ON THE DIRECT COMPUTATION OF CRITICAL EQUILIBRIUM STATES IN SOLID AND STRUCTURAL MECHANICS

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ABSTRACT

Determination of critical points along an equilibrium path of a structure is a non-linear eigenvalue problem. If the external load is parametrized by a single parameter such an eigenvalue problem consist of solving the critical state variables, the eigenmode and the critical load parameter from the equation system consisting of the equilibrium equations, the criticality condition and some auxiliary conditions depending what kind of a critical point is in question. The non-linear eigenvalue problem can be solved by using the Newton's method. To obtain the Jacobian of the augmented equation system, second order derivatives of the residual vector with respect to the state variables need to be evaluated. Since correct computation of the critical points require geometrically exact kinematics, such derivatives are tedious to obtain. In this paper, explicit formulas for those second order derivatives of a 3D geometrically exact beam element based on the Reissner's model are presented. Additionally, some alternative iterative schemes which avoid the formation of the second order derivatives are presented and compared to the full Newton's method.

1 INTRODUCTION

An important task in computational structural analysis is the determination of the critical loads. If the pre-buckling displacements are small, it is usually sufficient to solve the linearized eigenvalue problem, where the linearization is performed with respect to the undeformed configuration. However, in many cases the pre-buckling deflections have considerable effect on the buckling behaviour and cannot be neglected.

One way to compute the critical points is to use a path-following method to solve the non-linear equilibrium equations and as a byproduct detect the change in stability properties along the traced path. Such an an indirect method is considered e.g. in [1, 2, 3, 4]. Also solving a polynomial eigenvalue problem at a certain equilibrium state is an appealing approach, [5, 6].

A direct method to compute the critical point is so constituted that the equilibrium equations are augmented with a criticality condition. The criticality condition has in some sense to state the vanishing stiffness of the stucture and can be either a scalar equation [7, 8] or an eigenvalue problem [10, 9, 11, 13, 20]

The idea of augmenting the equilibrium equations with the criticality condition appears to be due to Keener and Keller [10], presented as early as in 1973. Most papers found in literature deal only with simple critical points, and the extension to multiple bifurcations, see Keener [14], will not be considered in this paper.

Direct solution of the critical points along the equilibrium path requires complete kinematical description of the underlying mechanical model. In particular, for dimensionally reduced models, like beam- and shell models, the description has to be capable to handle large rotations. Development of a geometrically exact model with large rotations is not a trivial task [21, 22, 23].

2 DETERMINATION OF CRITICAL POINTS

2.1 Non-linear eigenvalue problem

A critical point along an equilibrium path can be determined by solving the non-linear eigenvalue problem: find the critical value of $\mathbf{q}$, λ and the corresponding eigenvector $\mathbf{v}$ such that

$$\left\{ \begin{array}{rcl} \mathbf{f}(\mathbf{q},\lambda) & = & \mathbf{0} \\ \mathbf{f}'(\mathbf{q},\lambda)\boldsymbol{\phi} & = & \mathbf{0} \end{array} \right. \tag{1}$$

where $\mathbf{f}$ is the vector of unbalanced forces and $\mathbf{f}'$ denotes the Gateaux derivative (Jacobian matrix) with respect to the state variables $\mathbf{q}$. Equation $(1)_1$ is the equilibrium equation, which has to be satisfied at the critical point, and equation $(1)_2$ states the zero stiffness in the direction of the critical eigenmode ϕ, which is the actual criticality condition. Such a system is considered in Refs. [9], [12], [20]. Abbot [7] considers a different extended system where the criticality is identified by means of the determinant of the tangent stiffness matrix. The drawback of this procedure is that the directional derivative of the determinant is difficult to compute.

The equilibrium equation $(1)_1$ constitutes the balance of internal forces $\mathbf{r}$ and external loads $\mathbf{p}$, which is usually parametrized by a single variable λ, the load parameter, defining the intensity of the load vector:

$$\mathbf{f}(\mathbf{q},\lambda) \equiv \mathbf{r}(\mathbf{q}) - \lambda \mathbf{p}_{\mathrm{r}}(\mathbf{q}). \tag{2}$$

If the loads does not dependent on deformations, like in dead-weight loading, the reference load vector $\mathbf{p}_{\mathrm{r}}$ is independent of the displacement field $\mathbf{q}$.

