On the numerical solution of a micropolar continuum model

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Summary. A simple numerical test for two low-order standard Lagrange type elements for the polar-continuum model in 3-D is performed.

Key words: micropolar continuum, finite element method

Introduction

The ordinary Cauchy continuum model cannot describe effects originating from the microstructure of a material, such as the size-effect. After the first trials by Woldemar Voigt [14] and the brothers Eugene and Francois Cosserat [2] for generalizing the Cauchy continuum model, it took more than a half century for growing interest to the generalized continuum models [1, 5, 9, 6, 13].

Micropolar continuum

Equilibrium equations

For the micropolar continuum the following local forms of the equilibrium equations can be obtained

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = 0, \tag{1}$$

$$\frac{\partial \mu_{ji}}{\partial x_j} + \rho c_i + \epsilon_{ijk}\sigma_{jk} = 0, \tag{2}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ stand for force- and moment stress tensors, $\boldsymbol{b}$, $\boldsymbol{c}$ are the body force and moment per unit mass, respectively. In general, the force stress and moment stress tensors are not symmetric. The alternating tensor is denoted as ϵ_{ijk} and ρ is the mass density.

Constitutive equations

For centrosymmetric material the linear constitutive equations for the force-stress σ_{ij} and the couple-stress μ_{ij} can be written as [3, 12, 10]

$$\sigma_{ij} = C^{(\gamma)}_{ijkl}\gamma_{kl}, \qquad \mu_{ij} = C^{(\kappa)}_{ijkl}\kappa_{kl}, \tag{3}$$

where the Cosserat's first strain tensor γ_{ij} and microcurvature tensor κ_{ij} are defined as

$$\gamma_{ij} = u_{j,i} - \epsilon_{kij}\varphi_k, \quad \text{and} \quad \kappa_{ij} = \varphi_{j,i}, \tag{4}$$

in which $\boldsymbol{\varphi}$ is the independent microrotation field. The material stiffness tensors for an isotropic solid can be expressed as

$$C_{ijkl}^{(\gamma)} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \mu_{\mathrm{c}}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{5}$$

$$C_{ijkl}^{(\kappa)} = \alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{6}$$

In addition to the classical Lamé constants λ, μ, there are four additional material parameters: $\mu_{\mathrm{c}}, \alpha, \beta$ and γ. However, a more comprehensible set of constants are the Young's modulus $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$, the shear modulus μ, and [8, 10, 11]

$$\text{the characteristic length in torsion} \qquad \ell_{\mathrm{t}} = \sqrt{\frac{\beta}{\mu}}, \tag{7}$$

$$\text{the characteristic length in bending} \qquad \ell_{\mathrm{b}} = \sqrt{\frac{\beta + \gamma}{4\mu}}, \tag{8}$$

$$\text{the coupling number} \qquad N = \sqrt{\frac{\mu_{\mathrm{c}}}{\mu + \mu_{\mathrm{c}}}}, \tag{9}$$

$$\text{and the polar ratio} \qquad \psi = \frac{\beta}{\beta + \frac{1}{2}\alpha}. \tag{10}$$

The allowed range for the dimensionless parameters N and ψ are

$$0 \leq N \leq 1, \quad \text{and} \quad 0 \leq \psi \leq \frac{3}{2}, \tag{11}$$

Determination of the four additional constants is a major problem for practical applications. Neff et al. [10, 11] have inroduced so called conformally invariant curvature state, which reduces the number of additional material parameters to two and which also facilitates a stable estimation of them. For conformally invariant curvature state, the curvature tensor is purely deviatoric and symmetric, thus

$$\gamma = 0, \qquad \beta = \mu\ell_{\mathrm{t}}^2, \qquad \alpha = -\tfrac{2}{3}\mu\ell_{\mathrm{t}}^2. \tag{12}$$

The conformally invariant curvature state corresponds thus to the upper limit of the polar ratio, i.e. $\psi = \frac{3}{2}$. Notice that in this case $\ell_{\mathrm{b}} = \frac{1}{2}\ell_{\mathrm{t}}$. It also results in a non-sigular behaviour of stiffening in torsion and bending, see Figure 1, where the stiffening effect in pure bending of a straight beam with circular cross-section is shown. The singular behaviour of the non-conformally invariant curvature state is clearly seen. The analytical solution in given in [7].

Numerical solution

The virtual work expression for the polar continua can be written as

$$\int_V \left[\left(-\nabla \cdot \boldsymbol{\sigma}^T - \rho\boldsymbol{b}\right) \cdot \delta\boldsymbol{u} + \left(-\nabla \cdot \boldsymbol{\mu}^T - \boldsymbol{\epsilon} : \boldsymbol{\sigma} - \rho\boldsymbol{c}\right) \cdot \delta\boldsymbol{\phi}\right] \mathrm{d}V = 0. \tag{13}$$

After integration by parts and utilizing the divergence theorem, it transforms into the form suitable for finite element approximation

$$\int_V \left[\boldsymbol{\sigma} : \left((\nabla\delta\boldsymbol{u})^T - \mathrm{skew}(\delta\boldsymbol{\varphi})\right) + \boldsymbol{\mu} : (\nabla\delta\boldsymbol{\varphi})^T\right] \mathrm{d}V - \int_S (\boldsymbol{t} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\varphi})\,\mathrm{d}S = 0. \tag{14}$$

