} \end{pmatrix},$$

$$\boldsymbol{s}^2 = \begin{pmatrix} \frac{1}{9}\sigma_{\rm yT}^2 & 0 & 0 \\ 0 & \frac{4}{9}\sigma_{\rm yT}^2 & 0 \\ 0 & 0 & \frac{1}{9}\sigma_{\rm yT}^2 \end{pmatrix},$$

and

$$J_2 = \tfrac{1}{3}\sigma_{\rm yT}^2, \quad J_4 = -\tfrac{1}{3}\sigma_{\rm yT}, \quad J_5 = \tfrac{1}{9}\sigma_{\rm yT}^2.$$

Substituting these into the yield condition gives

$$\tfrac{1}{3}k_1\sigma_{\rm yT}^2 + \tfrac{1}{9}k_2\sigma_{\rm yT}^2 + \tfrac{1}{9}\sigma_{\rm yT}^2 = \sigma_{\rm yL}^2$$
$$\Rightarrow \quad \tfrac{1}{3}k_1 + \tfrac{1}{9}k_2 + \tfrac{1}{9}k_3 = \left(\frac{\sigma_{\rm yL}}{\sigma_{\rm yT}}\right)^2 \equiv \xi^2. \quad (6.73)$$

Let us now investigate shear in a plane containing the longitudinal direction, For simplicity we can choose either the 1-2 or 1-3 plane. Choosing $\tau_{12} = \tau_{\rm yL}$ we get

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \tau_{\rm yL} & 0 \\ \tau_{\rm yL} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{s}, \quad \boldsymbol{s}^2 = \begin{pmatrix} \tau_{\rm yL}^2 & 0 & 0 \\ 0 & \tau_{\rm yL}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

resulting in

$$J_2 = \tau_{\rm yL}^2, \quad J_4 = 0, \quad J_5 = \tau_{\rm yL}^2.$$

Substituting into the yield condition gives

$$k_1\tau_{\rm yL}^2 + k_3\tau_{\rm yL}^2 = \sigma_{\rm yL}^2, \quad \Rightarrow \quad k_3 = \left(\frac{\sigma_{\rm yL}}{\tau_{\rm yL}}\right)^2 - k_1 = \eta^2 - k_1.$$

Further substituting this in (6.72) and (6.73) we get

$$\begin{aligned} \tfrac{1}{3}k_1 + \tfrac{4}{9}k_2 + \tfrac{4}{9}(\eta^2 - k_1) &= 1, \\ \tfrac{1}{3}k_1 + \tfrac{1}{9}k_2 + \tfrac{1}{9}(\eta^2 - k_1) &= \xi^2, \end{aligned}$$

from which we obtain

$$\begin{aligned} -k_1 + 4k_2 &= 9 - 4\eta^2, \\ 2k_1 + k_2 &= 9\xi^2 - \eta^2, \end{aligned}$$

and the solution is

$$\begin{aligned} k_1 &= 4\xi^2 - 1 = 4\left(\frac{\sigma_{\rm yL}}{\sigma_{\rm yT}}\right)^2 - 1, \\ k_2 &= 2 + \xi^2 - \eta^2 = \left(\frac{\sigma_{\rm yL}}{\sigma_{\rm yT}}\right)^2 - \left(\frac{\sigma_{\rm yL}}{\tau_{\rm yL}}\right)^2, \\ k_3 &= 1 + \eta^2 - 4\xi^2 = 1 + \left(\frac{\sigma_{\rm yL}}{\tau_{\rm yL}}\right)^2 - 4\left(\frac{\sigma_{\rm yL}}{\sigma_{\rm yT}}\right)^2. \end{aligned}$$

If now $\sigma_{\mathrm{yT}} = \sigma_{\mathrm{yL}}$ and $\tau_{\mathrm{yL}} = \sigma_{\mathrm{yL}}/\sqrt{3}$, i.e. $\xi = 1$ and $\eta^2 = 3$, we get $k_2 = k_3 = 0$ and $k_1 = 3$, and the model reduces in the isotropic case to the standard von Mises yield condition.

The last question is related to the yield strength in the transverse plane. Now the stress and deviatoric tensors are

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\mathrm{yT}} \\ 0 & \tau_{\mathrm{yT}} & 0 \end{pmatrix} = \boldsymbol{s}, \quad \boldsymbol{s}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tau_{\mathrm{yT}}^2 & 0 \\ 0 & 0 & \tau_{\mathrm{yT}}^2 \end{pmatrix},$$

resulting in

$$J_2 = \tau_{\mathrm{yT}}^2, \quad J_4 = J_5 = 0.$$

When substituting into the yield condition we get

$$k_1 \tau_{\mathrm{yT}}^2 = \sigma_{\mathrm{yL}}^2 \quad \Rightarrow \quad \tau_{\mathrm{yT}}^2 = \frac{\sigma_{\mathrm{yL}}^2}{k_1} = \frac{\sigma_{\mathrm{yT}}^2}{4(\sigma_{\mathrm{yL}}/\sigma_{\mathrm{yT}})^2 - 1}.$$

Notice that $\sigma_{\mathrm{yL}}^2 > \frac{1}{4}\sigma_{\mathrm{yT}}^2$.

6.5.2 Orthotropy

For an orthotropic material the most general yield function is of the form

$$f(I_1, \dots, I_7) = 0, \tag{6.74}$$

where the invariants can be defined in the symmetric format as

$$I_1 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_1), \quad I_2 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_2), \quad I_3 = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{M}_3), \quad I_4 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_1),$$
$$I_5 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_2), \quad I_6 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{M}_3), \quad I_3 = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}^3). \tag{6.75}$$

For metals the yield can often be modelled to be independent of the mean stress, thus it is helpful to formulate the yield function in terms of the deviatoric stresses

$$J_1 = \mathrm{tr}(\boldsymbol{s}\boldsymbol{M}_1), \quad J_2 = \mathrm{tr}(\boldsymbol{s}\boldsymbol{M}_2), \quad J_3 = \mathrm{tr}(\boldsymbol{s}\boldsymbol{M}_3), \quad J_4 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{s}^2\boldsymbol{M}_1),$$
$$J_5 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{s}^2\boldsymbol{M}_2), \quad J_6 = \tfrac{1}{2}\mathrm{tr}(\boldsymbol{s}^2\boldsymbol{M}_3), \quad J_7 = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{s}^3). \tag{6.76}$$

As an example let us consider an orthotropic yield function which is independent of hydrostatic stress and has equal compressive and tensile yield stresses in the directions of the orthotropy. The yield function satisfying these requirements is of the form

$$f = \sigma_{\mathrm{eff}} - \sigma_{\mathrm{y1}} = 0,$$
$$\sigma_{\mathrm{eff}} = \sqrt{\alpha_1(J_1 - J_2)^2 + \alpha_2(J_2 - J_3)^2 + \alpha_3(J_3 - J_1)^2 + \alpha_4 J_4 + \alpha_5 J_5 + \alpha_6 J_6}, \tag{6.77}$$

where σ_{y1} is the yield strength in the direction of $\boldsymbol{m}_1$. There are six material parameters in the yield function (6.77), which can be determined from the following six tests for individual stress components:

- yield under normal stress state in the directions 1,2 and 3, yield stresses $\sigma_{y1}, \sigma_{y2}, \sigma_{y3}$, respectively, and
- yield in shear on planes 1-2, 2-3 and 3-1, with respective yield stresses $\tau_{y12}, \tau_{y23}, \tau_{y31}$.

For determining the parameters $\alpha_1, \ldots, \alpha_6$, it is convenient to write the yield condition in the form

$$\alpha_1(J_1 - J_2)^2 + \alpha_2(J_2 - J_3)^2 + \alpha_3(J_3 - J_1)^2 + \alpha_4 J_4 + \alpha_5 J_5 + \alpha_6 J_6 = \sigma_{y1}^2. \tag{6.78}$$

If we now associate the directions of orthotropy to coincide the coordinate axes.

- Stress in $\boldsymbol{m}_1$, i.e. x_1-axis direction $\sigma_{11} = \sigma_{y1}$ results in

  $$J_1 = \tfrac{2}{3}\sigma_{y1}, \quad J_2 = J_3 = -\tfrac{1}{3}\sigma_{y1}, \quad J_4 = \tfrac{4}{9}\sigma_{y1}^2, \quad J_5 = J_6 = \tfrac{1}{18}\sigma_{y1}^2.$$

  and substituting it into (6.78) gives

  $$\alpha_1 + \alpha_3 + \tfrac{2}{9}\alpha_4 + \tfrac{1}{18}\alpha_5 + \tfrac{1}{18}\alpha_6 = 1. \tag{6.79}$$

- Stress in $\boldsymbol{m}_2$, i.e. x_2-axis direction $\sigma_{22} = \sigma_{y2}$ results in

  $$J_2 = \tfrac{2}{3}\sigma_{y2}, \quad J_1 = J_3 = -\tfrac{1}{3}\sigma_{y2}, \quad J_5 = \tfrac{4}{9}\sigma_{y2}^2, \quad J_4 = J_6 = \tfrac{1}{18}\sigma_{y2}^2.$$

  and substituting these values into (6.78) gives

  $$\alpha_1 + \alpha_2 + \tfrac{1}{18}\alpha_4 + \tfrac{2}{9}\alpha_5 + \tfrac{1}{18}\alpha_6 = (\sigma_{y1}/\sigma_{y2})^2 \equiv \xi_2^2. \tag{6.80}$$

- Stress in $\boldsymbol{m}_3$, i.e. x_3-axis direction $\sigma_{33} = \sigma_{y3}$ results in

  $$J_3 = \tfrac{2}{3}\sigma_{y3}, \quad J_1 = J_2 = -\tfrac{1}{3}\sigma_{y3}, \quad J_6 = \tfrac{4}{9}\sigma_{y3}^2, \quad J_4 = J_5 = \tfrac{1}{18}\sigma_{y3}^2.$$

  and substituting these values into (6.78) gives

  $$\alpha_2 + \alpha_3 + \tfrac{1}{18}\alpha_4 + \tfrac{1}{18}\alpha_5 + \tfrac{2}{9}\alpha_6 = (\sigma_{y1}/\sigma_{y3})^2 \equiv \xi_3^2. \tag{6.81}$$

- Shear stress in the 1-2 plane: $\tau_{12} = \tau_{y12}$ gives

  $$J_1 = J_2 = J_3 = 0, \quad J_4 = J_5 = \tfrac{1}{2}\tau_{y12}^2, \quad J_6 = 0,$$

  and substituting these values into (6.78) gives

  $$\alpha_4 + \alpha_5 = 2(\sigma_{y1}/\tau_{y12})^2 \equiv \eta_{12}^2 \tag{6.82}$$

- Shear stress in the 2-3 plane: $\tau_{23} = \tau_{y23}$ gives

$$J_1 = J_2 = J_3 = 0, \quad J_5 = J_6 = \tfrac{1}{2}\tau_{y23}^2, \quad J_6 = 0,$$

and substituting these values into (6.78) gives

$$\alpha_5 + \alpha_6 = 2(\sigma_{y1}/\tau_{y23})^2 \equiv \eta_{23}^2. \tag{6.83}$$

- Shear stress in the 3-1 plane: $\tau_{31} = \tau_{y31}$ gives

$$J_1 = J_2 = J_3 = 0, \quad J_4 = J_6 = \tfrac{1}{2}\tau_{y31}^2, \quad J_6 = 0,$$

and substituting these values into (6.78) gives

$$\alpha_4 + \alpha_6 = 2(\sigma_{y1}/\tau_{y31})^2 \equiv \eta_{31}^2. \tag{6.84}$$

From the shear stress conditions (6.82), (6.83) and (6.84), it is obtained

$$\begin{aligned} \alpha_4 &= \tfrac{1}{2}(\eta_{12}^2 + \eta_{31}^2 - \eta_{23}^2), \\ \alpha_5 &= \tfrac{1}{2}(\eta_{23}^2 + \eta_{12}^2 - \eta_{31}^2), \\ \alpha_6 &= \tfrac{1}{2}(\eta_{31}^2 + \eta_{23}^2 - \eta_{12}^2). \end{aligned} \tag{6.85}$$

Observe the logic in the cyclic symmetry of the indexes. Substituting these expressions into (6.79), (6.79) and (6.81) results

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}(1 + \xi_2^2 - \xi_3^2 - \tfrac{5}{18}\eta_{12}^2 + \tfrac{1}{18}\eta_{23}^2 + \tfrac{1}{18}\eta_{31}^2), \\ \alpha_2 &= \tfrac{1}{2}(\xi_2^2 + \xi_3^2 - 1 - \tfrac{5}{18}\eta_{23}^2 + \tfrac{1}{18}\eta_{31}^2 + \tfrac{1}{18}\eta_{12}^2), \\ \alpha_3 &= \tfrac{1}{2}(1 - \xi_2^2 + \xi_3^2 - \tfrac{5}{18}\eta_{31}^2 + \tfrac{1}{18}\eta_{12}^2 + \tfrac{1}{18}\eta_{23}^2). \end{aligned} \tag{6.86}$$

For isotropic von Mises solid $\sigma_{y1} = \sigma_{y2} = \sigma_{y3} = \sigma_y$ and $\tau_{y12} = \tau_{y23} = \tau_{y31} = \sigma_y/\sqrt{3}$, gives $\xi_2 = \xi_3 = 1$ and $\eta_{ij}^2 = 6$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = \alpha_5 = \alpha_6 = 3$. The orthotropic yield function (6.77) reduces to

$$\begin{aligned} f &= \sqrt{3(J_4 + J_5 + J_6)} - \sigma_y \\ &= \sqrt{\tfrac{3}{2}\operatorname{tr}[\boldsymbol{s}^2(\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3)]} - \sigma_y = \sqrt{\tfrac{3}{2}\operatorname{tr}(\boldsymbol{s}^2)} - \sigma_y = 0. \end{aligned} \tag{6.87}$$

which is identical to the isotropic von Mises yield condition (6.9).

Notice that the linear deviatoric invariants J_1, J_2 and J_3 are not independent, since

$$\operatorname{tr}(\boldsymbol{s}\boldsymbol{M}_1) + \operatorname{tr}(\boldsymbol{s}\boldsymbol{M}_2) + \operatorname{tr}(\boldsymbol{s}\boldsymbol{M}_3) = \operatorname{tr}[\boldsymbol{s}(\boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3)] = \operatorname{tr}\boldsymbol{s} = 0, \tag{6.88}$$

and therefore $J_3 = -J_2 - J_1$. The effective stress (6.77) can thus be written as

$$\sigma_{\text{eff}} = \sqrt{\tilde{\alpha}_1 J_1^2 + \tilde{\alpha}_2 J_2^2 + 2\tilde{\alpha}_3 J_1 J_2 + \alpha_4 J_4 + \alpha_5 J_5 + \alpha_6 J_6}, \tag{6.89}$$

where

$$\begin{aligned} \tilde{\alpha}_1 &= \alpha_1 + \alpha_2 + 4\alpha_3 = 2 + 2\xi_3^2 - \xi_2^2 - \tfrac{1}{2}\eta_{31}^2, \\ \tilde{\alpha}_2 &= \alpha_1 + \alpha_3 + 4\alpha_2 = 2\xi_2^2 + 2\xi_3^2 - 1 - \tfrac{1}{2}\eta_{23}^2, \\ \tilde{\alpha}_3 &= 2\alpha_2 + 2\alpha_3 - \alpha_1 = -\tfrac{1}{2} + \tfrac{1}{2}\xi_2^2 + \tfrac{5}{2}\xi_3^2 + \tfrac{1}{4}(\eta_{12}^2 - \eta_{23}^2 - \eta_{31}^2). \end{aligned} \tag{6.90}$$

6.6 Determining material parameters

6.7 Solved example problems

Example 6.2. *It is assumed that a yield of a certain material is governed by the yield function*

$$f(I_1, J_2) = J_2 + \alpha(a_1 - I_1)(I_1 + a_2) = 0, \tag{6.91}$$

where $J_2 = \frac{1}{2}\operatorname{tr}\boldsymbol{s}^2$ is the second invariant of the deviatoric stress and $I_1 = \operatorname{tr}\boldsymbol{\sigma}$ is the first stress invariant and α, a_1, a_2 are parameters which can be determined from the three tests listed below.

In a triaxial loading device the following three stress states (a)-(c) cause yielding

1. *hydrostatic compression $\sigma_{11} = \sigma_{22} = \sigma_{33} = -2p_0$,*
2. *hydrostatic tension $\sigma_{11} = \sigma_{22} = \sigma_{33} = \frac{1}{3}p_0$,*
3. *under the cell pressure $\sigma_{22} = \sigma_{33} = -\frac{1}{2}p_0$ the yield occurs when the compressive stress in the 1-axis direction reaches the value $\sigma_{11} = -2p_0$.*

Above p_0 is a positive stress value. Determine the material parameters α, a_1, a_2 such that $a_1, a_2 > 0$. Notice that α is dimensionless while a_1 and a_2 has a dimension of stress.

Determine the shear strength as a function of hydrostatic pressure $p = -\frac{1}{3}I_1$ and its maximum value. Draw a figure.

Solution. The loading case 1 and 2 are purely hydrostatic, that is $J_2 = 0$ and the first stress invariant I_1 has values $-6p_0$ and p_0, respectively. Substituting these values to the yield function gives

$$\alpha(a_1 + 6p_0)(a_2 - 6p_0) = 0, \tag{6.92}$$

$$\alpha(a_1 - p_0)(a_2 + p_0) = 0. \tag{6.93}$$

If the parameters a_1 and a_2 are assumed to be positive, it is obtained $a_1 = p_0$ and $a_2 = 6p_0$.

For the loading case 3: $\sigma_{11} = -2p_0, \sigma_{22} = \sigma_{33} = -\frac{1}{2}p_0$, then $I_1 = -3p_0$ and the deviatoric stress tensor has non-zero components $s_{11} = -2p_0 + p_0 = -p_0, s_{22} = s_{33} = -\frac{1}{2}p_0 + p_0 = \frac{1}{2}p_0$. As a check, notice that $s_{11} + s_{22} + s_{33} = 0$, as it should be. The second invariant of the deviatoric stress has now the value $J_2 = \frac{1}{2}(s_{11}^2 + s_{22}^2 + s_{33}^2) = \frac{3}{4}p_0^2$. Substituting the values of J_2 and I_1 into the yield function (6.91) gives

$$\tfrac{3}{4}p_0^2 + \alpha(p_0 + 3p_0)(-3p_0 + 6p_0) = \tfrac{3}{4}p_0^2 + 12p_0^2\alpha = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{16}. \tag{6.94}$$

To compute the shear strength as a function of hydrostatic pressure $p = -\frac{1}{3}I_1$, we can use the following stress state

$$\boldsymbol{\sigma} = \begin{pmatrix} -p & \tau & 0 \\ \tau & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \quad \Rightarrow \quad J_2 = \tau^2, \quad I_1 = -3p_0. \tag{6.95}$$

Substituting these values to the yield condition (6.91) gives

$$\tau^2 = \tfrac{1}{16}(p_0 + 3p)(6p_0 - 3p) = \tfrac{9}{16}(\tfrac{1}{3}p_0 + p)(2p_0 + p). \tag{6.96}$$

Let us determine the extremum value of the function

$$g(p) = (\tfrac{1}{3}p_0 + p)(2p_0 + p) = -p^2 + \tfrac{5}{3}p_0 p + \tfrac{2}{3}p_0^2,$$
$$g'(p) = -2p + \tfrac{5}{3}p_0 = 0 \quad \Rightarrow \quad p = \tfrac{5}{6}p_0. \tag{6.97}$$

Substituting this value into (6.96) results in $\tau^2 = (7/8)p_0^2$, thus the maximum shear strenght occurs at the hydrostatic pressure value $p = (5/6)p_0$ and it is $\tau_{\max} = \sqrt{7/8}p_0$.