The system (1) consists of $2N+1$ unknowns, the displacement vector $\mathbf{q}$, the eigenmode $\mathbf{v}$ and the load parameter value λ at the critical state. Since the eigenvector $\mathbf{v}$ is defined uniquely up to a constant, the normalizing condition can be added to the system (1):

$$\mathbf{g}(\mathbf{q},\boldsymbol{\phi},\lambda) = \left\{ \begin{array}{c} \mathbf{f}(\mathbf{q},\lambda) \\ \mathbf{f}'(\mathbf{q},\lambda)\boldsymbol{\phi} \\ N(\boldsymbol{\phi},\lambda) \end{array} \right\} = \mathbf{0}, \tag{3}$$

where the Jacobian matrix $\mathbf{f}' = \partial \mathbf{f}/\partial \mathbf{q}$ is usually denoted by $\mathbf{K}$ in structural applications, and $N(\boldsymbol{\phi})$ defines some normalizing condition to the eigenvector. The Jacobian matrix of the extended system (3) has the form

$$\mathbf{J} = \left[\begin{array}{ccc} \mathbf{K} & \mathbf{0} & -\mathbf{p}_r \\ \mathbf{Z} & \mathbf{K} & \mathbf{z} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{n}^{\mathrm{T}} & a \end{array} \right], \tag{4}$$

where

$$\mathbf{Z} = \left[\mathbf{f}'\boldsymbol{\phi}\right]', \quad \mathbf{z} = \partial(\mathbf{f}'\boldsymbol{\phi})/\partial\lambda, \quad \mathbf{n}^{\mathrm{T}} = \partial N/\partial\boldsymbol{\phi}, \quad \text{and } a = \partial N \partial\lambda. \tag{5}$$

One problem in using Newton's method to the system (3) is the computation of the matrix $\mathbf{Z}$ and vector $\mathbf{z}$. Finite difference approximations are usually used [15, 16, 8, 17, 18, 19, 20].

In order to obtain asymptotically quadratic convergence for the Newton's iteration, the Jacobian should be non-singular at the solution point. For Jacobian of the form (4) this is true only Therefore the use of system (3) is efficient for the computation of limit points only [12]. However, it has been used also to compute bifurcation points in Refs. [9].

Wriggers and Simo [20] regularized the system (3) with penalty terms by appending the constraint $\mathbf{e}_i^T\mathbf{q} = \mu$ to it

$$\mathbf{g}(\mathbf{q}, \phi, \lambda, \mu) = \left\{ \begin{array}{c} \mathbf{f}(\mathbf{q}, \lambda) + \gamma(\mathbf{e}_i^{\mathrm{T}}\mathbf{q} - \mu)\mathbf{e}_i \\ \mathbf{f}'(\mathbf{q}, \lambda)\phi + \gamma(\mathbf{e}_i^{\mathrm{T}}\phi - \phi_0)\mathbf{e}_i \\ \mathbf{e}_i^{\mathrm{T}}\phi - \phi_0 \\ \mathbf{e}_i^{\mathrm{T}}\mathbf{q} - \mu \end{array} \right\} = \mathbf{0}, \tag{6}$$

where γ is the non-negative regularizing penalty parameter and $\mathbf{e}_i$ is a unit vector having the unit value at position i corresponding to the smallest diagonal entry of the factorized tangent stiffness matrix.

2.2 Polynomial eigenvalue problem

Assuming an equilibrium state $(\mathbf{q}_*, \lambda_*)$ with a regular tangent matrix, a Taylor expansion of the non-linear eigenvalue problem (1) with respect to the load parameter λ has the form

$$\mathbf{q} = \mathbf{q}_* + \Delta\lambda\mathbf{q}_1 + \tfrac{1}{2}(\Delta\lambda)^2\mathbf{q}_2 + \cdots, \tag{7}$$

$$\mathbf{f} = \mathbf{f}_* + \Delta\lambda \left.\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\lambda}\right|_* + \frac{1}{2}(\Delta\lambda)^2 \left.\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\lambda^2}\right|_* + \cdots = \mathbf{0} \tag{8}$$

$$\left(\mathbf{f}'_* + \Delta\lambda \left.\frac{\mathrm{d}\mathbf{f}'}{\mathrm{d}\lambda}\right|_* + \frac{1}{2}(\Delta\lambda)^2 \left.\frac{\mathrm{d}^2\mathbf{f}'}{\mathrm{d}\lambda^2}\right|_* + \cdots\right)\phi = \mathbf{0} \tag{9}$$

where $\Delta\lambda = \lambda - \lambda_*$. Expressions for the derivatives are [1]

$$\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\lambda} = \frac{\partial\mathbf{f}}{\partial\mathbf{q}}\frac{\partial\mathbf{q}}{\partial\lambda} + \frac{\partial\mathbf{f}}{\partial\lambda} = \mathbf{f}'\dot{\mathbf{q}} + \dot{\mathbf{f}}, \tag{10}$$