As an example a straight beam with square cross-section is solved with both 27-node tri-quadratic and 8-node trilinear standard C_0-elements. The beam is divided in ten equal elements and loading is a uniform traction in the vertical direction at the free end of the beam. The
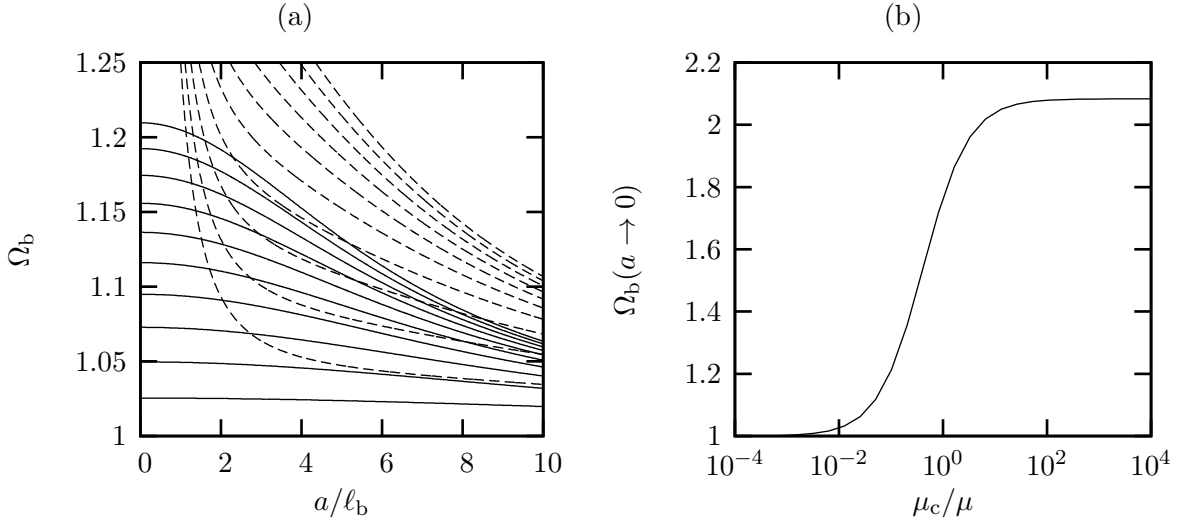
Figure 1. (a) Ratio of bending rigidity of the polar continuum versus standard Cauchy's continuum model Ω_{b} as a function of the radius a of the beam's cross-section with different values of the coupling modulus μ_{c}; from bottom to top $\mu_{\mathrm{c}}/\mu = 0.01, 0.02, 0.03, \ldots, 0.09, 0.1$. Conformally invariant curvature case ($\beta = 4\mu\ell_{\mathrm{b}}^2, \gamma = 0$) shown by solid lines and the case with parameters $\gamma = 0.01\beta$ indicated by dashed lines. (b) Limit value of Ω_{b} when $a \to 0$ as a function of the coupling modulus μ_{c} for the conformally invariant curvature state.
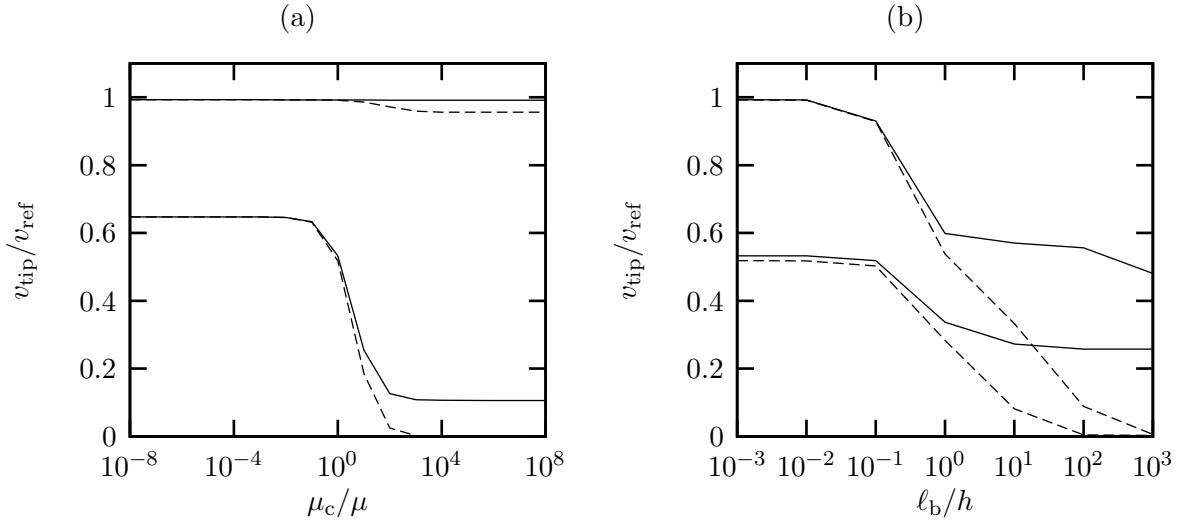

Figure 2. (a) The effect of the coupling modulus μ_{c} to the tip displacement of the cantilever beam. The reference value is the tip deflection according to the Euler-Bernoulli beam model: $v_{\mathrm{ref}} = FL^3/3EI$. The material length in bending is $\ell_{\mathrm{b}} = h/200$, where h is the cross-section height. The upper two curves correspond to solutions with triquadratic elements and the lower ones with trilinear elements. Solid lines indicate solutions where the rotations are free at the clamped edge and the dashed line where they are supressed. (b) The effect of internal length scale ℓ_{b} on the FE solution: $\mu_{\mathrm{c}} = \mu$.

length to height ratio of the beam is $L/h = 10$. Two different types of boundary conditions are computed; both the rotations and displacements are supressed at the clamped edge or only the displacements are supressed. Only conformally invariant curvature state is considered.

It is clearly seen from the Figure 2(a) that the trilinear elements lock also earlier as expected from standard continuum in the limit $\mu_{\mathrm{c}} \to \infty$. The behaviour of the quadratic element is also peculiar for the fully clamped case, Figure 2(b). However, it should be noted that the analytical solution is unknwon for this case.

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