Example 6.3. *Hydrostatic pressure does not influence to yielding of metals in the early phase of plastic deformation. However, if the material has unequal yield stresses in compression and tension, the yield function has to depend also from the third invariant of the deviatoric stress as*

$$f(J_2, J_3) = \sqrt{3J_2} + \alpha J_3 - \beta = 0 \tag{6.98}$$

where α and β are material parameters and the deviatoric invariants are $J_2 = \frac{1}{2}\operatorname{tr}\boldsymbol{s}^2 = \frac{1}{2}s_{ij}s_{ji}$ and $J_3 = \det\boldsymbol{s} = \frac{1}{3}\operatorname{tr}(\boldsymbol{s}^3) = \frac{1}{3}s_{ij}s_{jk}s_{ki}$. Determine the parameters α and β when the uniaxial tensile and compressive yield strengths are $\sigma_{\rm t}$ and $\sigma_{\rm c}$, respectively. Write the yield function also in terms of ρ and $\cos 3\theta$, which are defined as

$$\rho = \sqrt{s_{ij}s_{ij}}, \quad \cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}.$$

Solution. At the uniaxial tensile yield we have

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{\rm t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{s} = \boldsymbol{\sigma} - \tfrac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \begin{pmatrix} \frac{2}{3}\sigma_{\rm t} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{\rm t} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{\rm t} \end{pmatrix} \tag{6.99}$$

$$\Rightarrow \quad J_2 = \tfrac{1}{3}\sigma_{\rm t}^2, \quad J_3 = \tfrac{2}{27}\sigma_{\rm t}^3 \tag{6.100}$$

Correspondingly at the uniaxial compressive yield we have

$$\boldsymbol{\sigma} = \begin{pmatrix} -\sigma_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{s} = \boldsymbol{\sigma} - \tfrac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \begin{pmatrix} -\frac{2}{3}\sigma_c & 0 & 0 \\ 0 & \frac{1}{3}\sigma_c & 0 \\ 0 & 0 & \frac{1}{3}\sigma_c \end{pmatrix} \quad (6.101)$$

$$\Rightarrow \quad J_2 = \tfrac{1}{3}\sigma_c^2, \quad J_3 = -\tfrac{2}{27}\sigma_c^3. \quad (6.102)$$

Inserting this data into the yield function (6.98) results in

$$\sigma_t + \alpha\tfrac{2}{27}\sigma_t^3 - \beta = 0,$$
$$\sigma_c - \alpha\tfrac{2}{27}\sigma_c^3 - \beta = 0,$$

and the solution is

$$\alpha = \frac{27}{2}\frac{\sigma_c - \sigma_t}{\sigma_c^3 + \sigma_t^3}, \quad \beta = \frac{\sigma_t\sigma_c^3 + \sigma_c\sigma_t^3}{\sigma_c^3 + \sigma_t^3}. \quad (6.103)$$

Defining $\sigma_c = m\sigma_t$ nicer expressions are obtained

$$\alpha = \frac{27}{2}\frac{m-1}{m^3+1}\frac{1}{\sigma_t^2}, \quad \beta = \frac{m^3+m}{m^3+1}\sigma_t. \quad (6.104)$$

Since $J_2 = \frac{1}{2}\rho^2$ and

$$J_3 = \frac{2}{3\sqrt{3}}J_2^{3/2}\cos 3\theta = \frac{1}{3\sqrt{6}}\rho^3\cos 3\theta,$$

the yield function (6.98) can be written in the form

$$f(\rho, \cos 3\theta) = \sqrt{\frac{3}{2}}\rho + \alpha\frac{\rho^3}{3\sqrt{6}}\cos 3\theta - \beta = 0. \quad (6.105)$$

Draw the locus in the deviatoric plane!

6.8 Exercises

1.

Chapter 7
Failure of brittle materials

Plastic behaviour is characteristic to metals and polymers. For ceramics, rock, concrete and even for cast iron the material usually fails without significant plastic deformations. Several failure criteria with different level of complexity have been proposed for different brittle materials. In this lecture notes, only the most simple ones will be dealt with.

7.1 Rankine's maximum principal stress criterion

According to the Rankine's failure criterion, dating back to the year 1876, the material fails when the maximum principal stress attains a critical value, i.e. the uniaxial tensile strength of the material in question. The failure criterion is thus expressed simply as

$$\max(\sigma_1, \sigma_2, \sigma_3) = f_{\mathrm{t}}. \tag{7.1}$$

Using the Heigh-Westergaard coordinates ξ, ρ, θ or the invariant set I_1, J_2, θ, the failure criterion has the forms

$$f(\xi, \rho, \theta) = \sqrt{2}\rho\cos\theta + \xi - \sqrt{3}f_{\mathrm{t}} = 0, \tag{7.2}$$

or

$$f(I_1, J_2, \theta) = 2\sqrt{3J_2}\cos\theta + I_1 - 3f_{\mathrm{t}} = 0, \tag{7.3}$$

On the deviatoric plane the shape of the Rankine's failure surface is a triangle, and the meridian curves are straight lines, see fig. 7.1a and b. The ratio between the tensile and compressive meridians is $\rho_{\mathrm{t}}/\rho_{\mathrm{c}} = 0.5$.

In the plane stress case the Rankine's criterion is shown in fig. 7.1c. For the plane-strain case, the failure surface is similar in the to the plane-stress in the (σ_1, σ_2)-stress plane if the Poisson's ratio is positive, i.e. in the range $0 \leq \nu \leq 0.5$ ($\sigma_z = \nu(\sigma_1 + \sigma_2)$).

If the only non-zero stress components are σ and τ, the failure criterion has the form

$$\tau^2 = f_{\mathrm{t}}(f_{\mathrm{t}} - \sigma), \tag{7.4}$$

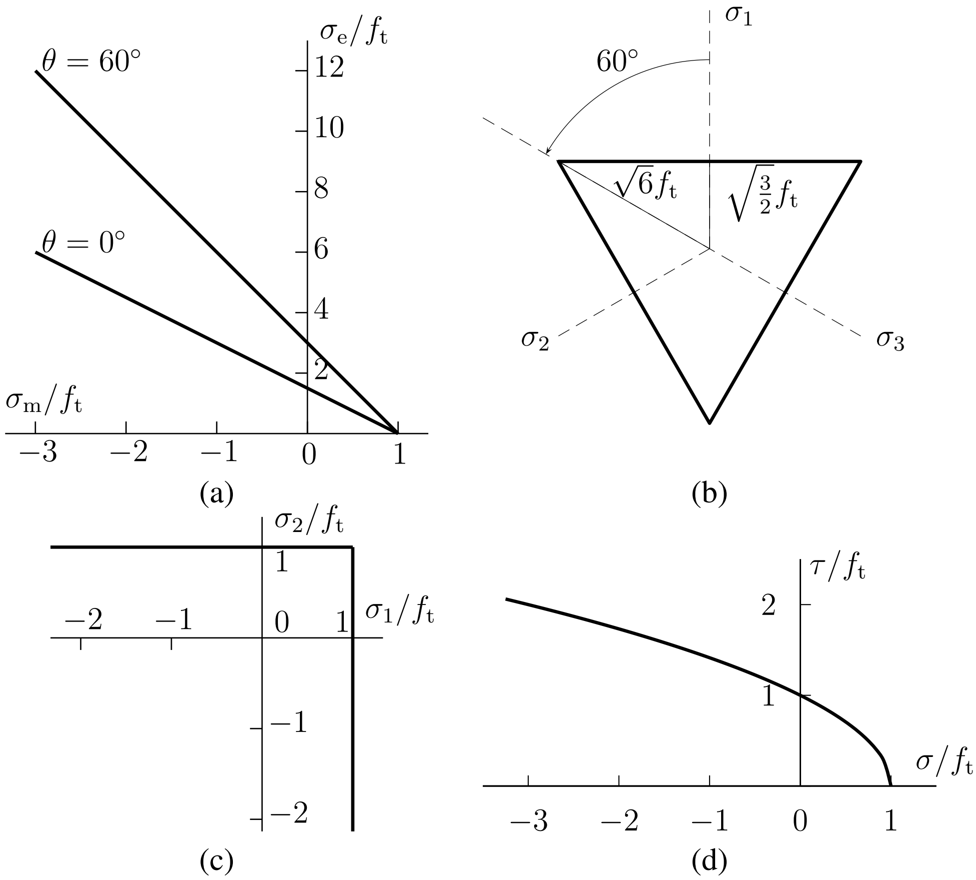


Figure 7.1: Rankine's maximum principal stress criterion: (a) comressive- and tensile meridian lines, (b) π-plane, (c) state of plane-stress, (d) for (σ, τ)-stress state.

and it is shown in fig. 7.1d.

Simplicity is the most important advantage of the Rankine's criterion, it has only one material parameter, $f_{\rm t}$, to be determined.

7.2 Maximum principal strain criterion

The maximum principal strain criterion, which is also called Saint-Venant's criterion, is completely analogous to Rankine's maximum principal stress criterion. It is assumed that the material fails when the maximum principal strain attains a critical value

$$\max(\varepsilon_1, \varepsilon_1, \varepsilon_3) = \varepsilon_{\rm t}. \tag{7.5}$$

For isotropic material, the directions of principal stresses and strains coincide, thus the material parameter $\varepsilon_{\rm t}$ can be written by using the uniaxial tensile stress $f_{\rm t}$ as

$$f_{\rm t} = E\varepsilon_{\rm t}. \tag{7.6}$$

On the meridian plane, the failure condition can be written as

$$2\sqrt{3J_2}\cos\theta + \frac{1-2\nu}{1+\nu}I_1 - \frac{3}{1+\nu}f_{\rm t} = 0, \tag{7.7}$$

which is similar to Rankin's maximum principal stress criterion (7.3). In pure hydrostatic tension the maximum principal strain criterion predicts the value $\sigma_{\rm mt} = f_{\rm t}/(1-2\nu)$, which with the value of the Poisson's ratio $\nu = 0.2$ results in the value $1.667 f_{\rm t}$.

In the plane-stress state $(\sigma_3 \equiv 0)$ the principal strain can be written in terms of principal stresses as

$$\varepsilon_1 = (\sigma_1 - \nu\sigma_2)/E, \tag{7.8}$$
$$\varepsilon_1 = (\sigma_2 - \nu\sigma_1)/E, \tag{7.9}$$
$$\varepsilon_3 = -\nu(\sigma_1 + \sigma_2)/E, \tag{7.10}$$

and the failure curve in the (σ_1, σ_2)-plane is composed of straight lines

$$\sigma_1 - \nu\sigma_2 = f_{\rm t}, \qquad \varepsilon_1 \geq \varepsilon_1, \varepsilon_3, \tag{7.11}$$
$$\sigma_2 - \nu\sigma_1 = f_{\rm t}, \qquad \varepsilon_1 > \varepsilon_1, \varepsilon_3, \tag{7.12}$$
$$\sigma_1 + \sigma_2 = -f_{\rm t}/\nu, \qquad \varepsilon_3 > \varepsilon_1, \varepsilon_1. \tag{7.13}$$

It is unfortunate that in many books the maximum strain failure locus in the plane-stress state is incorrectly drawn.

The maximum principal strain criterion is illustrated in fig. 7.2.

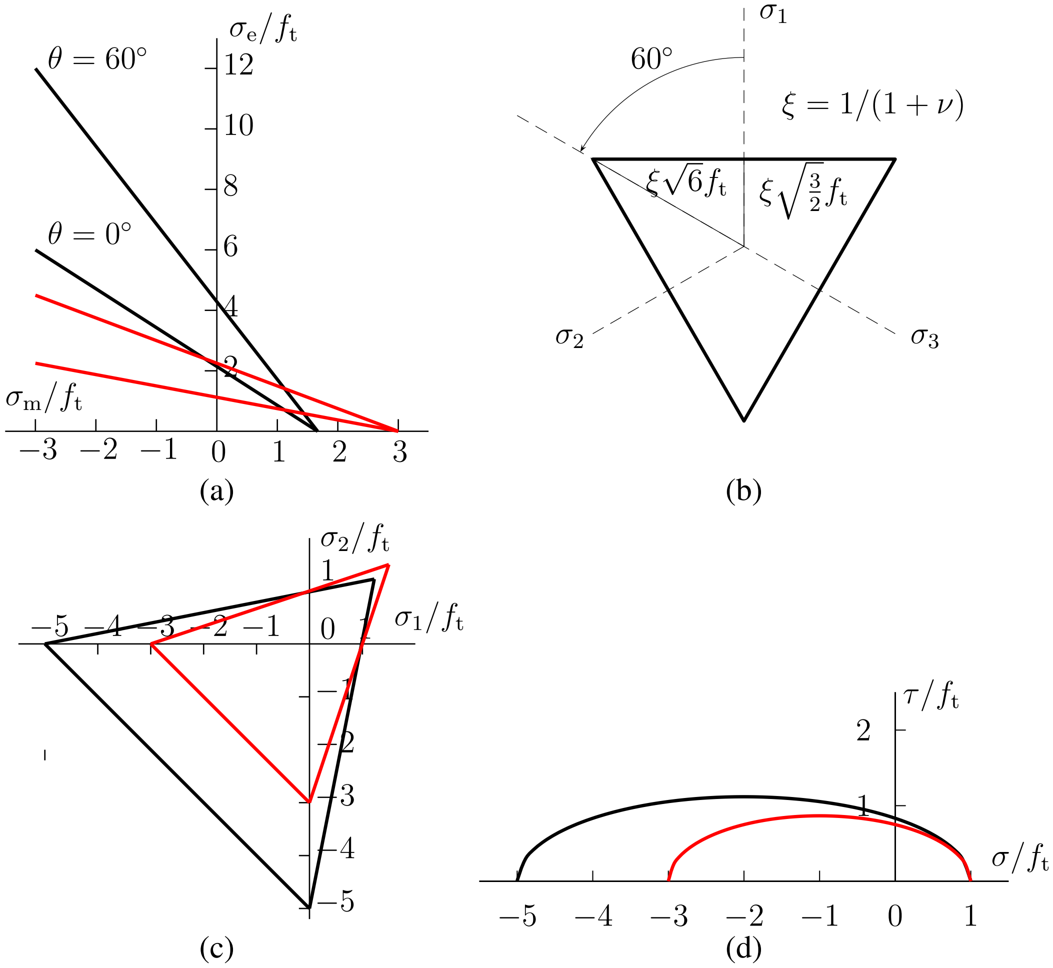


Figure 7.2: Maximum principal strain criterion: (a) compressive- and tensile meridians, (b) π-plane, (c) plane-stress state, (d) for (σ, τ)-stress state. The black line corresponds to Poisson's ratio 0.2 and red to $1/3$, respectively.

7.3 Continuum damage mechanics

7.3.1 Introduction

To model continuous degradation of a material Kachanov introduced in 1958 a formulation where evolution of a single internal variable continuously reduces the elastic properties [20]. Physically such variable, which he called damage index or integrity ϕ, can be interpreted as a ratio of the differential intact area element to the original area element, i.e.

$$\phi = \frac{\mathrm{d}A - \mathrm{d}A_{\mathrm{dam}}}{\mathrm{d}A}. \tag{7.14}$$

In uniaxial case, the constitutive equation is

$$\sigma = \phi E \varepsilon^{\mathrm{e}}, \tag{7.15}$$

where ε^{e} stands for the elastic strain, which in the small strain case can be written as

$$\varepsilon^{\mathrm{e}} = \varepsilon - \varepsilon^{\mathrm{th}} - \varepsilon^{\mathrm{in}}, \tag{7.16}$$

where $\varepsilon^{\mathrm{th}}$ and $\varepsilon^{\mathrm{in}}$ are thermal and inelastic strains, respectively. In the literature, it is quite customary to work with the damage D, defined as

$$D = \frac{\mathrm{d}A_{\mathrm{dam}}}{\mathrm{d}A} = 1 - \phi. \tag{7.17}$$

For the evolution of the integrity ϕ, Kachanov proposed the following kinetic law

$$\dot{\phi} = A\left(\frac{\sigma}{\phi}\right)^n, \tag{7.18}$$

where the superimposed dot denotes time rate and A, n are material parameters which can depend on e.g. temperature. For an undamaged material $\phi = 1$ (or $D = 0$) and during the damaging process it decreases monotonically to the value 0 in the fully damaged state (or increases monotonically to the value $D = 1$). The ratio $\sigma/\phi = \sigma/(1 - D)$ is called the effective stress, which is the net stress acting on the undamaged area. Kachanov used his theory in predicting creep failure times, see also [21]. Rabotnov [37] generalized Kachanov's evolution equation (7.18) to the form

$$\dot{\phi} = -\frac{A}{\phi^p}\left(\frac{\sigma}{\phi}\right)^n, \tag{7.19}$$

where p is an additional material parameter. Since then, continuum damage mechanics has developed into an important and active field of continuum mechanics exemplified by numerous scientific articles and books, e.g. [5, 24, 25, 29, 42].

