Finite Rotations in View of Structural Stability Analysis

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Summary

In this paper some formulations of the incremental equilibrium equations of thin bodies are described and the significance of specific terms in the discretized equations are discussed, particularly in view of structural stability analyses.

Introduction

The necessity to deal with finite rotations is a common feature for non-linear deformation problems of solid and structural mechanics. Such tasks are encountered e.g. when integrating constitutive equations in large strain problems, in kinematical descriptions of thin bodies and in some non-classical continuum models. In this paper special emphasis is given to the kinematical relations of thin bodies like beams and shells in which the geometrical non-linearity is mostly due to large rigid-body rotations, and the restriction to small strains is a relevant assumption.

The principal difficulty in formulating the equilibrium equations for non-linear analysis of thin bodies is the complex character of finite three-dimensional rotations, which are not elements of a linear space and thus their parametrization and the pertinent computational aspects are far from trivial. Many alternative derivations and possible parametrizations of the finite rotation formula are available. For computational point of view, many of the parametrizations require a special updating procedure for rotational degrees of freedom, which is an additional inconvenience.

To determine the final displacements of a point of a rigid body undergoing a finite rotation around a fixed axis in a three dimensional space is a classical problem. Many alternative derivations and possible parametrizations of the finite rotation formula are available. Spring [1] tabulates basic formulas of seven different representations: (a) rotation matrix, (b) Euler parameters (quaternions), (c) linear parameters, (d) Rodrigues parameters (Gibbs vector), (e) Euler rotation, (f) Euler angles (3-1-3 angles) and (g) Bryant or Gardan angles (1-2-3 Euler angles). At least this list should be added by the Cayley-Klein parameters. It should be noted that the basis of all these representations lies on Euler's theorem of rigid-body rotations: the attitude of a body after having undergone any sequence of rotations is equivalent to a single rotation of that body through an angle ϑ and an axis $\hat{\boldsymbol{n}}$. This direct representation is called Euler rotation.

Probably the most well-known of these parameter sets listed above are the 3-1-3 Euler angles, described in most of graduate text-books of non-relativistic mechanics, cf. [2]. This is somewhat unfortunate, since Euler angles are neither invariant, nor well behaved for arbitrary rotations. A possible explanation for this frequent existence of Euler angles in the texts of classical mechanics lies in the dominance of celestial mechanics in the past; the 3-1-3 Euler angles may be interpreted as precession, nutation and spin angles, respectively.

Some of these parametrizations suffer from singularities for certain angles of rotation. However, parametrizations exist which exhibit non-singular mapping, e.g. Euler parameters which form a unit quaternion and can be manipulated using a quaternion algebra and Cayley-Klein parameters forming a unit spinor and can be handled with its respective algebra.

Stuelpnagel [3] has shown, that it is impossible to have a global 3-dimensional parametrization without singularity points for the rotation group. He also pointed out that five is the minumum number of parameters which suffices to represent the rotation group in a one-to-one manner. For practical purposes the 4-parameter representations, like quaternions, which describes a non-singular two-to-one mapping, are adequate. However, the four parametric representations need special updating procedure for rotational quantities and are thus more cumbersome than some of the three parameter representations.

Rotational vector is rather common choice to parametrize rotations, see ref. [4]. Formulations which can handle all displacement quantities in a similar way without using a special updating procedure for rotational degrees of freedom have recently gained a lot of interest [5, 6, 7].

Equilibrium formulation for thin bodies which completely avoid the use of rotational parameters have also been presented, see e.g. [8].

