ANOMALOUS BUCKLING BEHAVIOUR OF TRUSS BEAMS

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ABSTRACT

A thorough stability analysis of a simple but representative truss beam, modeled as a finite degree of freedom system, is carried out. Depending on the main dimensions of the truss, different buckling modes are possible. One possible buckling mode is characterized by rigid body motion of the compressed member. It will be shown that this buckling mode is not included in the spectrum of the linearized buckling eigenvalue problem. Also the postbuckling behavior is studied in detail.

1 INTRODUCTION

Truss beams are widely used constructive elements due to lightness and high load carrying capacity. Although it is used in very basic building types, such as parking lots, shopping centres and industrial buildings, there seems to be no clear understanding of the structural behaviour or specific regulation in most design codes. Indeed, the design codes for steel construction focus mainly on the dimensioning and design of individual nodes and bars of the truss beam. Typically the designer would calculate resultant forces in each member of the truss using linear theory and compare the obtained effort to the non-linear resistance capacity of the member. In most design codes this non-linear resistance capacity accounts for combined buckling and yielding. The design procedure then assumes that the structure is safe when each compressed member is checked against buckling and each tension member is checked against yield failure.

To ensure stability of the truss against lateral buckling, the designer would restrict lateral movement of the upper chord typically with roof sheeting or purlins. In most situations the lower chord is left free to move laterally. This design choice is based on the assumption that a solid beam does not buckle laterally if the upper chord lateral movement is restricted. However, is this assumption true for a truss beam? In this article, it will be shown with a simple example case that there exists, indeed, a previously unreckoned buckling mode characterized by the rotation and sway of the compressed member as a rigid body while the upper chord undergoes torsion along its axis.

A thorough stability analysis of a simple but representative truss beam, modelled as a finite degree of freedom system, is carried out. Depending on the main dimensions of the truss, different buckling modes are possible. One possible buckling mode is characterized by lateral movement of the lower chord while the compressed vertical member buckles as a rigid body. It will be shown that this buckling mode is not included in the spectrum of the linearized buckling eigenvalue problem. Also the post buckling behaviour is studied in detail.

2 A TRUSS BEAM MODEL

The structure we are about to investigate consists of five members, as per figure 1: the upper chord member (1) and (2), a compressed vertical member (3) and the diagonal tensile members (4) and (5). The application point D of the load F is modeled by a undeformable member (6). Due to the symmetry of the problem with respect to the axis BC we may investigate only half of the structure consisting of members (1), (3) and (4).

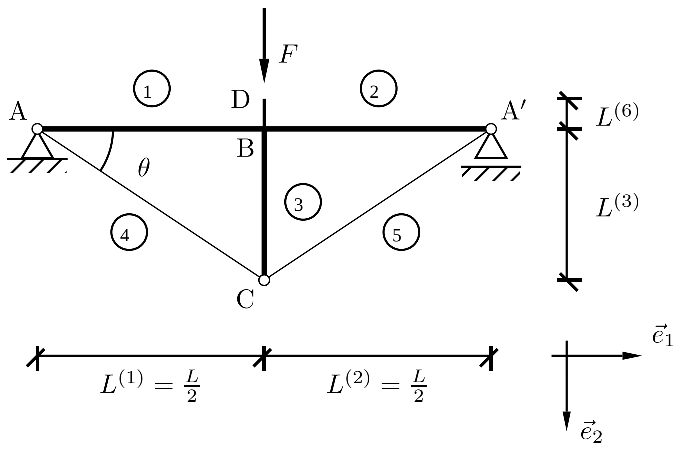

Figure 1: Truss Beam Model

2.1 Model Assumptions

The potential energy functional of the system may be formally written as the sum of internal energies due to each member of the assembly plus the work done by external forces

$$
\begin{aligned}
\Pi : \mathscr{C}([0, L^{(1)}]) \times \ldots \times \mathscr{C}([0, L^{(6)}]) \times \mathbb{R} \quad &\longrightarrow \quad \mathbb{R} \\
(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(6)}, F) \quad &\longmapsto \quad \sum_{e=1}^{6} U^{(e)}(\boldsymbol{u}^{(e)}) - F\,V(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(6)})
\end{aligned}
$$

The continuous functions $\boldsymbol{u}^{(e)} : [0, L^{(e)}] \longrightarrow \mathbb{R}^6$ such as $\boldsymbol{u}^{(e)} = (u_1^{(e)}, u_2^{(e)}, u_3^{(e)}, \phi_1^{(e)}, \phi_2^{(e)}, \phi_3^{(e)})$ describe the displacements and rotations for a given member e. The internal energies $U^{(e)} \in \mathscr{L}_2([0, L^{(e)}])$ are square integrable functions on their respective domain. Without a proof, we claim that for small displacements this continuous representation of potential energy can be equated to a discrete representation:

$$
\begin{aligned}
\Pi : \mathbb{R}^{M+1} \quad &\longrightarrow \quad \mathbb{R} \\
(\boldsymbol{x}, F) \quad &\longmapsto \quad \sum_{e=1}^{6} U^{(e)}(\boldsymbol{x}) - F\,V(\boldsymbol{x})
\end{aligned}
$$

Consider the vector $\boldsymbol{x} \in \mathbb{R}^M$, (here $M = 5$) where the meaning of the components is explained in table 1. Assuming linear elastic behaviour, we can then express the internal energy functionals and

Table 1: Discrete state variables.

x_1	$\phi_1^{(1)} := \phi_1^{(1)}(L^{(1)})$	torsion angle of member (1) at point B
x_2	$u_3^{(1)} := u_3^{(1)}(L^{(1)})$	lateral displacement of member (1) at point B
x_3	$u_2^{(1)} := u_2^{(1)}(L^{(1)})$	vertical displacement of member (1) at point B
x_4	$\epsilon_{11}^{(1)} := \partial_1 u_1^{(1)}(L^{(1)})$	axial strain of member (1) at point B
x_5	$\epsilon_{11}^{(3)} := \partial_1 u_1^{(3)}(L^{(3)})$	axial strain of member (3) at point B