7.3.2 Uniaxial behaviour

Let us consider a uniaxial constant strain-rate tensile/compression test in the absense of thermal and inelastic strains, then the stress-strain relation is

$$\sigma = \phi E \varepsilon = \phi E \dot{\varepsilon}_0 t, \tag{7.20}$$

where $\dot{\varepsilon}_0$ is the applied strain-rate. For the damage evolution equation the following form is chosen

$$\dot{\phi} = -\frac{1}{t_{\mathrm{d}}\phi^p}\left(\frac{\sigma^2}{\phi^2\sigma_{\mathrm{r}}^2}\right)^r, \tag{7.21}$$

where t_{d}, r and p are material parameters and σ_{r} is an arbitrary reference stress. Defining $\varepsilon_{\mathrm{r}} = \sigma_{\mathrm{r}}/E$ and using the constitutive equation (7.20), it is obtained

$$\dot{\phi} = -\frac{1}{t_{\mathrm{d}}\phi^p}\left(\frac{\varepsilon^2}{\varepsilon_{\mathrm{r}}^2}\right)^r = -\frac{1}{t_{\mathrm{d}}\phi^p}\left(\frac{\dot{\varepsilon}_0^2}{\varepsilon_{\mathrm{r}}^2}\right)^r, \tag{7.22}$$

which can easily be integrated

$$\int_1^t \phi^k \mathrm{d}\phi = -\int_0^t \frac{1}{t_{\mathrm{d}}}\left(\frac{\dot{\varepsilon}_0 t}{\varepsilon_{\mathrm{r}}}\right)^{2r} \mathrm{d}t, \tag{7.23}$$

resulting in

$$\phi = \left[1 - \frac{(p+1)\varepsilon_{\mathrm{r}}}{(2r+1)\dot{\varepsilon}_0 t_{\mathrm{d}}}\left(\frac{\varepsilon}{\varepsilon_{\mathrm{r}}}\right)^{2r+1}\right]^{1/(p+1)}, \qquad \text{if} \quad p \neq -1, \tag{7.24}$$

$$\phi = \exp\left[-\frac{1}{(2r+1)}\frac{\varepsilon_{\mathrm{r}}}{\dot{\varepsilon}_0 t_{\mathrm{d}}}\left(\frac{\varepsilon}{\varepsilon_{\mathrm{r}}}\right)^{2r+1}\right] \qquad \text{if} \quad p = -1. \tag{7.25}$$

Substituting it to the stress-strain relation (7.20) gives

$$\frac{\sigma}{\sigma_{\mathrm{r}}} = \left[1 - \frac{(p+1)\varepsilon_{\mathrm{r}}}{(2r+1)\dot{\varepsilon}_0 t_{\mathrm{d}}}\left(\frac{\varepsilon}{\varepsilon_{\mathrm{r}}}\right)^{2r+1}\right]^{1/(p+1)}\left(\frac{\varepsilon}{\varepsilon_{\mathrm{r}}}\right). \tag{7.26}$$

The ultimate tensile stress, i.e.the fracture stress σ_{frac} can be found to occur at strain

$$\frac{\varepsilon}{\varepsilon_{\mathrm{r}}} = \left[\frac{(2r+1)\dot{\varepsilon}_0 t_{\mathrm{d}}}{(2r+p+2)\varepsilon_{\mathrm{r}}}\right]^{1/(2r+1)}, \tag{7.27}$$

and the fracture stress is thus found from

$$\frac{\sigma_{\mathrm{frac}}}{\sigma_{\mathrm{r}}} = \left(\frac{2r+1}{2r+p+2}\right)^{\frac{1}{p+1}}\left(\frac{(2r+1)}{(2r+p+2)}\frac{\dot{\varepsilon}_0 t_{\mathrm{d}}}{\varepsilon_{\mathrm{r}}}\right)^{\frac{1}{2r+1}}$$
$$= \left(\frac{2r+1}{2r+p+2}\right)^{\frac{2r+p+2}{(p+1)(2r+1)}}\left(\frac{\dot{\varepsilon}_0 t_{\mathrm{d}}}{\varepsilon_{\mathrm{r}}}\right)^{\frac{1}{2r+1}}. \tag{7.28}$$

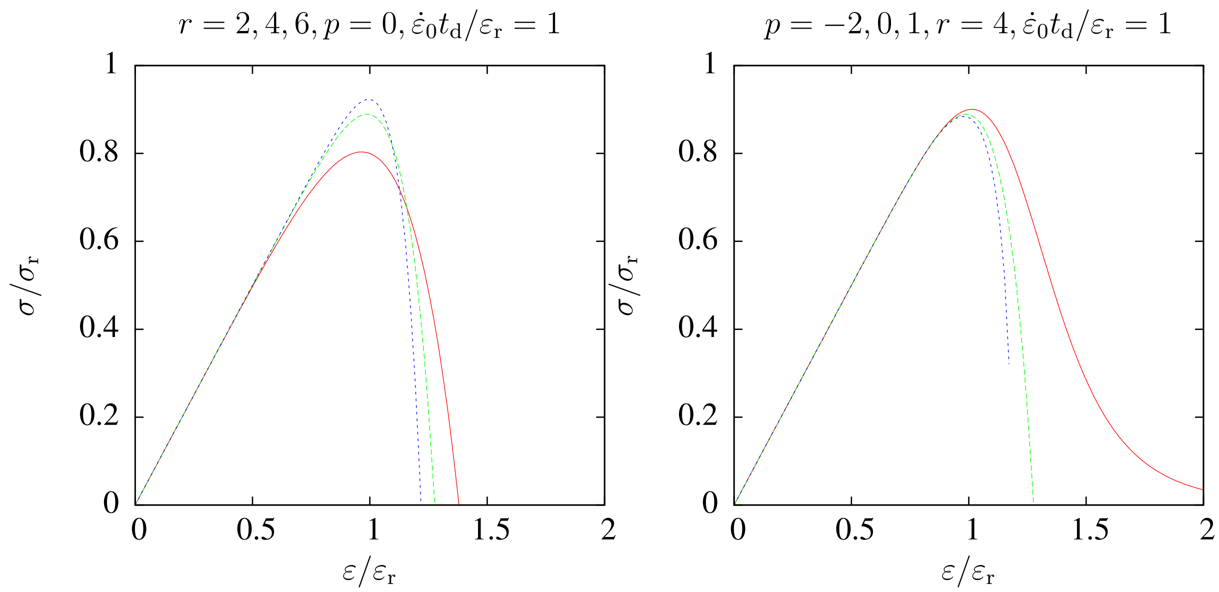


Figure 7.3: Stress-strain relation in a uniaxial constant strain-rate tensile test. Left-hand side effect of the r-parameter variation. Increase of the r-parameter makes the model more brittle, $r = 2$ red solid, $r = 4$, green dashed, $r = 6$ blue dotted curve. Right-hand side effect of the p-parameter variation. Increasing k-parameter makes the model more brittle, $p = -2$ red solid, $p = 0$, green dashed, $p = 1$ blue dotted curve.

In figure 7.3(left) the parameter r is varied while keeping the other parameters p and t_d fixed. Incresing the r-parameter increases the ultimate tensile strength, however, it also increases the "brittleness".

In figure 7.3(right) the parameter p is varied while keeping the other parameters r and t_d fixed. Incresing the p-parameter decreases the ultimate tensile strength, however, it also increases the "brittleness". It can be seen that if $p < -1$ the model shows terminal phase ductility, thus $\sigma \to 0$ when $\varepsilon \to \infty$.

If the loading rate is increased and the other parameters are constant, the behaviour is similar but the ultimate stress is increasing with increasing loading rate, see figure 7.4.

7.3.3 General elastic-damage model

A continuum damage model with a single damage variable can be generalised for a 3-dimensional continuum as

$$\boldsymbol{\sigma} = \phi \boldsymbol{C}^e \boldsymbol{\varepsilon}^e = (1 - D) \boldsymbol{C}^e \boldsymbol{\varepsilon}^e, \tag{7.29}$$

where $\boldsymbol{C}^e$ is the elastic stiffness matrix. Models with single damage parameter are also called isotropic damage models since the effect of damage is the same in all directions.

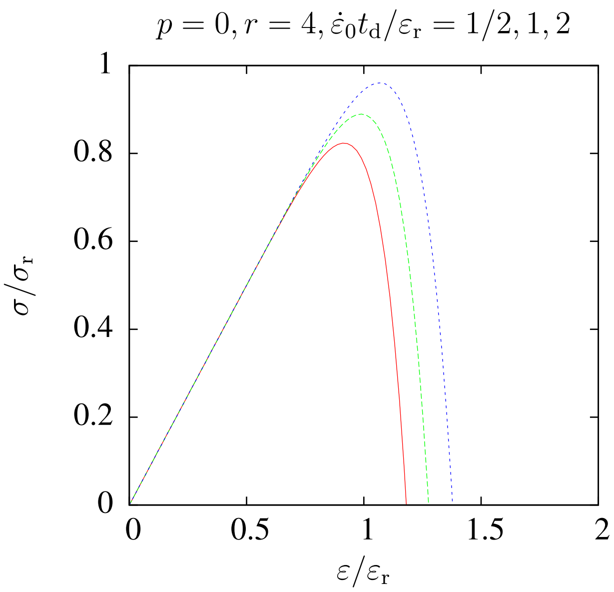


Figure 7.4: Stress-strain relation in a uniaxial constant strain-rate tensile test, $\dot{\varepsilon} t_\mathrm{d}$ varied, i.e. either t_d varied or the loadig rate $\dot{\varepsilon}_0$. Increasing loading rate increases the maximum stress, $\dot{\varepsilon}_0 t_\mathrm{d}/\varepsilon_\mathrm{r} = 1/2$ red solid, $\dot{\varepsilon}_0 t_\mathrm{d}/\varepsilon_\mathrm{r} = 1$, green dashed, $\dot{\varepsilon}_0 t_\mathrm{d}/\epsilon_\mathrm{r} = 2$ blue dotted curve.

7.3.4 On parameter estimation

Calibration of elasticity parameters has been discussed in Chapter 5, only the determination of parameters related to the damage evolution is explaned here. There are three parameters r, p and t_d to be calibrated. However, the p-parameter practically influences only the material's post-peak behaviour and near the region of complete failure. Thus the parameter p can be chosen in advance based purely on computational convenience. The remaining two "real" parameters r and t_d can be determined from two tensile/compression tests performed with different strain-rate. Denoting $\dot{\varepsilon}_{01}$ and $\dot{\varepsilon}_{02}$ the two test strain-rates and $\sigma_{\mathrm{frac},1}, \sigma_{\mathrm{frac},2}$ the correspnding fracture stresses, from (7.28) it is found that

$$r = \frac{1}{2}\left(\frac{\ln(\dot{\varepsilon}_{02}/\dot{\varepsilon}_{01})}{\ln(\sigma_{\mathrm{frac},2}/\sigma_{\mathrm{frac},1})} - 1\right). \tag{7.30}$$

Time parameter t_d is then obtained from either of the failure tests as

$$t_\mathrm{d} = \frac{\varepsilon_\mathrm{r}}{\dot{\varepsilon}_{0i}}\left(\frac{1}{\beta}\frac{\sigma_{\mathrm{frac},i}}{\sigma_\mathrm{r}}\right)^{1/(2r+1)}, i = 1 \text{ or } 2, \text{ and } \beta = \left(\frac{2r+1}{2r+p+2}\right)^{\frac{2r+p+2}{(p+1)(2r+1)}}. \tag{7.31}$$

Chapter 8
Linear viscoelasticity

8.1 Introduction

All the previously described material models have been time- or rate independent, even though the formulation of elasto-plastic constitutive models can conveniently be written in rate-form. However, most materials show a pronounced influence of the rate of loading, especially at high temperatures. For example increasing the strain-rate in a tensile or compression test will result an increase in measured stress. Other viscoelastic effects are (i) *creep*, i.e. increase of strain when the specimen is loaded by a constant stress and (ii) *stress relaxation* when the strain is prescribed.

To describe viscoelastic materials a linear elastic spring and a linear viscous dashpot are frequently used in deriving uniaxial constitutive equations, see Fig. 8.1.

For an elastic spring the length of the spring increases when a tensile force is applied and the spring returns to its original length when the load is removed. However, it is preferable to use the stress σ and strain ε to describe the material behaviour instead of force and displacement. A linear-elastic material is described by a linear relationship between the stress and strain

$$\sigma = E\varepsilon, \tag{8.1}$$

where E is the modulus of elasticity, or the Young's modulus.

For a linear viscous dashpot the force increases linearly with the rate of elongation. In terms of stress σ and strain-rate $\dot{\varepsilon}$ the constitutive model of a linear-viscous material is

$$\sigma = \eta \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \eta\dot{\varepsilon}, \tag{8.2}$$

where η is the viscosity of the material.[1] In fluid mechanics it is spesifically called the dynamic viscosity relating the shear stress to the rate of shear strain.[2]

[1] Usually in fluid mechanics the dynamic viscosity is denoted by μ.
[2] The kinematic viscosity of a fluid, usually denoted by ν, is the ratio of the dynamic viscosity to the density ρ: $\nu = \eta/\rho$.

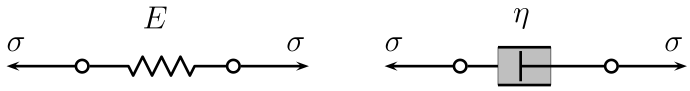


Figure 8.1: Basic viscoelastic elements: spring and dashpot.

More thorough exposition to viscoelasticty can be found in [13, 18, 35]

8.2 Some special functions

Before entering to the actual viscoelastic models some functions are described.

The *Heaviside step function*, or the unit step function, is a discontinuous function defined as

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}. \tag{8.3}$$

The value at $x = 0$ is not usually needed, however, to obtain an odd function the value $H(0) = 1/2$ can be chosen. However, there are other possibilities which are not discussed here. The Heaviside step function can also be written as an integral of the *Dirac delta function* $\delta(x)$

$$H(x) = \int_{-\infty}^{x} \delta(x)\mathrm{d}x. \tag{8.4}$$

The Dirac delta function can be loosely defined as

$$\delta(x) = \begin{cases} +\infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}, \tag{8.5}$$

and it is thus the derivative of the Heaviside step function[3]

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x). \tag{8.6}$$

An important property of the delta function is

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a), \tag{8.7}$$

for an arbitrary continuous function $f(x)$.

[3]In mathematical analysis the Dirac delta function and the Heaviside step functions are examples of *generalized functions* also known as *distributions*. Distributions facilitate differentiation of functions whose derivatives do not exist in the classical sense.

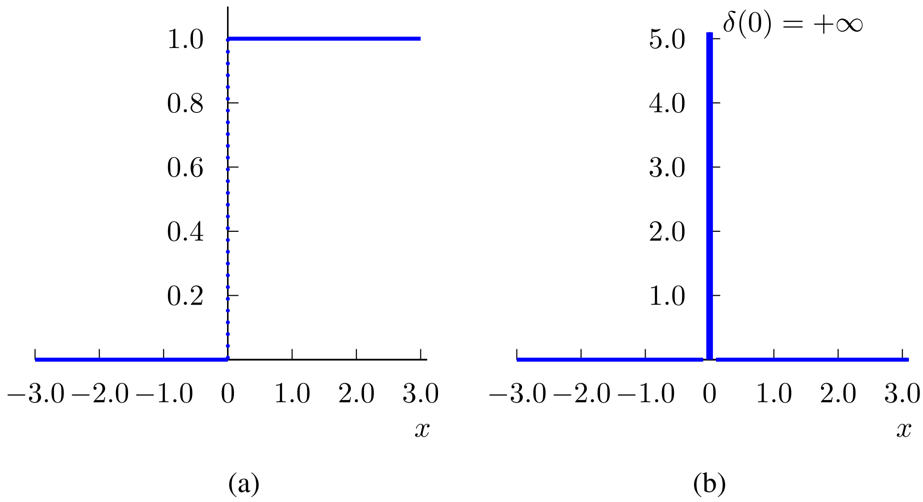


Figure 8.2: (a) The Heaviside unit step function and (b) the Dirac delta function.

8.3 Maxwell's model

A model in which a spring and a dashpot is combined in a series is known as the Maxwell's model of viscoelasticity, and it is illustrated in Fig. 8.3. The total strain ε is now additively divided into the elastic strain ε^{e} in the spring and viscous strain ε^{v} in the dashpot

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{v}}. \tag{8.8}$$

Since the stress in both elements is the same

$$\sigma = E\varepsilon^{\mathrm{e}} = \eta\dot{\varepsilon}^{\mathrm{v}}. \tag{8.9}$$

Taking time derivative by parts of the constitutive equation for the linear spring gives

$$\dot{\sigma} = E\dot{\varepsilon}^{\mathrm{e}} = E(\dot{\varepsilon} - \dot{\varepsilon}^{\mathrm{v}}). \tag{8.10}$$

Substituting now the constitutive equation of the dashpot to the equation (8.10) the final form of the of the Maxwell's viscoelastic model is obtained

$$\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon}. \tag{8.11}$$

Behaviour in a creep test. In a creep test a constant stress $\sigma = \sigma_0$ is imposed suddenly at time $t = 0$. Thus the stress rate $\dot{\sigma}$ is zero for $t > 0$, and the equation (8.11) gives directly the strain-rate

$$\dot{\varepsilon} = \eta^{-1}\sigma_0, \tag{8.12}$$

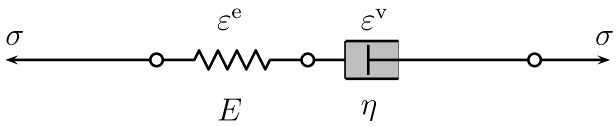


Figure 8.3: Maxwell's material model of viscoelasticity.

i.e. the creep strain-rate is a constant and depends linearly on the applied stress. Simple integration of the equation results in

$$\varepsilon(t) = \eta^{-1}\sigma_0 t + C, \tag{8.13}$$

where C is the inegration constant, which can be determined from the initial condition

$$\varepsilon(0) = E^{-1}\sigma_0. \tag{8.14}$$

Solution for a constant stress creep problem for the Maxwell model is thus

$$\varepsilon(t) = \frac{\sigma_0}{E}\left(1 + \frac{E}{\eta}t\right) = \frac{\sigma_0}{E}\left(1 + \frac{t}{\tau}\right) = \sigma_0 J(t), \tag{8.15}$$

where $\tau = \eta/E$ is the relaxation time and the function J is called the *creep compliance*. It defines the strain per unit applied stress and for $t > 0$ it is monotonously increasing function. For $t < 0$, $J(t) \equiv 0$.

Behaviour in a relaxation test. In a relaxation test the material is loaded by a suddenly applied constant strain ε_0 at time $t = 0$. Thus the strain rate $\dot{\varepsilon}$ vanish for times $t > 0$. When the strain is imposed at $t = 0$ the elastic component reacts immediately, therefore the initial value for the stress is $\sigma(0) = \sigma_0 = E\varepsilon_0$. The differential equation to be solved is

$$\dot{\sigma} + \frac{E}{\eta}\sigma = 0, \tag{8.16}$$

with the initial condition $\sigma(0) = \sigma_0 = E\varepsilon_0$. Trying to find the solution in the form $\sigma(t) = C\exp(rt)$, and substituting it into the equation (8.16) results in

$$C\mathrm{e}^{rt}(r + E/\eta) = 0, \tag{8.17}$$

which gives the value $r = -E/\eta$ and the solution of the homogeneous differential equation (8.16) is

$$\sigma(t) = C\mathrm{e}^{-Et/\eta}. \tag{8.18}$$

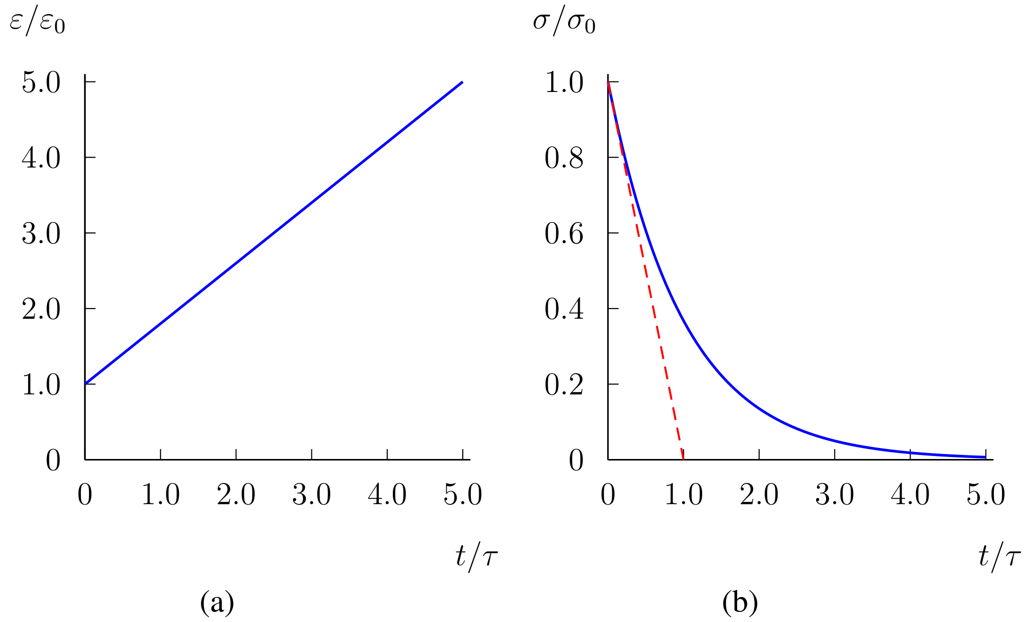


Figure 8.4: Behaviour of Maxwell's viscoelastic model in (a) creep and (b) relaxation tests. The dashed red line indicates hypothetical relaxation with initial rate giving the physical intepretation of the relaxation time $\tau = \eta/E$.