Finite rotations and their parametrizations

Rigid-body rotation of a particle in a continuum can be expressed in the form

$$\bar{\boldsymbol{r}} = \boldsymbol{Q}\boldsymbol{r},$$

where $\boldsymbol{Q}$ is an orthogonal rotation matrix and $\boldsymbol{r}, \bar{\boldsymbol{r}}$ are position vectors of a particle in initial and final configurations, respectively. This formula is the basis for kinematical relations needed in formulating discrete equilibrium equations, however, the representation of rotation by means of the nine parameters in $\boldsymbol{Q}$ is cumbersome since they are coupled by the orthogonality condition

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}, \quad \text{or in a component form} \quad Q_{ki}Q_{kj} = Q_{ik}Q_{jk} = \delta_{ij},$$

which defines six constraints; the sum of squares of direction cosines equals to unity (three constraints) and the axes are mutually perpendicular, i.e. distances of particles and angles between line elements remain constant during rigid rotation (three constraints). Mathematically spoken the rotation matrix is an element in the 3-dimensional special orthogonal non-commutative group SO(3):

$$\mathrm{SO}(3) = \left\{ \boldsymbol{Q} \in \mathbb{R}^9 | \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}, \det \boldsymbol{Q} = 1 \right\}.$$

Even the representation of rotation by means of the elements of a rotation matrix is not a reasonable one, the rotation matrix itself is an important quantity; and the relationship between the parameters of two common descriptions to the rotation matrix is described next.

A unit quaternion (also called versor) describing rotation ϑ about an axis $\hat{\boldsymbol{n}}$ is [9]:

$$\underline{\hat{\boldsymbol{q}}} = \left\{ \begin{array}{c} \cos(\vartheta/2) \\ \sin(\vartheta/2)\hat{\boldsymbol{n}} \end{array} \right\} = \left\{ \begin{array}{c} q_0 \\ \boldsymbol{q} \end{array} \right\}$$

where $q_0 = \cos(\vartheta/2)$ is the scalar part and $\boldsymbol{q}^T = (q_1, q_2, q_3) = \sin(\vartheta/2)\hat{\boldsymbol{n}}^T$ the vector part of a quaternion. The four quaternion parameters, also called Euler [1, 10, 11] or Euler-Rodrigues parameters [12], have to satisfy the normalizing condition

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \qquad (1)$$

The rotation matrix can now be written in terms of the Euler parameters [10, 11]

$$\boldsymbol{Q} = (2q_0^2 - 1)\boldsymbol{I} + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0 \text{skew}(\boldsymbol{q}),$$

where

$$\boldsymbol{q}\times = \text{skew}(\boldsymbol{q}) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix},$$

showing the correspondence of a skew-symmetric matrix and the cross product operation. In finite element analysis the components of the vector part of the quaternion $\boldsymbol{q}$ can be used as primary variables and the scalar part is determined from the constraint equation (1) [13].

A vector-like parametrization of finite rotations is considered by Cardona and Geradin [4], Ibrahimbegović *et al.* [5] and Pacoste and Eriksson [6]:

$$\boldsymbol{\theta} = \vartheta\hat{\boldsymbol{n}}, \qquad \vartheta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2},$$

for which the relationship to the rotation matrix is given through the exponential mapping formula:

$$\boldsymbol{Q} = \cos\vartheta\boldsymbol{I} + \frac{\sin\vartheta}{\vartheta}\text{skew}(\boldsymbol{\theta}) + \frac{1-\cos\vartheta}{\vartheta^2}\boldsymbol{\theta}\boldsymbol{\theta}^T = \exp(\text{skew}(\boldsymbol{\theta})).$$

An enlarged exploration for the matrix formulation of 3D-finite rotations can be found in ref. [14].

Consistent linearization

The most common formulations for the discretized non-linear equilibrium equations are the Eulerian, total Lagrangian and updated Lagrangian approaches, see ref. [4]. For the analysis of thin bodies some Eulerian type formulations are called as co-rotational [15]. However, a necessary step for all these formulations is the linearization of the virtual work equation [16]. At this step an additional complexity is encountered by $\boldsymbol{Q}$ as being a two-point tensor. Therefore clear distinction between the spatial and material representations of the admissible variations of the rotation tensor [4, 5, 6]:

$$\delta\boldsymbol{Q} = \delta\boldsymbol{\phi}\boldsymbol{Q} = \boldsymbol{Q}\delta\boldsymbol{\psi}$$