Table 2: Stiffness Coefficients

$c_1^{(1)}$	$:=$	$GI_t^{(1)}/L^{(1)}$	torsional stiffness of member (1)
$k_3^{(1)}$	$:=$	$3\,EI_{13}^{(1)}/{L^{(1)}}^3$	lateral stiffness of member (1)
$k_2^{(1)}$	$:=$	$3\,EI_{12}^{(1)}/{L^{(1)}}^3$	vertical stiffness of member (1)
$k_1^{(1)}$	$:=$	$EA^{(1)}/L^{(1)}$	axial stiffness of member (1)
$k_1^{(3)}$	$:=$	$EA^{(3)}/L^{(3)}$	axial stiffness of member (3)
$k_1^{(4)}$	$:=$	$EA^{(4)}/L^{(4)}$	axial stiffness of member (4)

work done by external forces can be expressed as follows:

$$
\begin{aligned}
U^{(1)}(\boldsymbol{x}) &= \frac{1}{2}\,c_1^{(1)}\ {\phi_1^{(1)}}^2 + \frac{1}{2}\,k_3^{(1)}\ {u_3^{(1)}}^2 + \frac{1}{2}\,k_2^{(1)}\ {u_2^{(1)}}^2 + \frac{1}{2}\,k_1^{(1)}\ \left(\epsilon_{11}^{(1)}\,L^{(1)}\right)^2 \\
U^{(2)}(\boldsymbol{x}) &= U^{(1)}(\boldsymbol{x}) \\
U^{(3)}(\boldsymbol{x}) &= \frac{1}{2}\,k_1^{(3)}\ \left(\epsilon_{11}^{(3)}\,L^{(3)}\right)^2 \\
U^{(4)}(\boldsymbol{x}) &= \frac{1}{2}\,k_1^{(4)}\ \left(\epsilon_{11}^{(4)}\,L^{(4)}\right)^2 \\
U^{(5)}(\boldsymbol{x}) &= U^{(4)}(\boldsymbol{x}) \\
U^{(6)}(\boldsymbol{x}) &= 0 \\
V(\boldsymbol{x}) &= L^{(6)}\,(1-\cos\phi_1^{(1)}) + u_2^{(1)}
\end{aligned}
$$

where the constraint equation that expresses the stain in member (4) is defined as follows: (assume $\cos^2\theta = L^{(1)}/L^{(4)}$, $\sin^2\theta = L^{(3)}/L^{(4)}$)

$$
\epsilon_{11}^{(4)} := \cos^2\theta\,\epsilon_{11}^{(1)} + \sin^2\theta\,\epsilon_{11}^{(3)} + \sin\theta\,\cos\theta\,(1+\epsilon_{11}^{(3)})\left(\frac{u_2^{(1)}}{L^{(1)}}\,\cos\phi_1^{(1)} + \frac{u_3^{(1)}}{L^{(1)}}\,\sin\phi_1^{(1)}\right)
$$

dividing the total potential energy functional Π by $2\,c_1^{(1)}$ yields the non dimensional form as follows:

$$
P(\boldsymbol{q},\lambda) = \sum_{t=1}^{5}\frac{1}{2}\,\alpha_t\,q_t^2 + \frac{1}{2}\,\alpha_6\,Q_6^2(\boldsymbol{q}) - \lambda(q_3 + \rho\,R(\boldsymbol{q}))
$$

such as $P := \Pi/(2\,c_1^{(1)})$, $\lambda := F\,L^{(1)}/(2\,c_1^{(1)})$, and where the non dimensional variables are defined as follows:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} := \begin{pmatrix} \phi_1^{(1)} \\ u_3^{(1)}/L^{(1)} \\ u_2^{(1)}/L^{(1)} \\ \epsilon_{11}^{(1)} \\ \epsilon_{11}^{(3)} \end{pmatrix}$$

the non-dimensional stiffnesses α_i are defined in table 3 and the constraint equations are $Q_6(\boldsymbol{q}) := \cos^2\theta\, q_4 + \sin^2\theta\, q_5 + \sin\theta\,\cos\theta\,(1+q_5)\,(q_3\,\cos q_1 + q_2\,\sin q_1)$ and $R(\boldsymbol{q}) := 1 - \cos q_1$. The non-dimensional eccentricity of load-application point $\rho = L^{(6)}/L^{(1)}$.

Table 3: Non-dimensional stiffness coefficients.

α_1	:=	1
α_2	:=	$(2\,k_3^{(1)}\,\left(L^{(1)}\right)^2)/(2\,c_1^{(1)})$
α_3	:=	$(2\,k_2^{(1)}\,\left(L^{(1)}\right)^2)/(2\,c_1^{(1)})$
α_4	:=	$(2\,k_1^{(1)}\,\left(L^{(1)}\right)^2)/(2\,c_1^{(1)})$
α_5	:=	$(\,k_1^{(3)}\,\left(L^{(3)}\right)^2)/(2\,c_1^{(1)})$
α_6	:=	$(2\,k_1^{(4)}\,\left(L^{(4)}\right)^2)/(2\,c_1^{(1)})$

3 ASYMPTOTIC ANALYSIS

In this section we develop the reduced potential energy taking inspiration from Koiter reduction theory [1, 2]. Although Koiter's works mainly focused on continuous functionals, here we are in presence of a discrete formulation. Hence we can observe some minor differences when comparing to the results obtained by Koiter [3, 4]. Essentially the Koiter's initial post-buckling theory is a kind of a Liapunov-Schmit type reduction technique [5, 6]. Assume $P : \mathbb{R}^{M+1} \to \mathbb{R} : (\boldsymbol{q}, \lambda) \mapsto P(\boldsymbol{q}, \lambda)$ a functional that describes the total potential energy of a discrete system which exhibits pitch-fork type symmetric bifurcation behaviour. Additionally we assume that $P(\boldsymbol{q}, \cdot) : \mathbb{R} \to \mathbb{R}$ is linear. The internal energy part of a linear elastic system can always be expressed as a diagonal quadratic form, $U(\boldsymbol{q}) = \sum_{t\geq 1} \frac{1}{2}\,\alpha_t\,Q_t^2(\boldsymbol{q})$ and the work done by external forces as a linear form $\lambda\,V(\boldsymbol{q}) = \lambda \sum_{t\geq 1} R_t(\boldsymbol{q})$. We can always find a coordinate system, at least locally, such as $Q_t(\boldsymbol{q}) = q_t$ for $1 \leq t \leq M$, and $R_1(\boldsymbol{q}) = q_{m+1}$ where q_{m+1} is the first non-zero coordinate in the equilibrium set on primary path. We note also $R(\boldsymbol{q}) := \sum_{t\geq 2} R_t(\boldsymbol{q})$. Consider the general equilibrium set $E := \left\{(\boldsymbol{q}, \lambda) \in \mathbb{R}^{M+1} \;:\; P_{\boldsymbol{q}}(\boldsymbol{q}, \lambda) = 0\right\}$ and the equilibrium set on the primary path $E_I := \{(\boldsymbol{q}, \lambda) \in E \;:\; q_i = 0 \; for\; 1 \leq i \leq m < M\} \subset E$. With these assumptions we can write a potential energy expression for discrete linear elastic system with distinct primary path:

$$P(\boldsymbol{q}, \lambda) = \sum_{t=1}^{M} \frac{1}{2}\,\alpha_t\,q_t^2 + \sum_{t>M} \frac{1}{2}\,\alpha_6\,Q_6^2(\boldsymbol{q}) - \lambda(q_{m+1} + R(\boldsymbol{q})) \qquad (1)$$

We can make a few remarks on the form of the potential energy as expressed in equation (1) and its derivatives. The equilibrium set on the primary path E_I is a 1-manifold in a $\mathbb{R}^{M-m+1}$ space, which implies that m of the M equations $P_{\boldsymbol{q}}(\hat{\boldsymbol{q}}, \hat{\lambda})_i = 0$ have to hold identically. It comes out that the first derivative is of the form:

$$\hat{P}_{\boldsymbol{q}} = \begin{pmatrix} 0 \\ \hat{P}_{\boldsymbol{q}}\,(m+1..M) \end{pmatrix} \tag{2}$$

Similarly we get from the second derivative a block diagonal matrix form:

$$\hat{P}_{\boldsymbol{qq}} = \begin{pmatrix} \hat{P}_{\boldsymbol{qq}}\,(1..m,1..m) & 0 \\ 0 & \hat{P}_{\boldsymbol{qq}}\,(m+1..M,m+1..M) \end{pmatrix} \tag{3}$$

These properties propagate of course to higher order derivatives also. Further, since $\hat{P}_{\boldsymbol{qq}}$ is symmetric, it is diagonalizable. Hence $\exists\, \boldsymbol{\Phi} \in \mathbb{R}^{M\times M}$ unitary ($\boldsymbol{\Phi}\,\boldsymbol{\Phi}^T = \boldsymbol{\Phi}^T\,\boldsymbol{\Phi} = \boldsymbol{I}$) such as $\boldsymbol{\Phi}^T\,\hat{P}_{\boldsymbol{qq}}\,\boldsymbol{\Phi} =$ diag$(0,...,0,d_{R+1},...,d_M)$ assuming that $\dim \mathrm{Ker}\,\hat{P}_{\boldsymbol{qq}} = R \leq m$. Due to the block matrix shape of $\hat{P}_{\boldsymbol{qq}}$ in equation (3) we get the following form for the unitary matrix $\boldsymbol{\Phi}$:

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}\,(1..m,1..m) & 0 \\ 0 & \boldsymbol{\Phi}\,(m+1..M,m+1..M) \end{pmatrix} \tag{4}$$

Considering $\boldsymbol{\phi}_p$ as the p-th column vector of $\boldsymbol{\Phi}$ we get that $\{\boldsymbol{\phi}_1,...,\boldsymbol{\phi}_M\}$ is a orthonormal basis of $\mathbb{R}^M = \mathrm{Ker}\,\hat{P}_{\boldsymbol{qq}} \oplus \mathrm{Ker}\,\hat{P}_{\boldsymbol{qq}}{}^{\perp}$. Hence $\{\boldsymbol{\phi}_1,...,\boldsymbol{\phi}_R\}$ would be an orthonormal basis for Ker $\hat{P}_{\boldsymbol{qq}}$ and $\{\boldsymbol{\phi}_{R+1},...,\boldsymbol{\phi}_M\}$ for Ker $\hat{P}_{\boldsymbol{qq}}{}^{\perp}$.

Now we seek to expand the potential energy in Taylor series around the critical point $(\hat{\boldsymbol{q}},\hat{\lambda}) \in E_I$ defined such as $P_{\boldsymbol{qq}}(\hat{\boldsymbol{q}},\hat{\lambda})$ is singular. Although it is not generally the case, for simplicity we assume that the geometric multiplicity equals to one. This assumption holds particularly for the system we are investigating. The Taylor's series expansion of the potential energy up to order four is given by equation (5):

$$P(\boldsymbol{q},\lambda) = \sum_{k=0}^{4} P_k(\Delta\boldsymbol{q},\hat{\lambda}) + \sum_{k=0}^{3} \Delta\lambda\,\dot{P}_k(\Delta\boldsymbol{q},\hat{\lambda}) + \mathscr{O}(\|(\Delta\boldsymbol{q},\Delta\lambda)\|^5) \tag{5}$$

where the $P_k(\Delta\boldsymbol{q},\hat{\lambda})$ represent the k-th order forms and $\dot{P}_k(\Delta\boldsymbol{q},\hat{\lambda})$ their derivatives with respect to λ:

$$P_k(\Delta\boldsymbol{q},\hat{\lambda}) := \frac{1}{k!}\,(\hat{P}_{\boldsymbol{q}...\boldsymbol{q}})_{i_1...i_k}\,\Delta q_{i_1}...\Delta q_{i_k} \tag{6}$$

further we define $P_{(k-l)l}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})$ as $(k-l)$-th order forms in $\boldsymbol{u}$ and l-th order forms in $\boldsymbol{v}$ such as $P_k(\boldsymbol{u}+\boldsymbol{v},\hat{\lambda}) = \sum_{0\leq l\leq k} P_{(k-l)l}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})$ holds. Since $\Delta\boldsymbol{q} \in \mathbb{R}^M$ it can be uniquely decomposed as $\Delta\boldsymbol{q} = \boldsymbol{u}+\boldsymbol{v}$ such as $\boldsymbol{u} \in \mathrm{Ker}\,\hat{P}_{\boldsymbol{qq}}$ and $\boldsymbol{v} \in \mathrm{Ker}\,\hat{P}_{\boldsymbol{qq}}{}^{\perp}$. Consequently we have the expressions of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in their respective basis: $\boldsymbol{u} = \sum_{1\leq p\leq R} a_p\,\boldsymbol{\phi}_p$ and $\boldsymbol{v} = \sum_{R+1\leq p\leq M} b_p\,\boldsymbol{\phi}_p$, where a_p are the amplitudes of each direction of the eigenmode and where b_p are second order amplitudes in directions orthogonal to the eigenmode. Define the coefficients b_p as:

$$b_p := \Delta\lambda \sum_{1\leq s\leq R} a_s\,\xi_{sp} + \sum_{1\leq s,t\leq R} a_s\,a_t\,\eta_{stp} \tag{7}$$