The integration constant C is determined from the initial condition

$$\sigma(0) = C = \sigma_0. \tag{8.19}$$

Solution for the relaxation problem of the Maxwell viscoelastic model is

$$\sigma(t) = \sigma_0 \mathrm{e}^{-Et/\eta} = \sigma_0 \mathrm{e}^{-t/\tau} = \sigma_0 G(t), \tag{8.20}$$

where the function $G(t)$ is called the *relaxation modulus*, which is the stress developed in a relaxation test when loaded by a unit strain. This form gives also a simple physical meaning for the relaxation time τ. It is a hypothetical time after which the stress is relaxed to zero if the complete relaxation takes place with the initial rate, see Fig. 8.4.

It is also seen from (8.20) that the stress will tend to zero in the limit $t \to \infty$. Therefore the Maxwell model is often considered as a fluid model. However, distinction between a fluid and a solid is is not a trivial task.

Uniaxial tensile test, influence of strain rate. If the strain is increased with a constant rate, i.e. $\varepsilon(t) = \dot{\varepsilon}_0 t$, the constitutive equation (8.11) has the form

$$\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon}_0. \tag{8.21}$$

The solution for the homogeneous equation is given in (8.18). A general solution for the non-homogeneous equation (8.21) is a sum of the general solution of the homogeneous equation and a *particular solution*, which in this case can be chosen to be a constant, thus

$$\sigma(t) = C\exp(-Et/\eta) + B. \tag{8.22}$$

Substituting it into (8.21) gives $B = \eta\dot{\varepsilon}_0$. The integration constant C can be solved from the initial condition $\sigma(0) = 0$, giving $C = -B$, and the complete solution is

$$\sigma(t) = \eta\dot{\varepsilon}_0\left(1 - \mathrm{e}^{-Et/\eta}\right). \tag{8.23}$$

It is seen that the limiting value when $t \to \infty$ is $\eta\dot{\varepsilon}_0$.

To obtain a stress-strain relationship, time can be eliminated from $\varepsilon(t) = \dot{\varepsilon}_0 t$, giving

$$\sigma(\varepsilon) = \eta\dot{\varepsilon}_0\left(1 - \mathrm{e}^{-E\varepsilon/\eta\dot{\varepsilon}_0}\right). \tag{8.24}$$

From this equation, it can be verified that the modulus of elasticity for the Maxwell model does not depend on strain rate

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}_{|\varepsilon=0} = E. \tag{8.25}$$

Defining an arbitrary reference stress σ_r and a reference strain $\varepsilon_\mathrm{r} = \sigma_\mathrm{r}/E$, the stress-strain relation can be written in the form

$$\frac{\sigma}{\sigma_\mathrm{r}} = \frac{\eta\dot{\varepsilon}_0}{\sigma_\mathrm{r}}\left[1 - \exp\left(\frac{\sigma_\mathrm{r}}{\eta\dot{\varepsilon}_0}\frac{\varepsilon}{\varepsilon_\mathrm{r}}\right)\right]. \tag{8.26}$$

In Fig. 8.5 the stress-strain is shown for various values of the strain-rate $\dot{\varepsilon}_0$.

8.4 Kelvin model

Another basic viscoelastic model is the Kelvin model[4], where spring and dashpot are placed in parallel, see Fig. 8.6. Now the stress σ is divided into components

$$\sigma = \sigma_1 + \sigma_2, \qquad \text{where} \qquad \sigma_1 = E\varepsilon, \qquad \text{and} \qquad \sigma_2 = \eta\dot{\varepsilon}, \tag{8.27}$$

and the costitutive equation for the viscoelastic Kelvin model is readily obtained in the form

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}. \tag{8.28}$$

[4] Also known as the Kelvin-Voigt model.

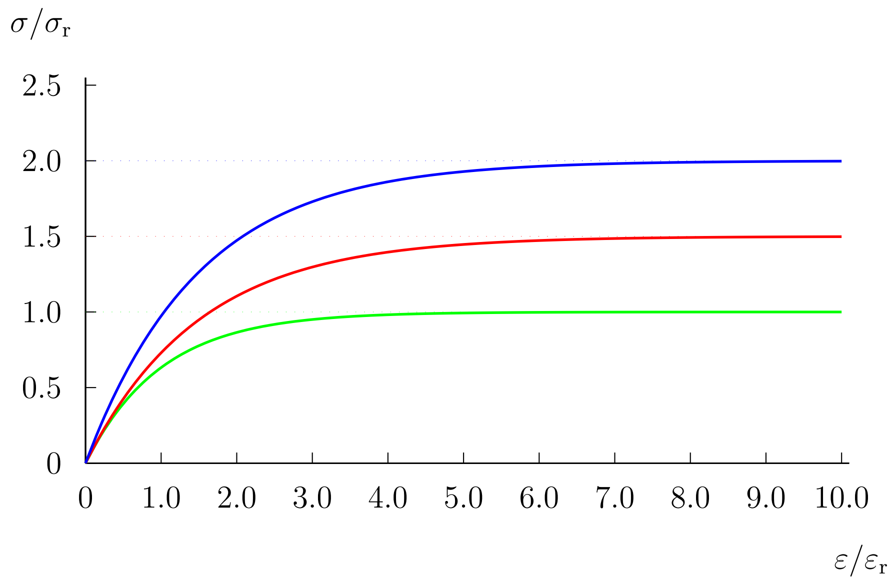


Figure 8.5: Behaviour of Maxwell's viscoelastic model in a tensile test perform with prescribed strain rates $\dot{\varepsilon}_0 = \sigma_r/\eta$ (green line), $1.5\sigma_r/\eta$ (red line) and $2\sigma_r/\eta$ (blue line). The limiting stress is shown by dotted lines.

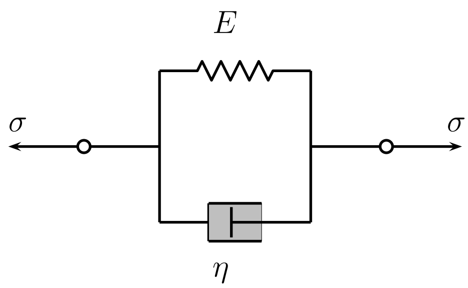


Figure 8.6: Viscoelastic Kelvin model.

Behaviour in a creep test. In a creep test a constant stress $\sigma = \sigma_0$ is imposed suddenly at time $t = 0$. From (8.28) it is obtained

$$\dot{\varepsilon} + \frac{E}{\eta}\varepsilon = \frac{\sigma_0}{\eta}. \tag{8.29}$$

Analoguous to (8.22) general solution is of the form

$$\varepsilon(t) = C\exp(-E/\eta t) + B. \tag{8.30}$$

Substituting the particular solution (constant B) into (8.29) gives $B = \sigma_0/E$, thus

$$\varepsilon(t) = C\exp(-Et/\eta) + \sigma_0/E. \tag{8.31}$$

The integration constant C can be determined from the initial condition. However, it is not as obvious as in the case of the Maxwell model. For the Kelvin model there is *no instantaneous elasticity* due to the parallel conbination of the spring and dashpot. Therefore the proper initial condition for the creep test is $\varepsilon(0) = 0$, which results in $C = -\sigma_0/E$, and the solution for the creep problem of the Kelvin model is

$$\varepsilon(t) = \frac{\sigma_0}{E}\left(1 - \mathrm{e}^{-Et/\eta}\right), \tag{8.32}$$

which is shown in Fig. 8.7a. Thus the creep compliance for the Kelvin model is

$$J(t) = \frac{1}{E}\left(1 - \mathrm{e}^{-Et/\eta}\right). \tag{8.33}$$

At the limit the creep strain of the Kelvin models approaches

$$\lim_{t\to\infty}\varepsilon(t) = \frac{\sigma_0}{E} = \varepsilon_\infty. \tag{8.34}$$

Relaxation test. If the suddenly imposed constant strain function $\varepsilon(t) = \varepsilon_0 H(t)$ is substituted into (8.28) results in

$$\sigma(t) = E\varepsilon_0 H(t) + \eta\varepsilon_0\delta(t), \tag{8.35}$$

which shows no stress relaxation when $t > 0$ and the graph is shown in Fig. 8.7b. The infinite stress at the jump is due to the viscous dashpot.

Behaviour of the Kelvin model in uniaxial constant strain-rate loading is unrealistic for most materials and will not be discussed here. Determination of the response in constant strain-rate loading is left as an exercise.

As a conclusion, the viscoelastic Kelvin model alone is a poor description of actual material, either solid or fluid.

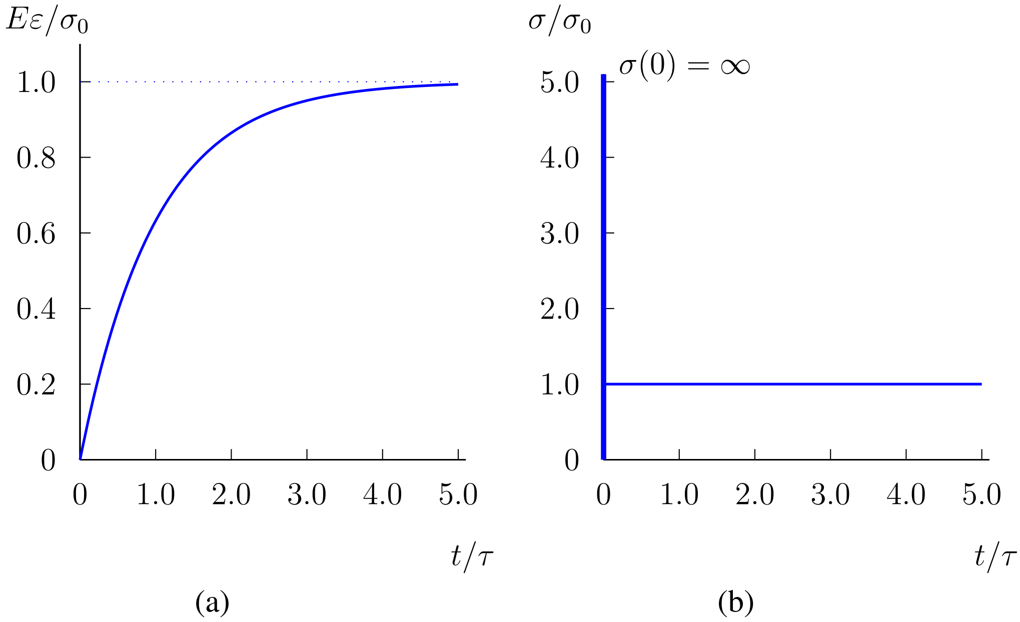


Figure 8.7: Behaviour of Kelvin's viscoelastic model in (a) creep and (b) relaxation tests.

8.5 Linear viscoelastic standard model

In Fig. 8.8 a three parameter model where an elastic spring and a Kelvin element is in series. Such a model is known as the *standard linear viscoelastic solid model*, also known as the *Zener model*.[5] The same behaviour can also be obtained if the linear spring is in parallel with the Maxwell model.[6] Derivation of the constitutive equation is much more involved in comparison to Maxwell and Kelvin models.

[5] Flügge [13] calls the standard viscoelastic model simply as a 3-parameter model.
[6] There exist also a standard linear viscoelastic *fluid* model.

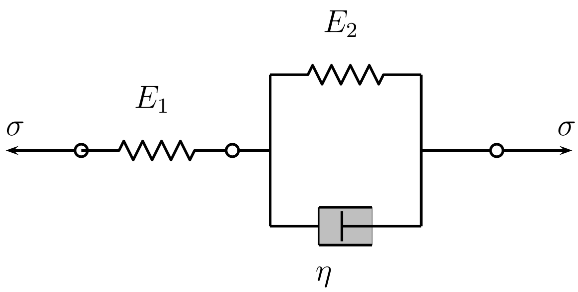


Figure 8.8: Viscoelastic linear standard solid model.

The constitutive law in the linear spring on the rigth hand side is

$$\sigma = E_1\varepsilon_1, \tag{8.36}$$

and the stress σ is divided into two components in the Kelvin element

$$\sigma = \sigma_1 + \sigma_2, \qquad \text{where} \qquad \begin{cases} \sigma_1 = E_2\varepsilon_2, \\ \sigma_2 = \eta\dot{\varepsilon}_2. \end{cases} \tag{8.37}$$

The total strain of the three-parameter element is

$$\varepsilon = \varepsilon_1 + \varepsilon_2. \tag{8.38}$$

Differentiating by parts w.r.t. time, Eq. (8.36) gives

$$\dot{\sigma} = E_1\dot{\varepsilon}_1 = E_1(\dot{\varepsilon} - \dot{\varepsilon}_2). \tag{8.39}$$

The strain-rate in the Kelvin element is

$$\begin{aligned} \dot{\varepsilon}_2 &= \frac{\sigma_2}{\eta} = \frac{\sigma - \sigma_1}{\eta} \\ &= \frac{1}{\eta}[\sigma - E_2(\varepsilon - \varepsilon_1)] = \frac{1}{\eta}\left[\sigma - E_2\left(\varepsilon - \frac{\sigma}{E_1}\right)\right] \\ &= \frac{1}{\eta}\left[\left(1 + \frac{E_2}{E_1}\right)\sigma - E_2\varepsilon\right]. \end{aligned} \tag{8.40}$$

Substituting this expression into Eq. (8.39) gives the final form of the constitutive equation for the standard solid

$$\dot{\sigma} + \frac{E_1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = E_1\dot{\varepsilon} + \frac{E_1E_2}{\eta}\varepsilon. \tag{8.41}$$

Creep test. In the creep test the stress σ_0 is suddenly applied at time $t = 0$, thus $\sigma(t) = \sigma_0 H(t)$ and substituting it into eq. (8.41) gives

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma_0 + \delta(t)\frac{\sigma_0}{E_1}, \tag{8.42}$$

where δ is the Dirac delta function. A trial function for the particular solution is

$$\varepsilon_{\mathrm{p}}(t) = BH(t). \tag{8.43}$$

Substituting this expression into Eq. (8.42) gives the value

$$B = \left(1 + \frac{E_2}{E_1}\right)\frac{\sigma_0}{E_2}. \tag{8.44}$$

$$\varepsilon(t) = A\exp(-E_2t/\eta) + \left(1 + \frac{E_2}{E_1}\right)\frac{\sigma_0}{E_2} \tag{8.45}$$

Due to the spring element on the left hand side, the model can show an *instantaneous elastic strain*. The initial condition is thus

$$\lim_{t\to 0+} \varepsilon(t) = \varepsilon_0 = \frac{\sigma_0}{E_1}, \tag{8.46}$$

the integration constant A can be solved, resulting in

$$A = -\frac{\sigma_0}{E_2}. \tag{8.47}$$

Solution to the creep problem for the viscoelastic standard solid is

$$\varepsilon(t) = \frac{\sigma_0}{E_2}\left[1 + \frac{E_2}{E_1} - \exp(-E_2t/\eta)\right] = \frac{\sigma_0}{E_1}\left[1 + \frac{E_1}{E_2}\left(1 - \exp(-E_2t/\eta)\right)\right]. \tag{8.48}$$

The creep compliance is thus

$$J(t) = \frac{1}{E_1}\left[1 + \frac{E_1}{E_2}\left(1 - \exp(-E_2t/\eta)\right)\right]. \tag{8.49}$$

It is easily seen that the limiting strain when $t \to \infty$ is

$$\varepsilon_\infty = (1 + E_1/E_2)\frac{\sigma_0}{E_1}. \tag{8.50}$$

Relaxation test. In the relaxation test the strain is prescribed as $\varepsilon(t) = \varepsilon_0 H(t)$, thus to obtain the relaxation function the following differential equation has to be solved

$$\dot{\sigma} + \frac{E_1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = \frac{E_1E_2}{\eta}\varepsilon_0, \tag{8.51}$$

with the initial condition

$$\sigma(0) = E_1\varepsilon_0. \tag{8.52}$$

When the strain is suddenly imposed, the left elastic spring can only respond instantaneously, while the Kelvin element on the right hand side is initially infintely stiff, see Fig. 8.7. Solution of the homogeneous part of Eq. (8.51) has the form

$$\sigma_{\mathrm{h}}(t) = A\exp(-(E_1 + E_2)t/\eta), \tag{8.53}$$

and the particular solution is simply a constant $\sigma_{\mathrm{p}} = C$, the value of which can be found to be

$$C = \frac{E_1E_2}{E_1 + E_2}\varepsilon_0. \tag{8.54}$$

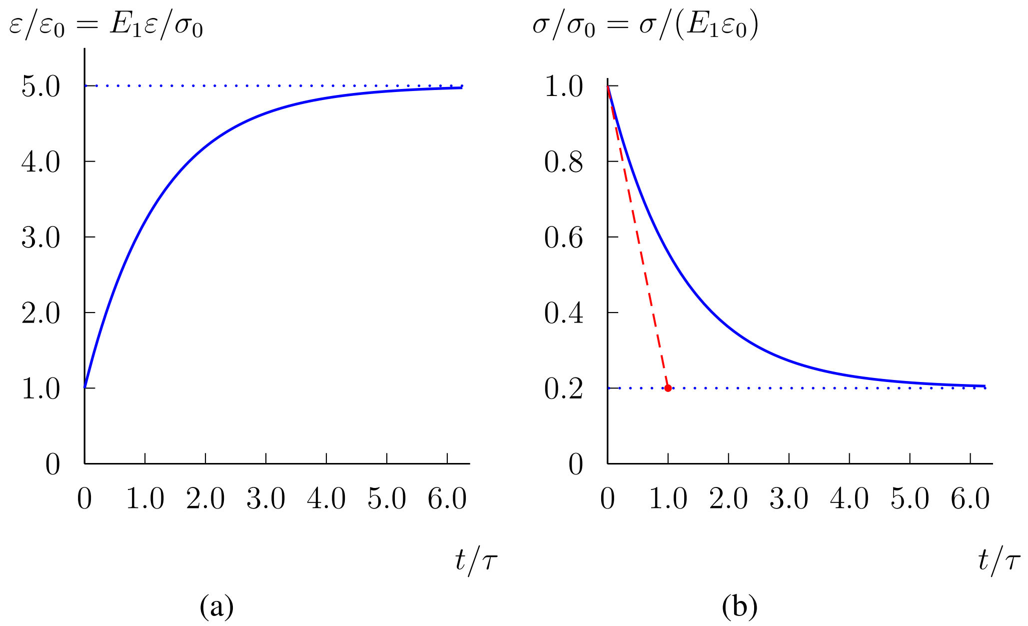


Figure 8.9: Behaviour of standard viscoelastic model in (a) creep and (b) relaxation tests ($E_2 = E_1/4$).