should be made. In order to obtain a consistent parametrization, $\delta\boldsymbol{\phi}$ and $\delta\boldsymbol{\psi}$ have to be projected onto the paramater space adopted for $\boldsymbol{Q}$. The spatial and material angular variations are

$$\delta\boldsymbol{\phi} = \boldsymbol{T}_s(\boldsymbol{\theta})\delta\boldsymbol{\theta}, \qquad \delta\boldsymbol{\psi} = \boldsymbol{T}_m(\boldsymbol{\theta})\delta\boldsymbol{\theta},$$

where $\boldsymbol{T}_s = \boldsymbol{T}_m^T$ and

$$\boldsymbol{T}_m = \frac{\sin\vartheta}{\vartheta}\boldsymbol{I} + \left(1 - \frac{\sin\vartheta}{\vartheta}\right)\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^T - \frac{1}{2}\left(\frac{\sin(\vartheta/2)}{(\vartheta/2)}\right)\text{skew}(\boldsymbol{\theta}).$$

Therefore it is easily understood, that the consistent linearization of the non-linear expression of the internal virtual work

$$\delta W_{int} = -\int \boldsymbol{\sigma} \cdot \delta\boldsymbol{\epsilon} \mathrm{d}V$$

will result in an extremely complex dependency of the displacement quantities for the discrete internal force vector. Detailed expressions for the internal force vector and tangent operator matrix for different formulations can be found from refs. [4, 5, 6, 7].

Stability analysis

For thin bodies, structural stability analyses are of utmost importance. A critical point along the equilibrium path can be determined by solving the non-linear eigenvalue problem: find $\boldsymbol{u}$, λ, $\boldsymbol{v}$ such that

$$D\boldsymbol{f}(\boldsymbol{u},\lambda)\boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{f}(\boldsymbol{u},\lambda) = \boldsymbol{0} \tag{2}$$

where $\boldsymbol{f}$ is the vector of unbalanced forces and $D\boldsymbol{f}$ denotes the Gateaux derivative with respect to the state variables $\boldsymbol{u}$. The non-linear mapping $\boldsymbol{f}$ is usually expressed as a balance between external loads and internal forces $\boldsymbol{f} = \boldsymbol{r} - \lambda\boldsymbol{p}_r$, where the internal force vector $\boldsymbol{r}$ is assembled from the element contributions:

$$\boldsymbol{r}^{(e)} = \int_{V^{(e)}} \boldsymbol{B}^T\boldsymbol{\sigma}\mathrm{d}V,$$

where $\boldsymbol{\sigma}$ denotes the stresses and $\boldsymbol{B}$ is the strain-displacement operator matrix, defined via the variation of strain tensor $\delta\boldsymbol{\epsilon} = \boldsymbol{B}\delta\boldsymbol{u}$. In the stability analysis of thin bodies the strains are usually small, thus for linear elastic solids the tangent constitutive matrix is independent of the displacements. For large strains the definition of appropriate tangent matrix is more involved, see ref. [17].

Assuming an equilibrium state $(\boldsymbol{u}_*, \lambda_*)$ with a regular tangent matrix, a Taylor expansion of the non-linear eigenvalue problem (2) has the form [18]

$$\left(\boldsymbol{K}_* + \Delta\lambda\boldsymbol{K}'_* + \frac{1}{2}(\Delta\lambda)^2\boldsymbol{K}''_* + \cdots\right)\boldsymbol{v} = \boldsymbol{0}, \tag{3}$$

where $\Delta\lambda = \lambda - \lambda_*$ and the prime denotes the differentiation with respect to the load parameter λ. Expanding the displacement vector $\boldsymbol{u} = \boldsymbol{u}_* + \Delta\boldsymbol{u} = \boldsymbol{u}_* + \Delta\lambda\boldsymbol{u}' + \frac{1}{2}(\Delta\lambda)^2\boldsymbol{u}'' + \cdots$ the matrices in (3) are