Assume from now on the kernel is one dimensional (R = 1), which leads to a simple expression for the coefficients b_p:

$$b_p := \Delta\lambda\,a_1\,\xi_{1p} + a_1^2\,\eta_{11p} \tag{8}$$

Comparing the norms of $\boldsymbol{u}$ and $\boldsymbol{v}$ we can say the following $\|\boldsymbol{v}\| = \mathscr{O}(\|(\Delta\lambda\,a_1, a_1^2)\|)$, while $\|\boldsymbol{u}\| = \mathscr{O}(|a_1|)$ which is the reason why we call $\boldsymbol{v}$ the second order field. Substituting $\boldsymbol{u}$ and $\boldsymbol{v}$ in the potential energy Taylor series expansion, equation (5), yields:

$$\begin{aligned}P(\boldsymbol{q},\hat{\lambda}) &= P_0(\boldsymbol{u},\hat{\lambda})+\\&\quad P_1(\boldsymbol{u},\hat{\lambda})+P_1(\boldsymbol{v},\hat{\lambda})+\\&\quad P_2(\boldsymbol{u},\hat{\lambda})+P_{11}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+P_2(\boldsymbol{v},\hat{\lambda})+\\&\quad P_3(\boldsymbol{u},\hat{\lambda})+P_{21}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+\mathscr{O}(\left\|(\Delta\lambda\, a_1,a_1^2)\right\|^5)+\\&\quad P_4(\boldsymbol{u},\hat{\lambda})+\mathscr{O}(\left\|(\Delta\lambda\, a_1,a_1^2)\right\|^5)\end{aligned}$$

and

$$\begin{aligned}\Delta\lambda\,\dot{P}(\boldsymbol{q},\hat{\lambda}) &= \Delta\lambda\,\dot{P}_0(\boldsymbol{u},\hat{\lambda})+\\&\quad \Delta\lambda\,\dot{P}_1(\boldsymbol{u},\hat{\lambda})+\Delta\lambda\,\dot{P}_1(\boldsymbol{v},\hat{\lambda})+\\&\quad \Delta\lambda\,\dot{P}_2(\boldsymbol{u},\hat{\lambda})+\Delta\lambda\,\dot{P}_{11}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+\mathscr{O}(\left\|(\Delta\lambda\, a_1,a_1^2)\right\|^5)+\\&\quad \Delta\lambda\,\dot{P}_3(\boldsymbol{u},\hat{\lambda})+\mathscr{O}(\left\|(\Delta\lambda\, a_1,a_1^2)\right\|^5)\end{aligned}$$

Now we know from the previous discussion that $P_1(\boldsymbol{u},\hat{\lambda})=P_1(\boldsymbol{v},\hat{\lambda})=\dot{P}_1(\boldsymbol{u},\hat{\lambda})=0$ as well as $P_2(\boldsymbol{u},\hat{\lambda})=\boldsymbol{u}^T\,\hat{P}_{\boldsymbol{qq}}\,\boldsymbol{u}=0$. Likewise $P_{11}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})=\boldsymbol{v}^T\,\hat{P}_{\boldsymbol{qq}}\,\boldsymbol{u}=0$. For symmetry reasons $P_3(\boldsymbol{u},\hat{\lambda})=0$, which in turn implies $\dot{P}_3(\boldsymbol{u},\hat{\lambda})=0$. Discarding the zero terms we get the following representation for the potential energy:

$$\begin{aligned}P(\boldsymbol{q},\lambda) &= P_0(\boldsymbol{u},\hat{\lambda})+\Delta\lambda\,\dot{P}_0(\boldsymbol{u},\hat{\lambda})+\Delta\lambda\,\dot{P}_2(\boldsymbol{u},\hat{\lambda})+P_4(\boldsymbol{u},\hat{\lambda})\\&\quad \Delta\lambda\,\dot{P}_1(\boldsymbol{v},\hat{\lambda})+\Delta\lambda\,\dot{P}_{11}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+P_{21}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+P_2(\boldsymbol{v},\hat{\lambda})+\mathscr{O}(\left\|(\Delta\lambda\, a_1,a_1^2)\right\|^5)\end{aligned}$$

We seek now to minimize the functional $F(\boldsymbol{v}):=\Delta\lambda\,\dot{P}_1(\boldsymbol{v},\hat{\lambda})+\Delta\lambda\,\dot{P}_{11}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+P_{21}(\boldsymbol{u},\boldsymbol{v},\hat{\lambda})+P_2(\boldsymbol{v},\hat{\lambda})$ by solving the system $\delta F(\boldsymbol{v})=0$. Proceeding with the differentiation yields:

$$\Delta\lambda\,\dot{P}_1(\delta\boldsymbol{v},\hat{\lambda})+\Delta\lambda\,\dot{P}_{11}(\boldsymbol{u},\delta\boldsymbol{v},\hat{\lambda})+P_{21}(\boldsymbol{u},\delta\boldsymbol{v},\hat{\lambda})+P_{11}(\boldsymbol{v},\delta\boldsymbol{v},\hat{\lambda})=0 \tag{9}$$

Let's substitute in equation (9) the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by their component descriptions in the basis $\{\boldsymbol{\phi}_1,...,\boldsymbol{\phi}_M\}$.

$$\sum_{2\le p\le M}\Bigg(\Delta\lambda\,\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})+\Delta\lambda\,a_1\,\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})+a_1^2\,P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})+ \\ +\sum_{2\le q\le M}P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_q,\hat{\lambda})\,b_q\Bigg)\,\delta b_p=0 \tag{10}$$