Using the initial condition, the following value for A can be obtained

$$A = \frac{E_1^2}{E_1 + E_2}\varepsilon_0. \tag{8.55}$$

The complete solution for the relaxation problem is thus

$$\sigma(t) = \frac{E_1\varepsilon_0}{E_1 + E_2}\left[E_1 \mathrm{e}^{-t/\tau} + E_2\right] = \sigma_0\left[\frac{E_1}{E_1 + E_2}\mathrm{e}^{-t/\tau} + \frac{E_2}{E_1 + E_2}\right]\sigma_0 G(t), \tag{8.56}$$

where $\tau = \eta/(E_1 + E_2)$ is the relaxation time of the standard linear solid model.

It is easy to see that the limiting stress when $t \to \infty$ is

$$\sigma_\infty = \frac{E_1E_2\varepsilon_0}{E_1 + E_2} = \frac{E_1\varepsilon_0}{1 + E_1/E_2} = \frac{\sigma_0}{1 + E_1/E_2}. \tag{8.57}$$

Uniaxial tensile test, influence of strain rate. If a uniaxial tensile test is performed with a prescribed strain rate, i.e. $\varepsilon(t) = \dot{\varepsilon}_0 t$, the response can be solved from the equation (8.41) after substituting the prescribed strain into it, rsulting in

$$\dot{\sigma} + \frac{E}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = E_1\dot{\varepsilon}_0 + \frac{E_1E_2}{\eta}\dot{\varepsilon}_0. \tag{8.58}$$

Solution of this linear ordinary constant coefficient differential equation can be obtained as a sum of the general solution of the homogeneous equation and a particular solution satisfying the full equation (8.58):

$$\sigma_{\rm h} = A\exp(-(E_1+E_2)t/\eta), \qquad \sigma_{\rm p} = C_1 + C_2 t. \tag{8.59}$$

Substituting the particular solution into (8.58) results in

$$C_2 + \frac{E_1+E_2}{\eta}(C_1+C_2t) = E_1\dot{\varepsilon}_0 + \frac{E_1E_2}{\eta}\dot{\varepsilon}_0 t.$$

$$C_1 = \frac{E_1^2}{(E_1+E_2)^2}\dot{\varepsilon}_0\eta, \qquad C_2 = \frac{E_1E_2}{E_1+E_2}\dot{\varepsilon}_0.$$

The coefficient A is solved from the initial condition $\sigma(0)=0$, and it is

$$A = -C_1 = -\frac{E_1^2}{(E_1+E_2)^2}\dot{\varepsilon}_0\eta.$$

$$\sigma(t) = \frac{E_1^2}{(E_1+E_2)^2}\dot{\varepsilon}_0\eta\left(1-\exp(-(E_1+E_2)t/\eta)\right) + \frac{E_1E_2}{E_1+E_2}\dot{\varepsilon}_0 t. \tag{8.60}$$

Expressing the equation as a function of strain, the stress-strain relation in a constant strain rate tensile test is thus

$$\sigma(\varepsilon) = \frac{E_1^2}{(E_1+E_2)^2}\dot{\varepsilon}_0\eta\left(1-\exp(-(E_1+E_2)\varepsilon/\dot{\varepsilon}_0\eta)\right) + \frac{E_1E_2}{E_1+E_2}\varepsilon. \tag{8.61}$$

Notice that the Young's modulus of the linear standard viscoelastic solid is

$$E = \frac{{\rm d}\sigma}{{\rm d}\varepsilon}_{|\varepsilon=0} = \frac{E_1E_2}{E_1+E_2} + \frac{E_1^2}{(E_1+E_2)^2}\dot{\varepsilon}_0\eta\frac{E_1+E_2}{\dot{\varepsilon}_0\eta} = \frac{E_1E_2}{E_1+E_2} + \frac{E_1}{E_1+E_2} = E_1. \tag{8.62}$$

Defining an arbitrary reference stress $\sigma_{\rm r}$ and a reference strain $\varepsilon_{\rm r} = \sigma_{\rm r}/E_1$, the stress-strain relation can be written in the form

$$\frac{\sigma}{\sigma_{\rm r}} = \frac{E_1^2}{(E_1+E_2)^2}\frac{\dot{\varepsilon}_0\eta}{\sigma_{\rm r}}\left(1-\exp[-(1+E_2/E_1)(\sigma_{\rm r}/\dot{\varepsilon}_0\eta)(\varepsilon/\varepsilon_{\rm r})]\right) + \frac{1}{1+E_1/E_2}\frac{\varepsilon}{\varepsilon_{\rm r}}. \tag{8.63}$$

In Fig. 8.10 the stress-strain is shown for various values of the strain-rate $\dot{\varepsilon}_0$.

Notice that the stress-strain relation resembles of the strain hardening elasto-plastic model. The tangent modulus approaches the value E_2 with increasing strain. Due to linearity the strain increases linearly with increasing strain-rate.

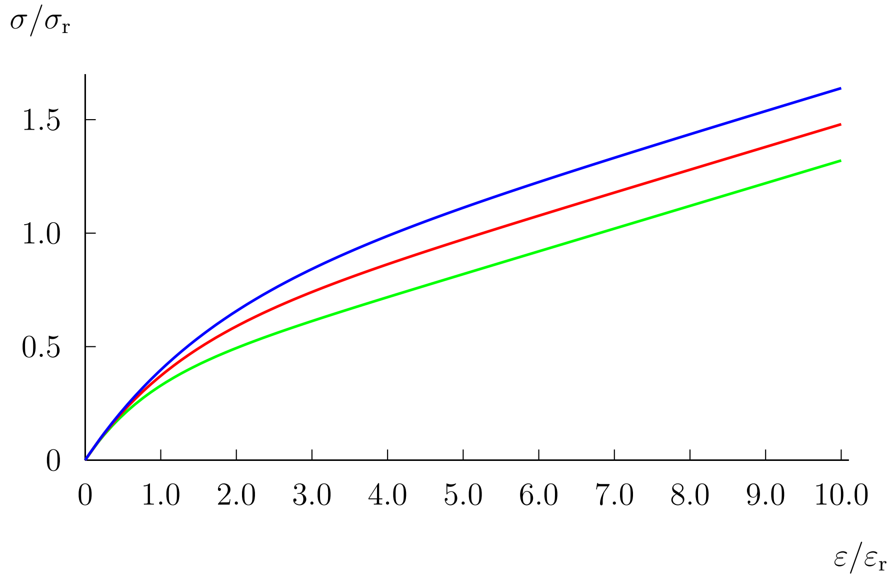


Figure 8.10: Behaviour of the standard linear viscoelastic solid model in a tensile test perform with prescribed strain-rates $\dot{\varepsilon}_0 = \sigma_{\rm r}/\eta$ (green line), $1.5\sigma_{\rm r}/\eta$ (red line) and $2\sigma_{\rm r}/\eta$ (blue line). Notice that the limiting tangent modulus $d\sigma/d\varepsilon$ is E_2.

8.6 Hereditary approach

8.7 Generalizations

Any linear model of the Maxwell or Kelvin type can be described by the differential equation in the form

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \cdots = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \cdots \tag{8.64}$$

or written in a concise form

$$\mathrm{P}\sigma = \mathrm{Q}\varepsilon \tag{8.65}$$

where P and Q are linear differential operators

$$\mathrm{P} = \sum_{k=1}^{m} p_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}, \quad \mathrm{Q} = \sum_{k=1}^{n} q_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}. \tag{8.66}$$

For isotropic three dimensional case, the viscoelastic formulation is conveniently done in the form where the stress and strain are decomposed in the spherical and deviatoric parts. The hydrostatic stress must produce a dilatation and not distortion, the linear elastic relationship between the mean stress $\sigma_{\rm m}$ and the volumetric strain $\varepsilon_{\rm v}$ (5.24) can be simply generalized to viscoelastic one as

$$\mathrm{P}''\sigma_{\rm m} = \mathrm{Q}''\varepsilon_{\rm v}. \tag{8.67}$$

In a similar way, the deviatoric parts of the stress and strain tensors are related as

$$\mathrm{P}'\boldsymbol{s} = \mathrm{Q}'\boldsymbol{e}. \tag{8.68}$$

The four operators $\mathrm{P}', \mathrm{Q}', \mathrm{P}''$ and Q'' describing the viscoelastic behaviour of a material are independent of each other. As a limiting case of a linear isotropic viscoelastic material is naturally the linear elastic isotropi material for which

$$\mathrm{P}' = \mathrm{P}'' = 1, \quad \mathrm{Q}' = 2G, \quad \mathrm{Q}'' = K. \tag{8.69}$$

For metals the dilatation can be assumed to be elastic and only the deviatoric part exhibits viscoelastic behaviour.

8.8 Solved example problems

Example 8.1. *Derive the constitutive equation for the following shown below where the Maxwell- and Kelvin elements are in parallel. Determine also the creep- and relaxation functions.*

Solution. Let us use the already solved constitutive equations for the Maxwell and Kelvin elements

$$\dot{\sigma}_M + \frac{E_2}{\eta_2}\sigma_M = E_2\dot{\varepsilon},$$
$$\dot{\sigma}_K = E_1\dot{\varepsilon} + \eta_1\varepsilon,$$

or

$$\dot{\sigma}_M = -\frac{E_2}{\eta_2}\sigma_M + E_2\dot{\varepsilon},$$
$$\dot{\sigma}_K = E_1\dot{\varepsilon} + \eta_1\varepsilon,$$

and substittin these into the equilibriu equation

$$\sigma = \sigma_M + \sigma_K, \quad \text{in rate form} \quad \dot{\sigma} = \dot{\sigma}_M + \dot{\sigma}_K,$$

which gives

$$\dot{\sigma} = -\frac{E_2}{\eta_1}\sigma_M + E_2\dot{\varepsilon} + E_1\dot{\varepsilon} + \eta_1\ddot{\varepsilon}.$$

From the equilibrium equation we get $\sigma_M = \sigma - \sigma_K = \sigma - E_1\varepsilon - \eta_1\dot{\varepsilon}$, hence

$$\dot{\sigma} + \frac{E_2}{\eta_2}\sigma = \eta_1\ddot{\varepsilon} + E_2\left(1 + \frac{E_1}{E_2} + \frac{\eta_1}{\eta_2}\right)\dot{\varepsilon} + \frac{E_1E_2}{\eta_2}\varepsilon. \tag{8.70}$$

Let us transform the equation to the form

$$\sigma + \frac{\eta_2}{E_2}\dot{\sigma} = E_1\varepsilon + \eta_2\left(1 + \frac{E_1}{E_2} + \frac{\eta_1}{\eta_2}\right)\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_2}\ddot{\varepsilon}$$

and putting it into the form

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon}, \tag{8.71}$$

where

$$\begin{aligned} p_1 &= \frac{\eta_2}{E_2}, \\ q_0 &= E_1, \\ q_1 &= \eta_2\left(1 + \frac{E_1}{E_2} + \frac{\eta_1}{\eta_2}\right) = \eta_1 + \left(1 + \frac{E_1}{E_2}\right)\eta_2, \\ q_2 &= \frac{\eta_1\eta_2}{E_2}. \end{aligned}$$

Creep response. The loading is now $\sigma(t) = \sigma_0 H(t)$, where $H(t)$ is the unit step function, thus $\dot{\sigma} \equiv 0$ when $t > 0$. The equation to be solved is

$$q_2\ddot{\varepsilon} + q_1\dot{\varepsilon} + q_0\varepsilon = \sigma_0, \quad t > 0. \tag{8.72}$$

Since the equation is a second order ordinary differential equation, two initial conditions are needed: $\varepsilon(0)$ and $\dot{\varepsilon}(0)$.

When investigating the model, it is observed that in the sudden stretching the Kelvin element cannot react and is infinitely rigid. Since the Kelvin element is in parallel with the Maxwell element, then $\varepsilon(0) = 0$. The initial strain rate is obtained by integrating the constitutive equation (8.71) over the initial time instant $t = 0$ and in the limit

$$\lim_{\delta\to 0}\int_{-\delta}^{\delta}(\sigma + p_1\dot{\sigma})\,\mathrm{d}t = \lim_{\delta\to 0}\int_{-\delta}^{\delta}(q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon})\,\mathrm{d}t,$$

resulting in

$$\lim_{\delta\to 0}\Big|_{-\delta}^{\delta}(\sigma_0\langle t\rangle + p_1\sigma_0 H(t)) = \lim_{\delta\to 0}\Big|_{-\delta}^{\delta}(q_1\varepsilon + q_2\dot{\varepsilon}), \tag{8.73}$$

since

$$\lim_{\delta\to 0}\int_{-\delta}^{\delta} q_0\varepsilon\,\mathrm{d}t = 0,$$

where $\langle t\rangle$ are the Macaulayn-brakets i.e. the ramp function

$$\langle t\rangle = \begin{cases} 0 & \text{kun} \quad t < 0 \\ t & \text{kun} \quad t \geq 0 \end{cases}.$$

From (8.73) it is obtained

$$p_1\sigma_0 = q_1\varepsilon(0) + q_2\dot{\varepsilon}(0) = q_2\dot{\varepsilon}(0),$$

and

$$\dot{\varepsilon}(0) = \frac{p_1}{q_2}\sigma_0 = \frac{\sigma_0}{\eta_1}.$$

The general solution for the homogeneous part of the equation (8.72) is looked in the form $\varepsilon = \exp(rt)$, and it results in the characteristic polynomial

$$q_2 r^2 + q_1 r + q_0 = 0,$$

and the solution for the roots is

$$r = -\frac{q_1}{2q_2}\left(1 \pm \sqrt{1 - 4\frac{q_0 q_2}{q_1^2}}\right).$$

Since the solution cannot be oscillatory, the discriminant $D = 1 - 4q_0q_2/q_1^2$ has to be positive resulting into inequality

$$1 + 2(e+n) + (e-n)^2 > 0,$$

where $e = E_1/E_2$ and $n = \eta_1/\eta_2$. It is observed that the inequality is satisfied for all parameter values.

The creep behaviour is thus of the form

$$\varepsilon(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t) + A,$$

where A some particular solution, for which a constant can be selected $A = \sigma_0/q_0 = \sigma_0/E_1$. The creep strain rate is

$$\dot{\varepsilon}(t) = r_1 c_1 \exp(r_1 t) + r_2 c_2 \exp(r_2 t),$$

and the second initial condition gives

$$\dot{\varepsilon}(0) = r_1 c_1 + r_2 c_2 = \frac{\sigma_0}{\eta_1}.$$

The constants c_1 and c_2 can be solved from the system

$$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where

$$b_1 = \sigma_0, \quad b_2 = \frac{\sigma_0}{\eta_1}.$$

The solution is

$$c_1 = \frac{r_2 b_1 - b_2}{r_2 - r_1}, \quad c_2 = \frac{b_2 - r_1 b_1}{r_2 - r_1}.$$

When $t \to \infty$ only the spring in the Kelvin part is active, hence

$$\lim_{t\to\infty} \varepsilon(t) = \frac{\sigma_0}{E_1}.$$

Relaxation. Now the loading is an instant straining $\varepsilon(t) = \varepsilon_0 H(t)$. The constitutive equation is now

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon_0, \quad t > 0, \quad \text{eli} \quad \sigma + \frac{\eta_2}{E_2}\dot{\sigma} = E_1\varepsilon_0,$$

with initial condition $\sigma(0+) = (E_1 + E_2)\varepsilon_0$. Solution of the homogeneous part can again be tried in the form $\sigma(t) = \exp(rt)$, and the characteristic equation is $1 + p_1 r = 0$, thus

$$\sigma(t) = C\exp(-t/p_1) + q_0\varepsilon_0,$$

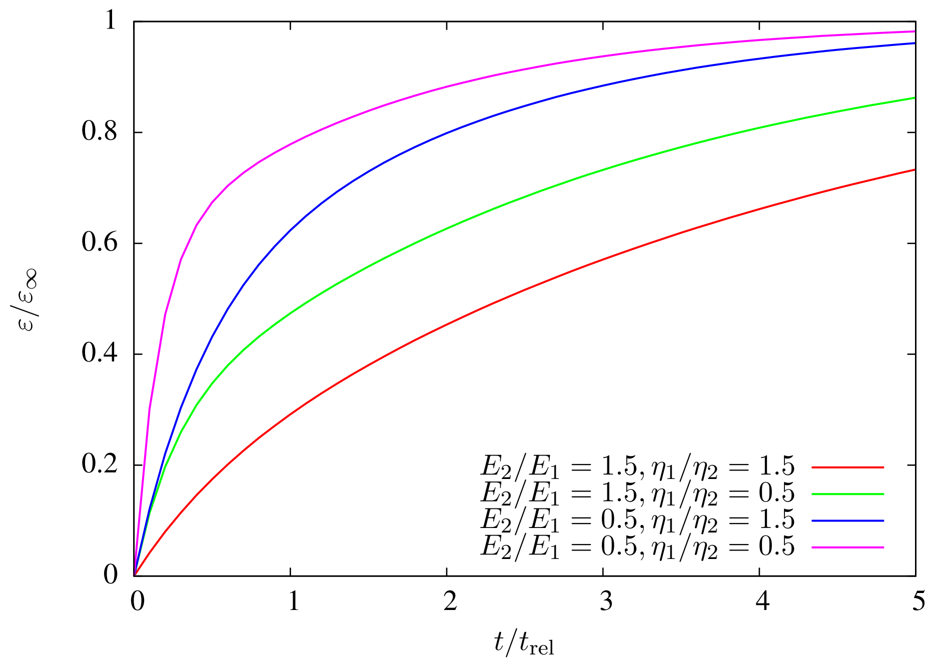
where the last part is the particular solution. From the initial condition the value for the C-parameter is obtained as $C = (E_2 - E_1)\varepsilon_0$. "The initial stress" is now $\sigma(0+) = (E_1 + E_2)\varepsilon_0$ and finally the relaxation function has the form

$$\sigma(t) = (E_1 + E_2\exp(-t/p_1))\,\varepsilon_0 = E_1\left(1 + \frac{E_2}{E_1}\exp(-t/p_1)\right)\varepsilon_0.$$

The relaxation time is equal to the Maxwell model

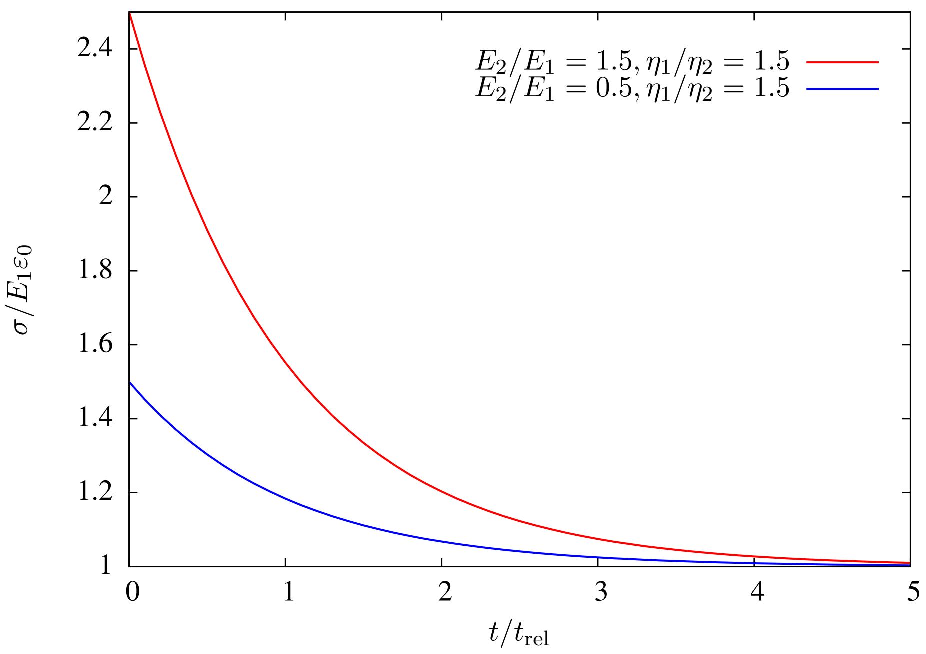
$$t_{\text{rel}} = p_1 = \frac{\eta_2}{E_2}.$$

In the figure below the creep behaviour is shown with some E_2/E_1 and η_1/η_2 values and $\varepsilon_\infty = \sigma_0/E_1$.