$$\begin{aligned}
\boldsymbol{K}_* &= D\boldsymbol{f}_*,\\
\boldsymbol{K}'_* &= D^2\boldsymbol{f}_*\boldsymbol{u}' + D\boldsymbol{f}'_*,\\
\boldsymbol{K}''_* &= D^2\boldsymbol{f}_*\boldsymbol{u}'' + D^3\boldsymbol{f}_*\boldsymbol{u}'\boldsymbol{u}' + 2D^2\boldsymbol{f}'_*\boldsymbol{u}' + D\boldsymbol{f}''_*,
\end{aligned}$$

where $\boldsymbol{f}_* = \boldsymbol{f}(\boldsymbol{u}_*, \lambda_*)$ etc. The Euler-predictor step $\boldsymbol{u}'$ can be solved from the linear system

$$\boldsymbol{K}_*\boldsymbol{u}' = -\boldsymbol{f}'_*,$$

and the higher order terms are found from differentiation of the above expression, e.g.

$$\boldsymbol{K}_*\boldsymbol{u}'' = -\left[D^2\boldsymbol{f}_*\boldsymbol{u}'\boldsymbol{u}' + 2D\boldsymbol{f}'_*\boldsymbol{u}' + \boldsymbol{f}''_*\right].$$

It is worthwhile to notice that the coefficent matrix to solve $\boldsymbol{u}'$, $\boldsymbol{u}''$... is the same for all cases.

In the classical linear stability analysis the reference state is the undeformed stress free configuration. Assuming dead weight loading, the term $D\boldsymbol{f}'_*$ and its higher order derivatives with respect to the load parameter will vanish. For the linear stability eigenvalue problem the matrices are simply the following:

$$\boldsymbol{K}_0 = D\boldsymbol{f}(\boldsymbol{0}, 0)$$
$$\boldsymbol{K}'_0 = D^2\boldsymbol{f}(\boldsymbol{0}, 0)\boldsymbol{u}',$$

where $\boldsymbol{K}_0\boldsymbol{u}' = \boldsymbol{p}_r$. Therefore the strains are linear functions of the displacements $\boldsymbol{u}'$ and the geometric stiffness matrix $\boldsymbol{K}'$ is a linear function of the displacements $\boldsymbol{u}'$.

It is seen from the definition of the $\boldsymbol{K}'$ matrix that the "initial stress" state to the linear eigenvalue problem has to be linear with respect to the load parameter change. This is not true if the linear stability eigenvalue problem is solved from

$$(\boldsymbol{K}_* + s(\boldsymbol{K}_{**} - \boldsymbol{K}_*))\,\boldsymbol{v} = \boldsymbol{0},$$

where $\boldsymbol{K}_*$ and $\boldsymbol{K}_{**}$ are the tangent stiffness matrices from two consecutive equilibrium states. It will be a correct approximation to the linear eigenvalue problem only if the load increment $\Delta\lambda_* = \lambda_{**} - \lambda_*$ is small, i.e. $\boldsymbol{K}'_* \approx (\Delta\lambda_*)^{-1}(\boldsymbol{K}_{**} - \boldsymbol{K}_*)$.

An example

Only few particularly suitable test cases for the analysis of three dimensional spatial framed structures can be found in the literature. One proper example is a point loaded right angle cantilever beam, already used by Argyris *et al.* in 1978 [19], and further examined e.g. in refs. [15, 20, 21]. The correct value for the lateral buckling load, computed as a linear stability eigenvalue problem, seems to be $1.3215\sqrt{EI_yGI_t}/L^2$ or $0.8276\sqrt{EI_yGI_t}/L^2$ depending on the direction of the point load (L denotes the length of the members, EI_y the bending rigidity in the lateral direction and GI_t the torsional rigidity). These numbers are the converged values when a two noded shear deformable beam element is used. However, if the rotation matrix is linearized prior the linearization of the virtual work expression, incorrect buckling predictions are obtained, which are about 40-50 % lower than the correct values [21]. On the other hand, if the initial stress field is incorrectly computed (contains non-linear contributions of the reference displacement field), the following values are obtained $1.788\sqrt{EI_yGI_t}/L^2$ and $0.7083\sqrt{EI_yGI_t}/L^2$.

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