Equation (10) has to hold $\forall(\delta b_p)\subset\mathbb{R}^{(M-1)}$. Also reminding that $P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_q,\hat{\lambda})=d_p\,\delta_{pq}$ and substituting b_q by its expression we get:

$$\begin{aligned}\Delta\lambda\,\left(\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})+a_1\,\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})+d_p\,a_1\,\xi_{1p}\right) &+\\ a_1^2\left(P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})+d_p\,\eta_{11p}\right) &= 0\end{aligned} \tag{11}$$

which has to hold $\forall\Delta\lambda\in\mathbb{R}$. Taking a closer look at the components $\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})$, $\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})$ and $P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})$ tells us that

$$\begin{aligned}\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})=0\,, &\qquad 2\le p\le m\\ \dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})=0\,, &\qquad m+1\le p\le M\\ P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})=0\,, &\qquad 2\le p\le m\end{aligned}$$

Taking this information into account yields the following expression for the minimized components $\tilde{b}_p$

$$\tilde{b}_p = \begin{cases} -\Delta\lambda\, a_1 \frac{\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} & , \quad 2 \le p \le m \\ -\left(\Delta\lambda \frac{\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} + a_1^2 \frac{P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})}\right) & , \quad m+1 \le p \le M \end{cases} \tag{12}$$

Consider any functional, which we want to minimize, consisting of a quadratic and linear part: $F_2(\boldsymbol{v}) + F_1(\boldsymbol{v})$. Let $\tilde{\boldsymbol{v}}$ be the point which minimizes the functional. Taking the first variation and evaluating it at $\boldsymbol{v} = \tilde{\boldsymbol{v}}$ gives $F_{11}(\tilde{\boldsymbol{v}}, \delta\boldsymbol{v}) + F_1(\delta\boldsymbol{v}) = 0$, which holds $\forall \delta\boldsymbol{v}$. In particular it holds for $\delta\boldsymbol{v} = \tilde{\boldsymbol{v}}$, hence $F_{11}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}) + F_1(\tilde{\boldsymbol{v}}) = 0$. The latter in turn can be simplified as $2\,F_2(\tilde{\boldsymbol{v}}) + F_1(\tilde{\boldsymbol{v}}) = 0$. Now the original functional at its minimum is expressed as $F_2(\tilde{\boldsymbol{v}}) + F_1(\tilde{\boldsymbol{v}}) = -F_2(\tilde{\boldsymbol{v}})$. Apply this to our case, and express the potential energy functional:

$$P(\boldsymbol{q},\lambda) = P_0(\boldsymbol{u},\hat{\lambda}) + \Delta\lambda\,\dot{P}_0(\boldsymbol{u},\hat{\lambda}) + \Delta\lambda\,\dot{P}_2(\boldsymbol{u},\hat{\lambda}) + P_4(\boldsymbol{u},\hat{\lambda}) - P_2(\tilde{\boldsymbol{v}}) + \mathscr{O}(\left\|(\Delta\lambda\,a_1, a_1^2)\right\|^5)$$

where

$$\begin{aligned} P_2(\tilde{\boldsymbol{v}}) =& \frac{1}{2}\sum_{p=2}^{m} \tilde{b}_p^2\, d_p + \frac{1}{2}\sum_{p=m+1}^{M} \tilde{b}_p^2\, d_p \\ =& \frac{1}{2}\sum_{p=2}^{m} \Delta\lambda^2\, a_1^2 \frac{\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} + \frac{1}{2}\sum_{p=m+1}^{M} \frac{\left(\Delta\lambda\,\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda}) + a_1^2\, P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})\right)^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} \end{aligned}$$

Rearranging the equation yields for the potential energy:

$$\begin{aligned} P(\boldsymbol{q},\lambda) \quad = \quad & P_0(\boldsymbol{u},\hat{\lambda}) + \Delta\lambda\,\dot{P}_0(\boldsymbol{u},\hat{\lambda}) - \Delta\lambda^2\,\frac{1}{2}\sum_{m+1\le p\le M} \frac{\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} + \\ & -\Delta\lambda^2\, a_1^2\,\frac{1}{2}\sum_{2\le p\le m} \frac{\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} + \\ & -\Delta\lambda\, a_1^2 \left(\sum_{m+1\le p\le M} \frac{\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})\,P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} - \dot{P}_2(\boldsymbol{\phi}_1,\hat{\lambda})\right) + \\ & +a_1^4 \left(P_4(\boldsymbol{\phi}_1,\hat{\lambda}) - \frac{1}{2}\sum_{m+1\le p\le M} \frac{P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})}\right) + \mathscr{O}(\left\|(\Delta\lambda\,a_1, a_1^2)\right\|^5) \end{aligned}$$

Deriving the potential energy with respect to the amplitude yields the reduced equilibrium equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a_1}P(\boldsymbol{q},\lambda) \quad = \quad & -a_1\Bigg\{\Delta\lambda^2 \sum_{2\le p\le m} \frac{\dot{P}_{11}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} + \\ & 2\,\Delta\lambda \left(\sum_{m+1\le p\le M} \frac{\dot{P}_1(\boldsymbol{\phi}_p,\hat{\lambda})\,P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})} - \dot{P}_2(\boldsymbol{\phi}_1,\hat{\lambda})\right) + \\ & +4\,a_1^2 \left(P_4(\boldsymbol{\phi}_1,\hat{\lambda}) - \frac{1}{2}\sum_{m+1\le p\le M} \frac{P_{21}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_p,\hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p,\boldsymbol{\phi}_p,\hat{\lambda})}\right)\Bigg\} + \mathscr{O}(\left\|(\Delta\lambda\,a_1, a_1^2)\right\|^4) \end{aligned}$$

From the equilibrium condition $\frac{\mathrm{d}}{\mathrm{d}a_1} P(\boldsymbol{q}, \lambda) = 0$ we solve the function $\lambda(a_1)$:

$$\lambda(a_1) = \hat{\lambda} + \frac{2\, P_4(\boldsymbol{\phi}_1, \hat{\lambda}) - \sum\limits_{p=m+1}^{M} \dfrac{P_{21}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_p, \hat{\lambda})^2}{P_{11}(\boldsymbol{\phi}_p, \boldsymbol{\phi}_p, \hat{\lambda})}}{\sum\limits_{p=m+1}^{M} \dfrac{\dot{P}_1(\boldsymbol{\phi}_p, \hat{\lambda})\, P_{21}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_p, \hat{\lambda})}{P_{11}(\boldsymbol{\phi}_p, \boldsymbol{\phi}_p, \hat{\lambda})} - \dot{P}_2(\boldsymbol{\phi}_1, \hat{\lambda})}\, a_1^2 + \mathscr{O}(|a_1|^4) \tag{13}$$