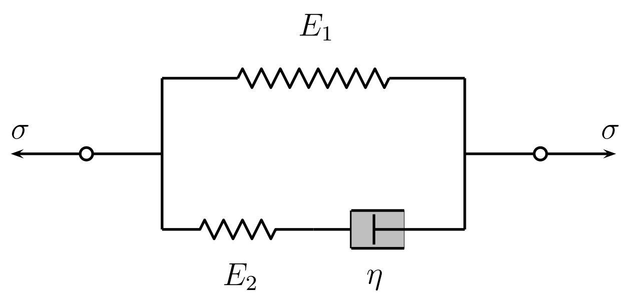
The model describes only the primary creep phase and thus cannot describe the instant elastic strain and secondary creep phase.

In the figure below the relaxation function is shown.



8.9 Exercises

1. Determine the constitutive equation for the standard linear solid model where the Maxwell element is parallel with the linear spring. Determine also the creep and relaxation functions and the relaxation time.



2. The Maxwell model, parameters E and η, is loaded by prescribed cyclic strain $\varepsilon(t) = \varepsilon_a \sin(\omega t)$, where ε_a is the strain amplitude. Solve the stress response $\sigma(t)$.

What can you say about the beaviour if $\omega \gg 1/\tau_{\mathrm{rel}}$, where $\tau_{\mathrm{rel}} = \eta/E$ the relaxation time of the Maxwell model.

Hint. Try the solution in the form $\sigma(t) = \sigma_{\mathrm{a}} \sin(\omega t + \phi)$. Solve the phase angle ϕ and stress amplitude σ_{a}. Some formulas which might be usefull:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\cos\alpha = \frac{1}{\sqrt{1 + \tan^2\alpha}}.$$

Chapter 9
Creep and creep fracture

9.1 Introduction

Creep is time dependent inelastic deformation which is usually divided into three phases shown schematically in Fig. 9.1[1]. For metals and ceramics the room-temperature behaviour can practically be considered as time independent. For metals creep starts to be significant when temperature exceeds 30 % of the melting temperature [3]. Therefore for structures used with energy conversion applications, like turbines, reactors, boilers creep has to be taken into account in their analysis and design.

In the primary phase, the creep strain-rate gradually decreases to a certain minimum value. This time instant where the minimum strain-rate is reached determines the change from primary to the secondary stage. During the primary phase dislocation movement is gradually slowed down at the "erkauma"particles and the material is hardening. A characteristic feature of the secondary creep phase is that the creep strain-rate is almost a constant, and at that stage the birth and annihilation of dislocations are balanced. Voids are formed at the grain boundaries, which starts to grow at the tertiary creep phase and weakens the material causing the increase of creep strain-rate. This phase ends to a rupture at t_{rup}, see 9.1.

The effect of temperature and stress is roughly speaking similar, i.e. increase of either stress or temperature increases the creep strain-rate and shortens time to rupture t_{rup}.

In Fig. (9.2) a typical deformation mechanism map for a metal alloy is shown.

9.2 Classical creep models

In classic books on creep the creep strain-rate is often decomposed multiplicatively into three separate functions depending on stress σ, time t and temperature T as

$$\dot{\varepsilon}_{\mathrm{c}} = f(\sigma)g(t)h(T). \tag{9.1}$$

[1] These three phases were first noticed by Costa Andrade in 1910.

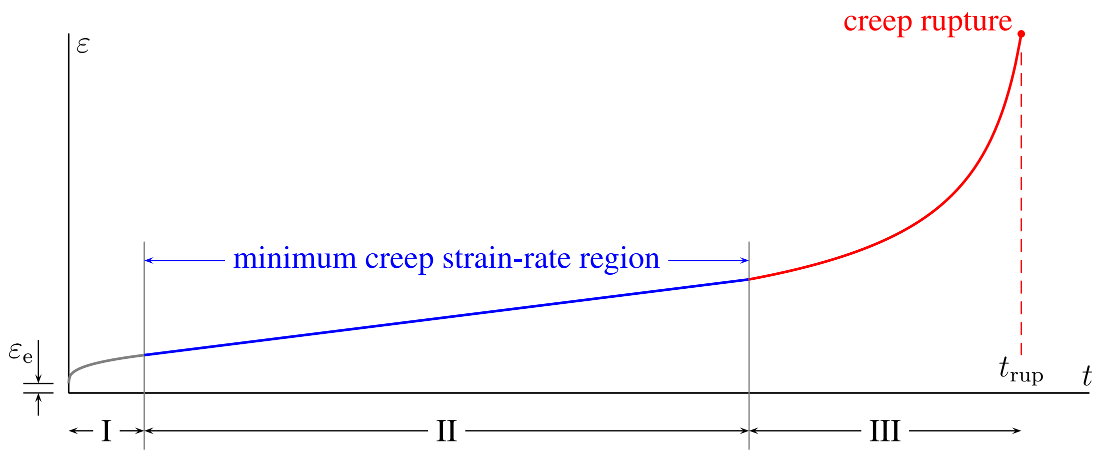


Figure 9.1: Constant stress creep test. After instantaneous elastic strain, threre can be noticed three phases: I primary creep, II secondary creep, i.e. steady-state creep where the creep strain rate is at its minimum, III tertiary creep.

The most well know empirical time and stress functions to decribe the primary and secondary creep are the following [32]:

Norton 1929	$f(\sigma) = C_1(\sigma/\sigma_r)^p,$	(9.2)
Soderberg 1936	$f(\sigma) = C_2(\exp(\sigma/\sigma_r) - 1),$	(9.3)
Dorn 1955	$f(\sigma) = C_3 \exp(\sigma/\sigma_r),$	(9.4)
Garofalo 1965	$f(\sigma) = \sinh^p(\sigma/\sigma_r),$	(9.5)
Andrade 1910	$g(t) = (1 + bt^{1/3})\exp(kt) - 1,$	(9.6)
Bailey 1935	$g(t) = (t/t_c)^n, \quad \text{usually} \quad \frac{1}{3} \le n \le \frac{1}{2},$	(9.7)
McVetty 1934	$g(t) = C_1(1 - \exp(-kt)) + C_2 t,$	(9.8)

where $C_1, C_2, C_3, b, p, k, t_c, n$ and σ_r are parameters. Often σ_r is called as the drag stress.

The effect of temperature is often taken into account by using the Arrhenius-type function

$$h(T) \propto \exp(-Q_c/RT), \tag{9.9}$$

where Q_c is the activation energy and R $(= 8,314\,\mathrm{J/mol\,K})$ is the universal gas constant. The product of strain-rate and the Arrhenius term

$$Z = \dot{\varepsilon}\exp(Q_c/RT) \tag{9.10}$$

is called as the Zener-Hollomon parameter.

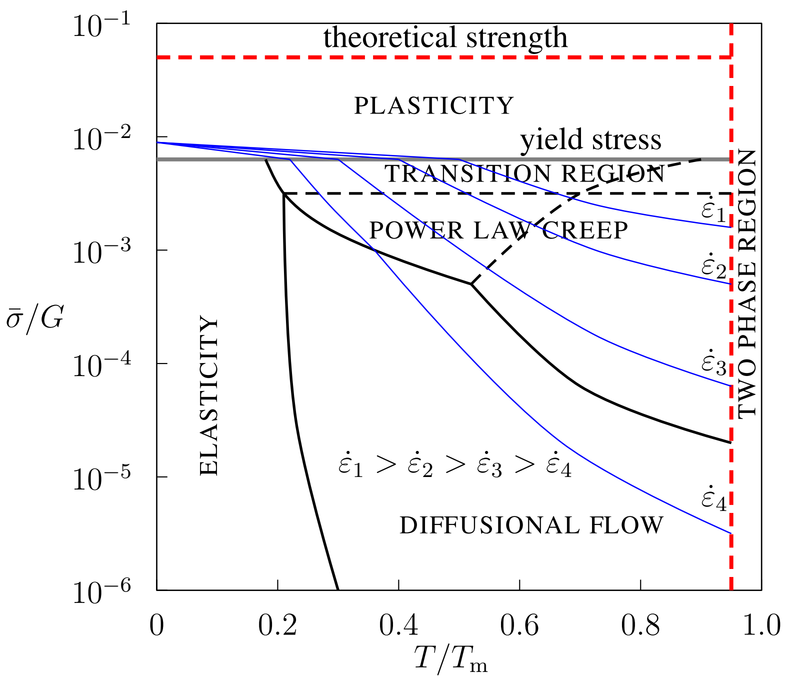


Figure 9.2: Schematic deformation mechanism map for a metal alloy.

9.2.1 Creep modelling using internal variables

Instead of classic creep equations (9.2)-(9.8) a modern approach to model creep phenomena is to use internal variables and evolution equations which describe their change. Typically the evolution equations for the internal variables κ_i, which are either scalars or second order tensors, are of the following form

$$\dot{\kappa}_i = h_i \dot{\varepsilon}^{\mathrm{c}} - r_i^{\mathrm{dyn}} \kappa_i \dot{\varepsilon}^{\mathrm{c}} - r_i^{\mathrm{st}} \kappa_i, \tag{9.11}$$

where the functions h_i, r^{dyn} and r^{st} describe strain-hardening, dynamic and static recovery [6].

A constitutive model which captures primary, secondary and tertiary creep phases can be written as

$$\boldsymbol{\sigma} = (1-D)\boldsymbol{C}^{\mathrm{e}}\boldsymbol{\varepsilon}^{\mathrm{e}} = (1-D)\boldsymbol{C}^{\mathrm{e}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{c}} - \boldsymbol{\varepsilon}^{\mathrm{th}}), \tag{9.12}$$

were the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is decomposed into elastic, creep and thermal parts

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{c}} + \boldsymbol{\varepsilon}^{\mathrm{th}}. \tag{9.13}$$

Continuum damage mechanics can be used to obtain correct behaviour in the tertiary creep phase and the following Kachanov-Rabotnov type damage evolution equation is often used

$$\dot{D} = \frac{1}{t_{\mathrm{d}}} \frac{\exp(-Q_{\mathrm{d}}/RT)}{(1-D)^k} \left(\frac{\bar{\sigma}}{(1-D)\sigma_0} \right)^{2r}, \tag{9.14}$$

where t_{d} is a time parameter, Q_{d} "damage activation energy", r is a dimensionless material parameter and σ_0 is an arbitray reference stress.

If the Norton-Bailey type stress function is chosen, the creep strain rate is

$$\dot{\varepsilon}^{\mathrm{c}} = \frac{1}{t_{\mathrm{c}}} \exp(-Q/RT) \left(\frac{\bar{\sigma}}{\sigma_{\mathrm{r}}} \right)^p \tag{9.15}$$

where t_{c} is a time parameter, related to the relaxation time and σ_{r} is the drag stress. In the above equation the temperature function is of Arrhenius[2] type $\exp(-Q/RT)$, where Q is the activation energy and R the gas constant which has the value 8.3145 J/(mol K). The scalar $\bar{\sigma}$ is an *effective stress* for which some commonly used expressions are

$$\bar{\sigma} = \begin{cases} \sigma_{\mathrm{eff}} = \sqrt{3J_2} & \text{von Mises stress,} \\ \alpha\sigma_{\mathrm{eff}} + (1-\alpha)\sigma_1 & \text{convex combination of vM and the largest principal stress,} \\ \alpha\langle\sigma_1\rangle + \beta I_1 + \gamma\sigma_{\mathrm{eff}} & \text{isochronous form, Hayhurst 1972.} \end{cases}$$

[2]Svante Arrhenius (1859-1927) was a Swedish physicist and the first Swedish laureate (1903 chemistry). He was also the first to use the basic rinciples of physical chemistry to estimate the the effect of the increase of carbon dioxide to the Earth's surface temperature.

In the isochronous case $\alpha + \beta + \gamma = 1$.

Primary creep can be modelled by setting the drag stress σ_r dependent on the effective creep strain

$$\bar{\varepsilon}^\mathrm{c} = \int \dot{\bar{\varepsilon}}^\mathrm{c} \mathrm{d}t, \quad \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^\mathrm{c} : \dot{\boldsymbol{\varepsilon}}^\mathrm{c}} \tag{9.16}$$

Similar kind of hardening rules like in plasticity, see Section 6.4, equation (6.56), i.e.

$$\sigma_\mathrm{r} = \sigma_\mathrm{r0} + K_\infty(1 - \exp(-h\bar{\varepsilon}^\mathrm{c}/K_\infty)). \tag{9.17}$$

Notice that the parameters $\sigma_\mathrm{r0}, K_\infty$ and h are usually temperature dependent as well as the powers r and p in (9.14) and (9.15), respectively. For high-temperature behaviour of metals usually $p \approx 2r$, see e.g. [22].

9.2.2 Some empirical rule of thumb relations

Monkman-Grant (1956) observed that the product of the minimum creep strain-rate and the failure time is a constant which is independent of the applied stress level and temperature

$$(\dot{\varepsilon}_\mathrm{min})t_\mathrm{f} = C_\mathrm{MC} \approx \varepsilon_\mathrm{f}, \tag{9.18}$$

and it is roughly the strain at failure. A slighly better fit to experimental results for some materials can be obtained if it is written in the form

$$(\dot{\varepsilon}_\mathrm{min})^m t_\mathrm{f} = C_\mathrm{MC}, \tag{9.19}$$

where the exponent $m < 1$.

A rough estimate for the failure time can also be determined by using the Larson-Miller (1952) parameter P:

$$P_\mathrm{LM} = T(C + \ln(t_\mathrm{f})), \tag{9.20}$$

where $C \approx 20$ and the fracture time t_f is given in hours. However, a more recommendable form of the Larson-Miller relationship would be

$$\tilde{P}_\mathrm{LM} = T\left[p\ln\left(\frac{\sigma}{\sigma_0}\right) + \ln\left(\frac{t_\mathrm{f}}{t_\mathrm{d}}\right)\right] = \frac{Q}{R} \tag{9.21}$$

9.3 Solved example problems

9.4 Exercises

1. Assume that a bolt connecting two steel plates has pretension $\sigma_0 = 100$ MPa. Determine the time when the stress has relaxed to one half of its initial value when it its assumed that the bolt material obeys the Norton-Bailey type power law. The plates can be assumed purely elastic. The material parameters are $E = 200$ GPa, $\sigma_\mathrm{r} = 100$ MPa, $p = 4$ and $t_\mathrm{c} = 10^4$ h.

2. Determine the relaxation time of the Norton-Bailey creep model.

Chapter 10
Viscoplasticity

10.1 Introduction

Many materials show strain-rate dependency in their plastic behaviour, especially in the high strain rate regime. Viscoplastic models are used to capture this phenomena. For macroscopic modelling of viscoplasticity there are basicly two types of approaches: (i) the overstress and (ii) the consistency models. In the overstress models the stress can lie outside the yield surface and the viscoplastic strain rate depend on some way on the distance between the stress point and the yield surface.

10.2 Overstress viscoplasticity

10.2.1 Perzyna type overstress viscoplasticity

Perzyna [33, 34][1] proposed in 1963 an overstress type viscoplastic model where the viscoplastic strain rate is defined as

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{vp}} = \frac{1}{\eta}\phi(f)\frac{\partial g}{\partial \boldsymbol{\sigma}}, \tag{10.1}$$

where η is the viscosity parameter, ϕ is some function of the yield function f and g is the plastic potential. As in inviscid plasticity the model is called associative if $g = f$ and otherwise non-associative. Common choises for the overstress function ϕ are power laws

$$\phi(f) = \left\langle \frac{f}{\sigma_{\mathrm{y}0}} \right\rangle^p \quad \text{or} \quad \phi(f) = \left\langle \frac{f}{\sigma_{\mathrm{y}}} \right\rangle^p, \tag{10.2}$$

in which p is a material parameter and $\sigma_{\mathrm{y}}, \sigma_{\mathrm{y}0}$ are the currect yield stress and the initial value of it, respectively. The notation $\langle y \rangle$ refers to the Macaulay brackets, i.e. $\langle y \rangle = yH(y)$ where H is the Heaviside unit step function.

[1] The idea of viscoplastic models goes back to Hohenemser & Prager 1932 [15].

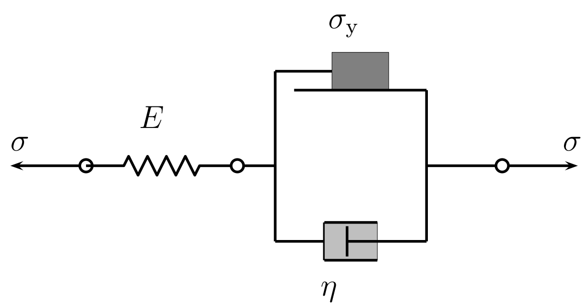


Figure 10.1: A spring, dashpot and frictional unit model of viscoplasticity.

10.2.2 Duvaut-Lions type overstress viscoplasticity

An alternative format for viscoplasticity was proposed by Duvaut and Lions in 1972 [11]. In their model the viscoplastic strain rate is based on the difference in the response between the rate-independent material model and the viscoplastic one. This is in contrast to the Perzyna model where the value of the yield surface determines the viscoplastic strain rate. In the Duvaut-Lions model the viscoplastic strain rate is defined as

$$\dot{\boldsymbol{\varepsilon}}^{\rm vp} = \frac{1}{t_{\rm vp}}\boldsymbol{D}^{\rm e} : (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm ep}), \quad \text{or} \quad \dot{\boldsymbol{\varepsilon}}^{\rm vp} = \frac{1}{\eta}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm ep}), \tag{10.3}$$

where $\boldsymbol{\sigma}^{\rm ep}$ is the solution of the rate-independent material model, also called as back-bone model, and $\boldsymbol{D}^{\rm e}$ is the elastic compliance. The model has only one additional parameter to the inviscid plasticity model, i.e. the viscosity η or time parameter $t_{\rm vp}$, depending which of the forms in (10.3) is used.

10.3 Consistency viscoplasticity

In both the Perzyna and Duvaut-Lions approaches the current stress state can lie outside the yield surface. Therefore also the consistency condition and the Kuhn-Tucker conditions are not applicable in the overstress viscoplasticity. In consistency viscoplasticity the yield surface restricts the allowable stress states but it depends on the strain rate, i.e.

$$f(\boldsymbol{\sigma}, K^\alpha, R^\alpha) = 0, \tag{10.4}$$

where the hardening parameters K^α depend on hardening variables κ^α and the rate hardening parameters R^α depend on the rates $\dot{\kappa}^\alpha$. The plastic strain-rate and the hardening variables κ^α are obtained in a standard fashion from the plastic potential

$$\dot{\boldsymbol{\varepsilon}}^{\rm vp} = \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}}, \quad \dot{\kappa}^\alpha = -\dot{\lambda}\frac{\partial g}{\partial K^\alpha}, \tag{10.5}$$

where the plastic multiplier λ is obtained from the consistency condition

$$\begin{aligned}\dot{f} =& \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial K^\alpha}\dot{K}^\alpha + \frac{\partial f}{\partial R^\alpha}\dot{R}^\alpha \\ =& \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}\dot{\kappa}^\beta + \frac{\partial f}{\partial R^\alpha}\frac{\partial R^\alpha}{\partial \dot{\kappa}^\beta}\ddot{\kappa}^\beta = 0.\end{aligned} \tag{10.6}$$

Also the evolution equation for the $\ddot{\kappa}^\alpha$ is required. Since the plastic potential cannot depend on the rates $\dot{\kappa}^\alpha$,

$$\ddot{\kappa}^\alpha = \frac{\mathrm{d}\dot{\kappa}^\alpha}{\mathrm{d}t} = \ddot{\lambda}\frac{\partial g}{\partial \dot{K}^\alpha} \tag{10.7}$$

which now results in ordinary differential equation for the plastic multiplier λ:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} - \frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}\frac{\partial g}{\partial K^\beta}\dot{\lambda} - \frac{\partial f}{\partial R^\alpha}\frac{\partial R^\alpha}{\partial \dot{\kappa}^\beta}\frac{\partial g}{\partial K^\beta}\ddot{\lambda} = 0. \tag{10.8}$$

In the above equation we can define the strain hardening modulus H and the strain rate sensitivity parameter S

$$H = \frac{\partial f}{\partial K^\alpha}\frac{\partial K^\alpha}{\partial \kappa^\beta}\frac{\partial g}{\partial K^\beta}, \tag{10.9}$$

$$S = \frac{\partial f}{\partial R^\alpha}\frac{\partial R^\alpha}{\partial \dot{\kappa}^\beta}\frac{\partial g}{\partial K^\beta}. \tag{10.10}$$

Above format is similar to the inviscid plasticity, the strain-hardening modulus is identical to (6.38), an additional term is the strain-rate sensitivity term. However, the consistency condition (10.8) is now a differential equation in contrast to the algebraic equation (6.36).

For many materials the strain rate sensitivity S is positive, i.e. the material is hardening with increasing strain-rate. However, certain materials show negative strain rate sensitivity, which results in serrated stress-strain curve e.g. in a tensile test, which is known as the Portevin-Le Chatelier (PLC) effect. It is a material instability phenomena and should not be mixed with the formation of Lüders band which can be observed in strain-softening solids.

Chapter 11
Thermodynamic framework for materials modelling

11.1 Introduction

A good way to formulate constitutive equations is a generalized standard model, which is defined by two potential functions. The energy function, which depends on state variables, like elastic strain, temperature and internal variables, defines the reversible behaviour of a material, while the dissipation potential defines the irreversible response.

Let us define the set of state variables without temperature as $\mathcal{S}$ and the set of dissipative variables as $\mathcal{D}$. As an example we usually operate with the following spesific energy functions: (i) internal energy $e(\mathcal{S}, s)$, (ii) the Helmholtz free energy $\psi(\mathcal{S}, T)$ and (iii) the Gibbs function, or the Gibbs free energy $\psi^{\mathrm{c}}(\mathcal{S}', T)$. I have here denoted $\mathcal{S}'$ as an other set of state variables than $\mathcal{S}$ and s is the spesific entropy. These energy functions are related to each orther by the Legendre-Fenchel transformation (or usually having a sign change):

$$-\psi(\mathcal{S}, T) = \sup_{s} \left(Ts - e(\mathcal{S}, s) \right), \tag{11.1}$$

$$\psi^{\mathrm{c}}(\mathcal{S}', T) = \sup_{\mathcal{S}} \left(\mathcal{S} \cdot \mathcal{S} - \psi(\mathcal{S}, T) \right). \tag{11.2}$$

The notation above is somewhat vague, since the transformation from $\mathcal{S}$ to $\mathcal{S}'$ is a partial transformation, as an example if the Helmholtz free energy depends on state variables $\mathcal{S} = \{\boldsymbol{\varepsilon}^{\mathrm{e}}, \kappa^{\alpha}\}$, and T and where $\kappa^{\alpha}, \alpha = 1, 2, ...$ are internal variables which can be scalars or second-order tensors and the Gibb's function depends on $\mathcal{S}' = \{\boldsymbol{\sigma}, \kappa^{\alpha}\}$ and T, then

$$\psi^{\mathrm{c}}(\boldsymbol{\sigma}, \kappa^{\alpha}, T) = \sup_{\boldsymbol{\varepsilon}^{\mathrm{e}}} \left(\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^{\mathrm{e}} - \psi(\boldsymbol{\varepsilon}^{\mathrm{e}}, \kappa^{\alpha}, T) \right). \tag{11.3}$$

The irreversible part of the behaviour is modelled with either the dissipation potential $\varphi(\mathcal{F})$ depending on the dissipative fluxes in the set $\mathcal{F}$ or the complementary dissipation potential $\varphi^{\mathrm{c}}(\mathcal{Y})$, which depends on the thermodynamic forces (or the dissipative forces) in the set $\mathcal{Y}$.

Then the well known Coleman-Noll procedure results the general thermodynamically consistent constitutive equations.

Thus, for the ***definition of a material model*** we have to

1. define the state variables, this step requires good knowledge of the physics and of the reversible and irreversible mechanisms of the considered material
2. identify the expression of the free energy (or Gibb's function),
3. identification of dissipative fluxes and thermodynamic forces,
4. identify the expression of dissipation potential (or the complementary form).

In the phases 2 and 4 we have to take care of the symmetry properties of the material in question. It means that we have to define the symmetry group $\mathcal{G}$, which means that we have to have, e.g.

$$\psi(\boldsymbol{\varepsilon}^{\mathrm{e}}, ...) = \psi(\boldsymbol{Q}\boldsymbol{\varepsilon}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{T}}, ...) \quad \forall\, \boldsymbol{Q} \in \mathcal{G}. \tag{11.4}$$

In a similar way we have to treat the dissipation potential. Notice, that we can have different symmetry property for e.g. tensorial hardening parameters in the free energy and thermodynamic forces or dissipative fluxes in the dissipation potential.

11.1.1 Legendre transformation

The Legendre transformation, often called also Legendre-Fenchel transformation, transforms a function $f(x)$ to its dual function $f^*(k)$, where the arguments x and k are dual quantities. It is defined by the variational formula

$$f^*(k) = \sup_x \{kx - f(x)\}\,. \tag{11.5}$$

To have a geometric picture of the Legendre transformation think k as a real number and define a line $y = kx$. Then the Legendre dual function $f^*(k)$ is found by performing the following three steps.

1. Define a distance F between the line $y = kx$ and the curve $y = f(x)$ i.e. $F(k, x) = kx - f(x)$.
2. Find a point $x(k)$ where the distance $F(k, x)$ has a maximum with respect to x.
3. The dual function is $f^*(k) = F(k, x(k))$.

If $f(x)$ is a convex function, then $f^*(k)$ is also a convex function.

More detailed description of Legendre transformation can be found in e.g. [2, 39].

Example 11.1. *Find the Legendre dual function of $f(x) = \frac{1}{2}ax^2$.*

Solution. The distance function is

$$F(k,x) = kx - \tfrac{1}{2}ax^2.$$

To obtain the maximum, the necessary condition for the extremum point is

$$\frac{\partial F}{\partial x} = k - ax = 0 \quad \Rightarrow \quad x = k/a.$$

Notice that $f''(x) > 0$, thus the extremum point is a maximum. The dual function is now

$$f^*(k) = F(k, x(k)) = k \cdot k/a - \tfrac{1}{2}a(k/a)^2 = \frac{1}{2}\frac{k^2}{a}.$$

11.1.2 Free energy

In small strain setting, consider the specific free energy ψ (i.e. per unit mass) depending on the state variables, thermo-elastic strains $\boldsymbol{\varepsilon}^{\text{te}} = \boldsymbol{\varepsilon} - \varepsilon^{\text{p}}$, internal variable κ_α, absolute temperature T, and possibly also some other state variables which could be e.g. electro-magnetic quatities, like magnetic field strength and electric field. However, electromagnetic interactions are not considered in these notes. A usual form is to separate the elastic part from the part depending on the internal parameter as

$$\rho\psi(T, \boldsymbol{\varepsilon}^{\text{te}}, \kappa_\alpha) = \rho\psi^{\text{th}}(T) + \rho\psi^{\text{e}}(T, \boldsymbol{\varepsilon}^{\text{te}}) + \rho\psi^{\text{p}}(T, \kappa_\alpha). \tag{11.6}$$

Some specific forms of these functions will be discussed later on.

11.2 Energy balance

The energy balance, i.e. the first principle of thermodynamics can be stated as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E} + \mathcal{K}) = \mathcal{P}_{\text{mech}} + \mathcal{P}_{\text{heat}} + \mathcal{P}_{\text{oth}} \tag{11.7}$$

where $\mathcal{E}, \mathcal{K}$ are internal- and kinetic energy, which can be defined as

$$\mathcal{E} = \int_V \rho e \,\mathrm{d}V, \tag{11.8}$$

$$\mathcal{K} = \frac{1}{2}\int_V \rho \boldsymbol{v} \cdot \boldsymbol{v} \,\mathrm{d}V, \tag{11.9}$$

where the state function e is the specific internal energy, which depends on specific entropy s, strains $\boldsymbol{\varepsilon}$, internal variables κ_α. The power of external mechanical forces is

$$\mathcal{P}_{\text{mech}} = \int_V \rho \boldsymbol{b} \cdot \boldsymbol{v} \,\mathrm{d}V + \int_S \boldsymbol{t} \cdot \boldsymbol{v} \,\mathrm{d}S, \tag{11.10}$$

the power of thermal processes is

$$\mathcal{P}_{\text{heat}} = \int_V \rho \mathcal{R}_{\text{heat}} \, \mathrm{d}V - \int_S \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{11.11}$$

where $\mathcal{R}_{\text{heat}}$ is the rate of heat production per unit mass and $\boldsymbol{q}$ is the heat flux vector. In addition the power of other non-mechanical processes is

$$\mathcal{P}_{\text{oth}} = \int_V \rho \mathcal{R}_{\text{oth}} \, \mathrm{d}V - \int_S \boldsymbol{q}_{\text{oth}} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{11.12}$$

where $\mathcal{R}_{\text{oth}}$ is the rate of energy production of other dissipative mechanisms than heat and $\boldsymbol{q}_{\text{oth}}$ is the energy flux, respectively. After some intermediate steps

$$\int_V \rho \dot{e} \, \mathrm{d}V = \int_V \left[\boldsymbol{\sigma} : \operatorname{grad} \boldsymbol{v} + \rho(\mathcal{R}_{\text{heat}} + \mathcal{R}_{\text{oth}}) - \operatorname{div}(\boldsymbol{q} + \boldsymbol{q}_{\text{oth}}) \right] \mathrm{d}V. \tag{11.13}$$

The global energy balance equation (11.13) can be written in the local form

$$\rho \dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \rho(\mathcal{R}_{\text{heat}} + \mathcal{R}_{\text{oth}}) - \operatorname{div}(\boldsymbol{q} + \boldsymbol{q}_{\text{oth}}). \tag{11.14}$$

The specific Helmholtz free energy ψ is obtained from the specific internal energy by using the partial Legendre transformation $\psi = e - sT$ (notice the sign convention), which is a function of measurable quantities like temperature T and strain $\boldsymbol{\varepsilon}$, and the local energy equation (11.14) can be written as

$$\rho(\dot{\psi} + \dot{s}T + s\dot{T}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \rho(\mathcal{R}_{\text{heat}} + \mathcal{R}_{\text{oth}}) - \operatorname{div}(\boldsymbol{q} + \boldsymbol{q}_{\text{oth}}). \tag{11.15}$$

In the following only thermomechanical processes are considered and thus it is assumed that $\mathcal{R}_{\text{oth}} \equiv 0$ and $\boldsymbol{q}_{\text{oth}} \equiv \boldsymbol{0}$.

11.3 Entropy inequality

The second principle of thermodynamics puts restrictions to the possible processes. Defining the entropy as $S = \int \rho s \, \mathrm{d}V$, where s is the specific entropy. The entropy equation, i.e. the Clausius-Duhem inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho s \, \mathrm{d}V \geq \int_V \frac{\rho \mathcal{R}_{\text{heat}}}{T} \, \mathrm{d}V - \int_S \frac{\boldsymbol{q} \cdot \boldsymbol{n}}{T} \, \mathrm{d}S, \tag{11.16}$$

After some intermediate steps the specific dissipation power can be expressed as

$$\gamma = -\rho(\dot{\psi} + s\dot{T}) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - T^{-1} \operatorname{grad} T \cdot \boldsymbol{q}, \tag{11.17}$$

and the entropy inequality can be simply written as

$$\gamma \geq 0. \tag{11.18}$$

In geometrically linear setting, the strain tensor $\boldsymbol{\varepsilon}$ can be divided into elastic $\boldsymbol{\varepsilon}^{\mathrm{e}}$, plastic/viscoplastic ε^{p} and thermal strain $\boldsymbol{\varepsilon}^{\mathrm{th}}$ components as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}} + \boldsymbol{\varepsilon}^{\mathrm{th}} = \boldsymbol{\varepsilon}^{\mathrm{te}} + \boldsymbol{\varepsilon}^{\mathrm{p}}. \tag{11.19}$$

Since the specific free energy depends on temperature T, thermo-elastic strain $\boldsymbol{\varepsilon}^{\mathrm{te}}$ and internal variables κ_α, the power of specific dissipation can be written as

$$\gamma = -\rho\left(s + \frac{\partial\psi}{\partial T}\right)\dot{T} + \left(\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}}\right):\dot{\boldsymbol{\varepsilon}}^{\mathrm{te}} + \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \rho\frac{\partial\psi}{\partial\kappa_\alpha}\dot{\kappa}_\alpha - T^{-1}\mathrm{grad}\,T\cdot\boldsymbol{q}. \tag{11.20}$$

The dissipative mechanisms of the system are described by the dissiparion dissipation potential, or the complementary dissipation potential $\varphi^{\mathrm{c}} = \varphi^{\mathrm{c}}(K_\alpha, \boldsymbol{q}, \boldsymbol{\sigma}; T)$, depending on the thermodynamic forces K_α, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$, and is defined such that the specific dissipation power γ is

$$\gamma = \frac{\partial\varphi}{\partial\boldsymbol{q}}\cdot\boldsymbol{q} + \frac{\partial\varphi}{\partial\boldsymbol{\sigma}}:\boldsymbol{\sigma} + \frac{\partial\varphi}{\partial K_\alpha}K_\alpha \tag{11.21}$$

Defining the thermodynamic forces K_α as $K_\alpha = \rho\partial\psi/\partial\kappa_\alpha$, equating (11.20) to be equal to the definition (11.21), it is obtained

$$\begin{aligned} -\rho\left(s + \frac{\partial\psi}{\partial T}\right)\dot{T} + \left(\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}}\right) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{te}} + \left(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \frac{\partial\varphi}{\partial\boldsymbol{\sigma}}\right) : \boldsymbol{\sigma} \\ - \left(\dot{\kappa}_\alpha + \frac{\partial\varphi}{\partial K_\alpha}\right)K_\alpha - \left(T^{-1}\mathrm{grad}\,T + \frac{\partial\varphi}{\partial\boldsymbol{q}}\right)\cdot\boldsymbol{q} = 0. \end{aligned} \tag{11.22}$$

Since the equality has to be satisfied for all thermodynamically admissible processes $\dot{T}, \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$, $\boldsymbol{\sigma}$, K_α and $\boldsymbol{q}$, the following general constitutive equations are obtained

$$s = -\rho\frac{\partial\psi}{\partial T}, \tag{11.23}$$

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}}, \tag{11.24}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial\varphi}{\partial\boldsymbol{\sigma}}, \tag{11.25}$$

$$\dot{\kappa}_\alpha = -\frac{\partial\varphi}{\partial K_\alpha}, \tag{11.26}$$

$$T^{-1}\,\mathrm{grad}\,T = -\frac{\partial\varphi}{\partial\boldsymbol{q}}. \tag{11.27}$$

11.4 Examples

11.4.1 Elastic damaging material model

In section 7.3 the Kachanov-Rabotnov type continuum damage model was introcuded. In this section this model is derived from the two thermodynamic potential functions. There are many alternatives, the elastic reversible behaviour can be described either by the Helmholtz free energy written in terms of elastic strain and integrity/damage or by the Gibbs function where stress and integrity/damage are the state variables.