3.1 Application to Example Case

Consider the example truss beam defined in section 2. We shall simplify it further by saying that both the upper chord (1) and (3) and the vertical compressed member (3) are axially infinitely stiff, which results in the following constraints: $q_4 = q_5 = 0$. Assume also, for simplicity, that the eccentricity of load application point $\rho = 0$. The potential energy of such a system is given by expression

$$P(\boldsymbol{q}, \lambda) = \frac{1}{2}\alpha_1\, q_1^2 + \frac{1}{2}\alpha_2\, q_2^2 + \frac{1}{2}\alpha_3\, q_3^2 + \frac{1}{2}\alpha_6\, Q_6^2(\boldsymbol{q}) - \lambda\, q_3 \tag{14}$$

where the constraint equation is given by $Q_6(\boldsymbol{q}) := \sin\theta\, \cos\theta\, (q_3\, \cos q_1 + q_2\, \sin q_1)$. Application of the equation (13) yields the following result:

$$\lambda(a_1) = \hat{\lambda} + \frac{1}{3}\frac{\alpha_1\,(\alpha_3 + \bar{\alpha}_6)}{\hat{\lambda}\,(\alpha_2 + \bar{\alpha}_6)} \left(3\alpha_3 - 2\alpha_2 + \frac{\alpha_2\,\alpha_3}{\bar{\alpha}_6}\right) (\phi_{11}\, a_1)^2 + \mathscr{O}(|a_1|^4) \tag{15}$$

where the critical value of the load parameter is $\hat{\lambda} = \sqrt{\gamma\, \alpha_1\, \alpha_2}$ and $\gamma = \frac{(\alpha_3+\bar{\alpha}_6)^2}{\bar{\alpha}_6\,(\alpha_2+\bar{\alpha}_6)}$. The expression $\bar{\alpha}_6$ has been introduced for convenience and defiend as: $\bar{\alpha}_6 := \sin^2\theta\, \cos^2\theta\, \alpha_6$. It is worth noting the limit behaviour of the system as $\alpha_6 \to \infty$, which is physically relevant, since usually axial stiffnesses are much larger than flexural or torsional stiffnesses.

$$\lambda(a_1) \xrightarrow[\alpha_6\to\infty]{} \sqrt{\alpha_1\,\alpha_2} + \frac{1}{3}\sqrt{\alpha_1/\alpha_2}\; (3\alpha_3 - 2\alpha_2)\; (\phi_{11}\, a_1)^2 + \mathscr{O}(|a_1|^4) \tag{16}$$

It comes out that we have a finite critical load, and that if the condition $3\alpha_3 - 2\alpha_2 \geq 0$ holds, then we have a stable secondary path in the neighbourhood of the critical point. Assuming that α_2 (lateral flexural stiffness) and α_3 (vertical flexural stiffness) are of same magnitude, which is usually the case, the secondary path is at worst only slightly unstable.

Compare now to another example case, where we restrict lateral sway of the upper chord (point D). We are considering a different model, but it is physically closely related to the previous one. Assume now $\alpha_2 = 0$. The potential energy of this system is given by equation (17)

$$P(\boldsymbol{q}, \lambda) = \frac{1}{2}\alpha_1\, q_1^2 + \frac{1}{2}\alpha_3\, q_3^2 + \frac{1}{2}\alpha_6\, Q_6^2(\boldsymbol{q}) - \lambda\, q_3 \tag{17}$$

where the constraint equation is given by $Q_6(\boldsymbol{q}) := \sin\theta\, \cos\theta\, q_3\, \cos q_1$. Application of the equation (13) yields the following result:

$$\lambda(a_1) = \hat{\lambda} + \frac{1}{3}\frac{\alpha_1\,(\alpha_3 + \bar{\alpha}_6)}{\hat{\lambda}\,\bar{\alpha}_6}\; (\alpha_3 - 2\alpha_6)\; (\phi_{11}\, a_1)^2 + \mathscr{O}(|a_1|^4) \tag{18}$$

where the critical value of the load parameter is $\bar{\lambda} = \sqrt{\gamma\, \alpha_1}$ such as $\gamma = \frac{(\alpha_3+\bar{\alpha}_6)^2}{\bar{\alpha}_6}$. Again, consider the limit behaviour of the system as $\alpha_6 \to \infty$. It comes out that $\hat{\lambda} \xrightarrow[\alpha_6\to\infty]{} \infty$, and in the neighbourhood of infinity the expression of λ gives:

$$\lambda(a_1) = \hat{\lambda} + \left(-\frac{2}{3}\hat{\lambda} + \mathscr{O}(\alpha_6^{-1}) \right) (\phi_{11}\, a_1)^2 + \mathscr{O}(|a_1|^4) \tag{19}$$

We infer that in case of the laterally restricted system, for large axial stiffnesses of the diagonal members the critical load is large, but on the other hand the postbuckling behaviour is very unstable.

4 NUMERICAL ANALYSIS

The goal is now to find a numeric approximation of the critical load $\hat{\lambda}$ defined previously. One possible strategy is to expand the criticality condition as a Taylor series with respect to the load parameter, and solve the critical load parameter approximation from the polynomial equation given by the truncated series. Usually one truncates starting from the quadratic term. First, we assume that there exists a neighbourhood of $(\boldsymbol{q}^{(0)}, \lambda^{(0)})$, $\mathscr{U}_{(\boldsymbol{q}^{(0)}, \lambda^{(0)})}$ such as $(\hat{\boldsymbol{q}}, \hat{\lambda}) \notin \mathscr{U}_{(\boldsymbol{q}^{(0)}, \lambda^{(0)})}$ and $P_{\boldsymbol{qq}}(\boldsymbol{q}, \lambda)$ is not singular for $(\boldsymbol{q}, \lambda) \in \mathscr{U}_{(\boldsymbol{q}^{(0)}, \lambda^{(0)})}$ Then, by the implicit function theorem we infer that there exists a function $\boldsymbol{q} : \mathscr{U}_{\lambda^{(0)}} \to \mathscr{U}_{\boldsymbol{q}^{(0)}} : \lambda \mapsto \boldsymbol{q}(\lambda)$. Then the criticality condition can be expressed as follows:

$$\hat{P}_{\boldsymbol{qq}}\, \boldsymbol{\phi}_1 = \left(P_{\boldsymbol{qq}}^{(0)} + \Delta\lambda \left(P_{\boldsymbol{qqq}}^{(0)}\, \dot{\boldsymbol{q}}^{(0)} + \dot{P}_{\boldsymbol{qq}}^{(0)} \right) \right) \boldsymbol{\phi}_1 + \mathscr{O}(\Delta\lambda^2) = 0 \tag{20}$$

where $\Delta\lambda = \hat{\lambda} - \lambda^{(0)}$. The expression of the vector $\dot{\boldsymbol{q}}^{(0)}$ is given by the equilibrium equation at the critical point:

$$\hat{P}_{\boldsymbol{q}} = P_{\boldsymbol{q}}^{(0)} + \Delta\lambda \left(P_{\boldsymbol{qq}}^{(0)}\, \dot{\boldsymbol{q}}^{(0)} + \dot{P}_{\boldsymbol{q}}^{(0)} \right) + \mathscr{O}(\Delta\lambda^2) = 0\,, \ \ \forall \Delta\lambda \in \mathbb{R} \tag{21}$$

hence we get the expression $\dot{\boldsymbol{q}}^{(0)} = -\, {P_{\boldsymbol{qq}}^{(0)}}^{-1}\, \dot{P}_{\boldsymbol{q}}^{(0)}$. Since we need that the point $(\boldsymbol{q}^{(0)}, \lambda^{(0)})$ is on the equilibrium path, the only point we know for sure for any system is the origin $(\boldsymbol{q}^{(0)}, \lambda^{(0)}) = (0, 0)$. Now assume that the potential energy is of the form given in equation (1). Then we get the following expressions:

$$\begin{aligned}
\dot{P}_{\boldsymbol{q}}^{(0)} &= -\sum_{i=m+1}^{M} (\delta_{i\, m+1} + R_{,i}) \boldsymbol{e}_i \\
P_{\boldsymbol{qq}}^{(0)} &= \sum_{t=1}^{m} \alpha_t \boldsymbol{e}_t \boldsymbol{e}_t^T + \sum_{t \geq M+1} \alpha_t \left(\sum_{i,j=m+1}^{M} Q_{t,i}^{(0)} Q_{t,j}^{(0)}\, \boldsymbol{e}_i \boldsymbol{e}_j^T \right) \\
\dot{\boldsymbol{q}}^{(0)} &= \sum_{i=m+1}^{M} \dot{q}_i^{(0)} \boldsymbol{e}_i \\
\dot{P}_{\boldsymbol{qq}}^{(0)} &= -\rho \left(\sum_{i,j=1}^{m} R_{,ij} \boldsymbol{e}_i \boldsymbol{e}_j^T + \sum_{i,j=m+1}^{M} R_{,ij} \boldsymbol{e}_i \boldsymbol{e}_j^T \right) \\
P_{\boldsymbol{qqq}}^{(0)}\, \dot{\boldsymbol{q}}^{(0)} &= \sum_{t \geq M+1} \alpha_t \Bigg[\sum_{\substack{i,j=1 \\ m+1 \leq k \leq M}}^{m} Q_{t,ij}^{(0)} Q_{t,k}^{(0)} \dot{q}_k^{(0)}\, \boldsymbol{e}_i \boldsymbol{e}_j^T + \\
&\quad \sum_{\substack{i,j=m+1 \\ m+1 \leq k \leq M}}^{M} \left(Q_{t,jk}^{(0)} Q_{t,i}^{(0)} + Q_{t,jk}^{(0)} Q_{t,i}^{(0)} + Q_{t,jk}^{(0)} Q_{t,i}^{(0)} \right) \dot{q}_k^{(0)}\, \boldsymbol{e}_i \boldsymbol{e}_j^T \Bigg]
\end{aligned} \tag{22}$$

We can easily see that our linearized criticality condition $P^{(0)}_{\boldsymbol{qq}} + \Delta\lambda \left(P^{(0)}_{\boldsymbol{qqq}}\, \dot{\boldsymbol{q}}^{(0)} + \dot{P}^{(0)}_{\boldsymbol{qq}} \right)$ is also block diagonal. Actually only the upper left block is relevant if we seek for an approximation of the critical value $\hat{\lambda}$. Introduce the notation: $\boldsymbol{K}^{(0)}_0 := P^{(0)}_{\boldsymbol{qq}}$, $\boldsymbol{K}^{(0)}_1 := P^{(0)}_{\boldsymbol{qqq}}\, \dot{\boldsymbol{q}}^{(0)} + \dot{P}^{(0)}_{\boldsymbol{qq}}$. Solving the equation (23) for $\Delta\lambda$

$$\det \begin{pmatrix} \left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(1..m, 1..m) & 0 \\ 0 & \left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(m+1..M, m+1..M) \end{pmatrix} = 0 \quad (23)$$

yields $\det(\left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(1..m, 1..m)) = 0$ or $\det(\left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(m+1..M, m+1..M)) = 0$. Assume that $\boldsymbol{K}^{(0)}_1(1..m, 1..m) = 0$, then the equation $\det(\left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(m+1..M, m+1..M)) = 0$ gives an a result, but this result is not an approximation of $\hat{\lambda}$. Therefore we introduce the following definition: the 1-st order coefficient matrix $\boldsymbol{K}^{(0)}_1$ is degenerated if $\boldsymbol{K}^{(0)}_1(1..m, 1..m) = 0$. When the 1-st order coefficient matrix is degenerated linear approximation can not give accurate numerical approximation of the critical load parameter. In this case one has to solve quadratic approximation of the eigenvalue problem:

$$\hat{P}_{\boldsymbol{qq}}\, \boldsymbol{\phi}_1 = \left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1 + \frac{1}{2}\Delta\lambda^2\, \boldsymbol{K}^{(0)}_2 \right) \boldsymbol{\phi}_1 + \mathscr{O}(\Delta\lambda^3) = 0 \quad (24)$$

where $\boldsymbol{K}^{(0)}_2 = \left(P^{(0)}_{\boldsymbol{qqqq}}\, \dot{\boldsymbol{q}}^{(0)} + \dot{P}^{(0)}_{\boldsymbol{qqq}} \right) \dot{\boldsymbol{q}}^{(0)} + P^{(0)}_{\boldsymbol{qqq}}\, \ddot{\boldsymbol{q}}^{(0)} + \ddot{P}^{(0)}_{\boldsymbol{qq}}$