Case 1: Selecting elastic strain, now $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon}$, and damage D as state variables, the Helmholtz free energy in the present isothermal case can be written as

$$\psi(\boldsymbol{\varepsilon}, D) = \frac{1}{2}(1-D)\,\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}, \tag{11.28}$$

where $\boldsymbol{C}$ is the elasticity tensor. The dissipative behaviour is due to damage and the the potential function can be expressed either as a function of the damage rate or as the thermodynamic force dual to the damage rate. Let us first consider the latter case and choose the complementary dissipation potential as

$$\varphi^{\mathrm{c}}(Y; D) = \frac{1}{r+1}\frac{Y_{\mathrm{r}}}{t_{\mathrm{d}}\,(1-D)^p}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r+1}, \tag{11.29}$$

where Y_{r} is a reference value which can be chosen freely and r, k, t_{d} are material parameters. The notation $\varphi^{\mathrm{c}}(Y; D)$ means that the damage D acts in the dissipation potential as a parameter like quantity. The dissipation power can be written either

$$\gamma = -\rho\dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = -\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial D}\dot{D} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \tag{11.30}$$

or

$$\gamma = \frac{\partial\varphi^{\mathrm{c}}}{\partial Y}Y. \tag{11.31}$$

Equating (11.30) and definition (11.31) and defining the thermodynamic force as $Y = -\rho\partial\psi/\partial D$, results in equation

$$\left(\boldsymbol{\sigma} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} + \left(\dot{D} - \frac{\partial\varphi^{\mathrm{c}}}{\partial Y}\right)Y = 0. \tag{11.32}$$

Since the equation has to be valid for all thermodynamically admissible processes $\dot{\boldsymbol{\varepsilon}}$ and Y, the constitutive equations are obtained as

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = \phi\boldsymbol{C} : \boldsymbol{\varepsilon}. \tag{11.33}$$

$$\dot{D} = \frac{\partial\varphi^{\mathrm{c}}}{\partial Y} = \frac{1}{t_{\mathrm{d}}\,(1-D)^p}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r}, \tag{11.34}$$

and the thermodynamic force Y is

$$Y = -\frac{\partial \psi}{\partial D} = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon}. \tag{11.35}$$

The driving force of the damage process is thus the elastic energy.

Notice that from (11.30) we obtain

$$\gamma = Y\dot{D} \geq 0, \tag{11.36}$$

since damage D is an increasing function. Therefore, the model satisfies automatically the second law of thermodynamics i.e. the entropy inequality (11.18) if the time parameter t_{d} is positive and the power $r \geq 0$. Notice that there is no limitatation for the power p.

Case 2: Now the free energy is kept as it is in (11.28) but the dissipation potential is written in terms of the dissipative flux i.e. the damage rate $\dot{D}$. These two dissipation potentials are dual functions in the Legendre-Fenchel transformation and they satisfy

$$\varphi(\dot{D}) + \varphi^{\mathrm{c}}(Y) = Y\dot{D}. \tag{11.37}$$

The distance function to be maximized is now

$$F(Y, \dot{D}) = Y\dot{D} - \frac{1}{r+1}\frac{Y_{\mathrm{r}}}{t_{\mathrm{d}}\,(1-D)^p}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r+1}, \tag{11.38}$$

and

$$\frac{\partial F}{\partial Y} = \dot{D} - \frac{1}{t_{\mathrm{d}}\,(1-D)^p}\left(\frac{Y}{Y_{\mathrm{r}}}\right)^{r} = 0 \quad \Rightarrow \quad Y = Y_{\mathrm{r}}\left[(1-D)^p t_{\mathrm{d}}\dot{D}\right]^{1/r}. \tag{11.39}$$

The dual function $\varphi(\dot{D})$ is then

$$\begin{aligned}\varphi(\dot{D}) &= Y_{\mathrm{r}}\left[(1-D)^p t_{\mathrm{d}}\dot{D}\right]^{1/r}\dot{D} - \frac{1}{r+1}\frac{Y_{\mathrm{r}}}{t_{\mathrm{d}}(1-D)^p}\left[(1-D)^p t_{\mathrm{d}}\dot{D}\right]^{(r+1)/r} \\ &= \frac{r}{r+1}\frac{Y_{\mathrm{r}}}{(1-D)^p t_{\mathrm{d}}}\left[(1-D)^p t_{\mathrm{d}}\dot{D}\right]^{(r+1)/r}.\end{aligned} \tag{11.40}$$

The dissipation inequality (11.30) is now the same, but the definition of the dissipation potential is now changed to

$$\gamma = \frac{\partial \varphi}{\partial \dot{D}}\dot{D}. \tag{11.41}$$

Setting (11.30) and (11.41) equal, results in

$$\left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} + \left(Y - \frac{\partial \varphi}{\partial \dot{D}}\right)\dot{D} = 0, \tag{11.42}$$

and the constitutive equations are

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = (1-D)\boldsymbol{C}:\boldsymbol{\varepsilon}. \tag{11.43}$$

$$Y = \frac{\partial \varphi}{\partial \dot{D}} = Y_{\mathrm{r}}\left[(1-D)^p t_{\mathrm{d}} \dot{D}\right]^{1/r}, \tag{11.44}$$

together with (11.35).

It can be seen that equations (11.43) and (11.44) are identical to (11.33) and (11.34).

11.4.2 Elastic viscoplastic material model

If restricting into linear elasticity, and isochoric plastic flow, the specific free energy can be written as

$$\rho\psi(T, \boldsymbol{\varepsilon}^{\mathrm{e}}, \kappa_{\mathrm{iso}}, \boldsymbol{\kappa}_\alpha) = \rho\psi^{\mathrm{th}}(T) + \psi^{\mathrm{e}}(T, I_1(\boldsymbol{\varepsilon}^{\mathrm{e}}), J_2(\boldsymbol{\varepsilon}^{\mathrm{e}})) + \rho\psi^{\mathrm{p}}(T, \kappa_{\mathrm{iso}}, J_2(\boldsymbol{\kappa}_\alpha)), \tag{11.45}$$

where[1]

$$\rho\psi^{\mathrm{th}}(T) = \rho c_\varepsilon \left(T - T\ln\frac{T}{T_{\mathrm{r}}}\right), \tag{11.46}$$

$$\rho\psi^{\mathrm{e}}(T, \boldsymbol{\varepsilon}^{\mathrm{te}}) = \frac{1}{2}BI_1(\boldsymbol{\varepsilon}^{\mathrm{te}})^2 + 2GJ_2(\boldsymbol{\varepsilon}^{\mathrm{te}}) - B\alpha I_1(\boldsymbol{\varepsilon}^{\mathrm{te}})(T - T_{\mathrm{r}}), \tag{11.47}$$

$$\begin{aligned}\rho\psi^{\mathrm{p}}(T, \kappa^{\mathrm{iso}}, \boldsymbol{\kappa}_\alpha) = {} & K_\infty^{\mathrm{iso}}\left[\kappa^{\mathrm{iso}} + \frac{K_\infty^{\mathrm{iso}}}{h^{\mathrm{iso}}}\exp(-h^{\mathrm{iso}}\kappa^{\mathrm{iso}}/K_\infty^{\mathrm{iso}})\right] + \\ & + \sum_{k=1}^{n} \tfrac{2}{3} h_k J_2(\boldsymbol{\kappa}_k).\end{aligned} \tag{11.48}$$

Here, B and G are the bulk and shear moduli, α is the linear coefficient of thermal expansion, $K_\infty^{\mathrm{iso}}, h^{\mathrm{iso}}$ are related to the isotropic hardening and h_k to the kinematic hardening. All these material parameters can depend on temperature.

The plastic potential for Perzyna type overstress viscoplasticity can be written as[2]

$$\begin{aligned}&\varphi(T, \boldsymbol{q}, \boldsymbol{\sigma}, K^{\mathrm{iso}}, \boldsymbol{K}_k; \boldsymbol{\kappa}_k) = \\ &\frac{\sigma_0}{(p+1)t_{\mathrm{vp}}}\left\langle \frac{f(\boldsymbol{\sigma}, K^{\mathrm{iso}}, \boldsymbol{K}_k, T) + \sum_k (3J_2(\boldsymbol{K}_k) - \frac{2}{3}h_k^2 \boldsymbol{\kappa}_k : \boldsymbol{\kappa}_k)/K_{k,\infty}}{\sigma_0}\right\rangle^{p+1} + \\ &+ \varphi_{\mathrm{th}}(T, \boldsymbol{q}),\end{aligned} \tag{11.49}$$

[1]Now I have just selected a specific model for the isotropic hardening, i.e. the exponential saturating hardening. Other models are naturally possible.

[2]Notice that I have to add zero in the numerator of the Macaulay expression, since the model is non-associated w.r.t. the hardening parameters $\boldsymbol{K}_k$: $3J_2(\boldsymbol{K}_k) - \frac{2}{3}h_k^2\boldsymbol{\kappa}_k:\boldsymbol{\kappa}_k \equiv 0$ by definition.

where t_{vp} is a time parameter, σ_0 is usually called the drag stress (oft en initial yield stress) and p is an additional parameter. Notice that when $p \to \infty$, we obtain inviscid plasticity formulation. The "static" yield function is denoted as f and $\langle f \rangle$ denotes the Macaulay brackets, i.e. the ramp function $\langle f \rangle = \frac{1}{2}(f + |f|)$. I now skip the thermal part of the dissipation potential φ_{th}. For simplicity, the following short hand notation is used

$$\hat{f} = f(\boldsymbol{\sigma}, K^{\mathrm{iso}}, \boldsymbol{K}_k, T) + \sum_k (3J_2(\boldsymbol{K}_k) - \tfrac{2}{3}h_k^2 \boldsymbol{\kappa}_k \boldsymbol{:} \boldsymbol{\kappa}_k)/K_{k,\infty}. \tag{11.50}$$

The forces K^{iso} and $\boldsymbol{K}_k$ are obtained from

$$K^{\mathrm{iso}} = \rho \frac{\partial \psi}{\partial \kappa^{\mathrm{iso}}} = K^{\mathrm{iso}}_\infty (1 - \exp(-h^{\mathrm{iso}} \kappa^{\mathrm{iso}} / K^{\mathrm{iso}}_\infty)) \tag{11.51}$$

$$\boldsymbol{K}_k = \rho \frac{\partial \psi}{\partial \boldsymbol{\kappa}_k} = \tfrac{2}{3} h_k \boldsymbol{\kappa}_k, \quad k = 1, \ldots, n. \tag{11.52}$$

The von Mises type isotropic yield function is of the form

$$f(\boldsymbol{\sigma}, K^{\mathrm{iso}}, \boldsymbol{K}_k, T) = \sqrt{3J_2(\boldsymbol{s} - \boldsymbol{K})} - (\sigma_{\mathrm{y0}} + K^{\mathrm{iso}}), \tag{11.53}$$

where $\boldsymbol{s} = \boldsymbol{\sigma} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}$ is the deviatoric stress and

$$\boldsymbol{K} = \sum_{k=1}^{n} \boldsymbol{K}_k. \tag{11.54}$$

Now the constitutive equations (11.24)–(11.26) for this specific case are

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{te}}} = \rho \frac{\partial \psi^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{te}}} = B[\mathrm{tr}\, \boldsymbol{\varepsilon}^{\mathrm{te}} - \alpha(T - T_{\mathrm{r}})]\boldsymbol{I} + 2G\boldsymbol{\varepsilon}^{\mathrm{te}}, \tag{11.55}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} = \frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \frac{3(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\mathrm{y}}} \tag{11.56}$$

$$\dot{\kappa}_{\mathrm{iso}} = -\frac{\partial \varphi}{\partial K^{\mathrm{iso}}} = -\frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \frac{\partial f}{\partial K^{\mathrm{iso}}} = \frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p, \tag{11.57}$$

$$\begin{aligned} \dot{\boldsymbol{\kappa}}_k &= -\frac{\partial \varphi}{\partial \boldsymbol{K}_k} = -\frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \frac{\partial \hat{f}}{\partial \boldsymbol{K}_k} = \frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \left(\frac{3(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\mathrm{y}}} - \frac{3}{K_{k,\infty}} \boldsymbol{K}_k \right) \\ &= \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \frac{1}{t_{\mathrm{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \frac{3}{K_{k,\infty}} \boldsymbol{K}_k, \end{aligned} \tag{11.58}$$

where

$$\sigma_{\mathrm{y}} = \sigma_{\mathrm{y0}} + K^{\mathrm{iso}}. \tag{11.59}$$

The mechanical part of the dissipation power has the expression

$$
\begin{aligned}
\gamma_{\text{mech}} =& \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\text{p}} - K^{\text{iso}} \dot{\kappa}_{\text{iso}} - \sum_{k=1}^{N} \boldsymbol{K}_k : \dot{\boldsymbol{\kappa}}_k \\
=& \frac{1}{t_{\text{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \left[\frac{3\boldsymbol{\sigma}:(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\text{y}}} - K^{\text{iso}} - \sum_{k=1}^{N} \left(\frac{3\boldsymbol{K}_k:(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\text{y}}} - \frac{3}{K_{k,\infty}} \boldsymbol{K}_k : \boldsymbol{K}_k \right) \right] \\
=& \frac{1}{t_{\text{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \left[A + \sum_{k=1}^{N} \frac{3}{K_{k,\infty}} \boldsymbol{K}_k : \boldsymbol{K}_k \right], \qquad (11.60)
\end{aligned}
$$

where

$$
\begin{aligned}
A =& \frac{3\boldsymbol{\sigma}:(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\text{y}}} - \sum_{k=1}^{N} \frac{3\boldsymbol{K}_k:(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\text{y}}} - K^{\text{iso}} \\
=& \frac{3(\boldsymbol{\sigma} - \boldsymbol{K}):(\boldsymbol{s} - \boldsymbol{K})}{2\sigma_{\text{y}}} - K^{\text{iso}} = \frac{3J_2(\boldsymbol{s} - \boldsymbol{K})}{\sigma_{\text{y}}} - K^{\text{iso}} \geq \sigma_{\text{y}0}, \qquad (11.61)
\end{aligned}
$$

thus

$$
\gamma_{\text{mech}} \geq \frac{1}{t_{\text{vp}}} \left\langle \frac{\hat{f}}{\sigma_0} \right\rangle^p \left[\sigma_{\text{y}0} + \sum_{k=1}^{N} \frac{3}{K_{k,\infty}} \boldsymbol{K}_k : \boldsymbol{K}_k \right] \geq 0. \qquad (11.62)
$$

The rate form of (11.55) is

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}} =& \frac{\mathrm{d}}{\mathrm{d}t} \left(\rho \frac{\partial \psi^{\text{e}}}{\partial \boldsymbol{\varepsilon}^{\text{te}}} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(B[\operatorname{tr} \boldsymbol{\varepsilon}^{\text{te}} - \alpha(T - T_{\text{r}})] \boldsymbol{I} + 2G\boldsymbol{\varepsilon}^{\text{te}} \right), \\
=& B \operatorname{tr} \dot{\boldsymbol{\varepsilon}}^{\text{e}} \boldsymbol{I} + 2G\dot{\boldsymbol{\varepsilon}}^{\text{te}} + \left(\frac{\mathrm{d}B}{\mathrm{d}T} \left[\operatorname{tr} \boldsymbol{\varepsilon}^{\text{te}} - \alpha(T - T_{\text{r}}) \right] \boldsymbol{I} + 2\frac{\mathrm{d}G}{\mathrm{d}T} \boldsymbol{\varepsilon}^{\text{te}} \right) \dot{T} \\
& - B\frac{\mathrm{d}\alpha}{\mathrm{d}T} \dot{T}(T - T_{\text{r}}) \boldsymbol{I} - B\alpha\dot{T}\boldsymbol{I}. \qquad (11.63)
\end{aligned}
$$

Perhaps, it would be nicer to write the constitutive equation (11.55) as

$$
\boldsymbol{\sigma} = \boldsymbol{C}^{\text{e}} \boldsymbol{\varepsilon}^{\text{te}} - \boldsymbol{C}^{\text{e}} \boldsymbol{\alpha}(T - T_{\text{r}}) = \boldsymbol{C}^{\text{e}} (\boldsymbol{\varepsilon}^{\text{te}} - \boldsymbol{\varepsilon}^{\text{th}}) = \boldsymbol{C}^{\text{e}} \boldsymbol{\varepsilon}^{\text{e}}, \qquad (11.64)
$$

where $\boldsymbol{\alpha}$ is the tensor of thermal expansion, in isotropic case $\boldsymbol{\alpha} = \alpha \boldsymbol{I}$. Now the rate can be written as

$$
\dot{\boldsymbol{\sigma}} = \boldsymbol{C}^{\text{e}} \dot{\boldsymbol{\varepsilon}}^{\text{e}} + \frac{\mathrm{d}\boldsymbol{C}^{\text{e}}}{\mathrm{d}T} \dot{T} \boldsymbol{\varepsilon}^{\text{e}} = \boldsymbol{C}^{\text{e}} \dot{\boldsymbol{\varepsilon}}^{\text{e}} + \frac{\mathrm{d}\boldsymbol{C}^{\text{e}}}{\mathrm{d}T} (\boldsymbol{C}^{\text{e}})^{-1} \boldsymbol{\sigma} \dot{T}. \qquad (11.65)
$$

Another way

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\frac{\partial\psi^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}}\right) = \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial(\boldsymbol{\varepsilon}^{\mathrm{te}})^2}\dot{\boldsymbol{\varepsilon}}^{\mathrm{te}} + \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}\partial T}\dot{T} \\
=& \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial(\boldsymbol{\varepsilon}^{\mathrm{te}})^2}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{th}}) + \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}\partial T}\dot{T} \\
=& \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial(\boldsymbol{\varepsilon}^{\mathrm{te}})^2}(\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) + \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}\partial T}\dot{T}.
\end{aligned} \tag{11.66}
$$

The general form of a constitutive model for thermo-visco-plastic behaviour with kinematic and isotropic hardening could be written as

$$
\dot{\boldsymbol{\sigma}} = \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial(\boldsymbol{\varepsilon}^{\mathrm{te}})^2}(\dot{\boldsymbol{\varepsilon}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) + \rho\frac{\partial^2\psi^{\mathrm{e}}}{\partial\boldsymbol{\varepsilon}^{\mathrm{te}}\partial T}\dot{T}, \tag{11.67}
$$

$$
\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{\partial\varphi}{\partial\boldsymbol{\sigma}}, \tag{11.68}
$$

$$
\dot{\kappa}_{\mathrm{iso}} = -\frac{\partial\varphi}{\partial K^{\mathrm{iso}}}, \tag{11.69}
$$

$$
\dot{\boldsymbol{\kappa}}_k = -\frac{\partial\varphi}{\partial\boldsymbol{K}_k}, \quad k = 1, \ldots, n, \tag{11.70}
$$

with the definitions (11.51) –(11.52):

$$
K^{\mathrm{iso}} = \rho\frac{\partial\psi}{\partial\kappa^{\mathrm{iso}}}, \tag{11.71}
$$

$$
\boldsymbol{K}_k = \rho\frac{\partial\psi}{\partial\boldsymbol{\kappa}_k}, \quad k = 1, \ldots, n. \tag{11.72}
$$

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