4.1 Application to Example Case

Apply now the previously described numerical scheme to our example case presented in the asymptotic analysis part, where the potential energy is given by:

$$P(\boldsymbol{q}, \lambda) = \frac{1}{2}\alpha_1\, q_1^2 + \frac{1}{2}\alpha_2\, q_2^2 + \frac{1}{2}\alpha_3\, q_3^2 + \frac{1}{2}\alpha_6\, Q_6^2(\boldsymbol{q}) - \lambda\, q_3 \quad (25)$$

Compute the necessary components:

$$\dot{\boldsymbol{q}}^{(0)} = \left(\begin{array}{cc|c} \alpha_1^{-1} & 0 & 0 \\ 0 & \alpha_2^{-1} & 0 \\ \hline 0 & 0 & (\alpha_3 + \bar{\alpha}_6)^{-1} \end{array} \right) \left(\begin{array}{c} 0 \\ 0 \\ \hline -1 \end{array} \right) \Delta\lambda = \left(\begin{array}{c} 0 \\ 0 \\ \hline -(\alpha_3 + \bar{\alpha}_6)^{-1} \end{array} \right) \Delta\lambda \quad (26)$$

$$\boldsymbol{K}^{(0)}_0 = \left(\begin{array}{cc|c} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ \hline 0 & 0 & (\alpha_3 + \bar{\alpha}_6) \end{array} \right) \quad (27)$$

$$\boldsymbol{K}^{(0)}_1 = -\frac{\bar{\alpha}_6}{\alpha_3 + \bar{\alpha}_6} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \quad (28)$$

Hence $\det(\left(\boldsymbol{K}^{(0)}_0 + \Delta\lambda\, \boldsymbol{K}^{(0)}_1\right)(m+1..M, m+1..M)) = \alpha_1\alpha_2 - \Delta\lambda^2\bar{\alpha}_6^2(\alpha_3 + \bar{\alpha}_6)^{-2} = 0$ Solving $\Delta\lambda$ yields

$$\Delta\lambda = \frac{\alpha_3 + \bar{\alpha}_6}{\bar{\alpha}_6}\sqrt{\alpha_1\alpha_2} \quad (29)$$

Now apply this scheme to the restricted scheme defined by the portential energy expression:

$$P(\boldsymbol{q}, \lambda) = \frac{1}{2}\alpha_1\, q_1^2 + \frac{1}{2}\alpha_3\, q_3^2 + \frac{1}{2}\alpha_6\, Q_6^2(\boldsymbol{q}) - \lambda\, q_3 \quad (30)$$

Compute the necessary components:

$$\dot{\boldsymbol{q}}^{(0)} = \left(\begin{array}{c|c} \alpha_1^{-1} & 0 \\ \hline 0 & (\alpha_3 + \bar{\alpha}_6)^{-1} \end{array}\right) \left(\begin{array}{c} 0 \\ \hline -1 \end{array}\right) \Delta\lambda = \left(\begin{array}{c} 0 \\ \hline -(\alpha_3 + \bar{\alpha}_6)^{-1} \end{array}\right) \Delta\lambda \tag{31}$$

$$\boldsymbol{K}_0^{(0)} = \left(\begin{array}{c|c} \alpha_1 & 0 \\ \hline 0 & (\alpha_3 + \bar{\alpha}_6) \end{array}\right) \tag{32}$$

$$\boldsymbol{K}_1^{(0)} = -\frac{\bar{\alpha}_6}{\alpha_3 + \bar{\alpha}_6} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right) \tag{33}$$

We see that $\boldsymbol{K}_1^{(0)}$ is degenerated and it is not possible to compute $\Delta\lambda$ from the linearized buckling eigenvalue problem.

5 CONCLUSIONS

In this paper we have compared two closely related systems based on a truss beam model: a non-restricted system and a restricted one. Asymptotic analysis showed us that although the critical load for the non-restricted system is less than for the restricted one, the post-buckling path is quite stable. On the other hand, even if the critical load of the restricted system is high, the post-buckling path is very unstable. This instability grows with the critical load. In numerical analysis we pointed out that due to the block diagonal form of the linearized eigenvalue problem, degeneration of the first order coefficient leads to wrong estimate of the critical load. In the particular case of our non-restricted system the first order coefficient is not degenerated, and the linearized eigenvalue problem yields correct estimate of the critical load. However, in the restricted case, the first order coefficient degenerates, and we are unable to get any estimate from the linearized eigenvalue

6 NOTATIONS, DEFINITIONS

$F : \mathbb{R}^N \to \mathbb{R}$, $F_{\boldsymbol{x}\ldots\boldsymbol{x}} \in \mathbb{R}^{N^k}$:	$(F_{\boldsymbol{x}\ldots\boldsymbol{x}})_{i_1,\ldots,i_k} := \partial_1 \ldots \partial_k\, F$	k-th derivative of F
$F : \mathbb{R}^{(N+1)} \to \mathbb{R}$:	$\dot{F}(\boldsymbol{x}, \lambda) := \frac{\partial}{\partial\lambda}\, F(\boldsymbol{x}, \lambda)$	derivative w/r to λ
$F : \mathbb{R}^N \to \mathbb{R}$:	$\hat{F} := F(\hat{\boldsymbol{x}})$	value of F at $\boldsymbol{x} = \hat{\boldsymbol{x}}$
$\boldsymbol{A} \in \mathbb{R}^{m\times n}$:	$\boldsymbol{A}(p..q, r..s)_{ij} = \boldsymbol{A}_{ij}$	
	$p \le i \le q$, $r \le j \le s$	submatrix

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