$$\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\lambda^2} = \mathbf{f}'\ddot{\mathbf{q}} + \mathbf{f}''\dot{\mathbf{q}}\dot{\mathbf{q}} + 2\dot{\mathbf{f}}'\dot{\mathbf{q}} + \ddot{\mathbf{f}}, \tag{11}$$

$$\frac{\mathrm{d}\mathbf{f}'}{\mathrm{d}\lambda} = \mathbf{f}''\dot{\mathbf{q}} + \dot{\mathbf{f}}', \tag{12}$$

$$\frac{\mathrm{d}^2\mathbf{f}'}{\mathrm{d}\lambda^2} = \mathbf{f}''\ddot{\mathbf{q}} + \mathbf{f}'''\dot{\mathbf{q}}\dot{\mathbf{q}} + 2\dot{\mathbf{f}}''\dot{\mathbf{q}} + \ddot{\mathbf{f}}'. \tag{13}$$

Evaluating these quantities at the equilibrium state $(\mathbf{q}_*, \lambda_*)$, gives

$$\dot{\mathbf{q}}_* = \mathbf{q}_1, \qquad \text{and} \qquad \ddot{\mathbf{q}}_* = \mathbf{q}_2, \qquad \text{etc..} \tag{14}$$

and the expressions (10)-(13) result in

$$\left.\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\lambda}\right|_* = \mathbf{f}'_*\mathbf{q}_1 + \dot{\mathbf{f}}_*, \qquad \left.\frac{\mathrm{d}\mathbf{f}'}{\mathrm{d}\lambda}\right|_* = \mathbf{f}''_*\mathbf{q}_1 + \dot{\mathbf{f}}'_*, \tag{15}$$

$$\left.\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\lambda^2}\right|_* = \mathbf{f}'_*\mathbf{q}_2 + \mathbf{f}''_*\mathbf{q}_1\mathbf{q}_1 + 2\dot{\mathbf{f}}'_*\mathbf{q}_1 + \ddot{\mathbf{f}}_*, \qquad \left.\frac{\mathrm{d}^2\mathbf{f}'}{\mathrm{d}\lambda^2}\right|_* = \mathbf{f}''_*\mathbf{q}_2 + \mathbf{f}'''_*\mathbf{q}_1\mathbf{q}_1 + 2\dot{\mathbf{f}}''_*\mathbf{q}_1 + \ddot{\mathbf{f}}'_*, \tag{16}$$

[1]Notice the difference between derivatives $\mathrm{d}\mathbf{f}/\mathrm{d}\lambda$ and $\dot{\mathbf{f}} = \partial\mathbf{f}/\partial\lambda$, i.e. $\mathrm{d}\mathbf{f}/\mathrm{d}\lambda = \mathbf{f}'(\partial\mathbf{q}/\partial\lambda) + \partial\mathbf{f}/\partial\lambda$.

where $\mathbf{f}_* = \mathbf{f}(\mathbf{q}_*, \lambda_*)$ etc. In the expansion of the equilibrium equations (8) all terms $\mathrm{d}^p\mathbf{f}/\mathrm{d}\lambda^p, p = 1, 2, \ldots$ has to vanish, thus giving the equation to solve the fields $\mathbf{q}_i$

$$\mathbf{f}'_*\mathbf{q}_1 = -\dot{\mathbf{f}}_*, \qquad \mathbf{f}'_*\mathbf{q}_2 = -\left[\mathbf{f}''_*\mathbf{q}_1\mathbf{q}_1 + 2\dot{\mathbf{f}}'_*\mathbf{q}_1 + \ddot{\mathbf{f}}_*\right], \ldots \quad \text{etc.} \tag{17}$$

It is worthwhile to notice that the coefficent matrix to solve $\mathbf{q}_1, \mathbf{q}_2 \ldots$ is the same for all cases. In structural mechanics, the symbol $\mathbf{K}$ is usually used to denote the stiffness matrix, thus the matrices in (9) can be written as

$$\mathbf{K}_{0|*} = \mathbf{f}'_*, \qquad \mathbf{K}_{1|*} = \left.\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\lambda}\right|_* = \mathbf{f}''_*\mathbf{q}_1 + \dot{\mathbf{f}}'_*, \tag{18}$$

$$\mathbf{K}_{2|*} = \left.\frac{\mathrm{d}^2\mathbf{f}}{\mathrm{d}\lambda^2}\right|_* = \mathbf{f}''_*\mathbf{q}_2 + \mathbf{f}'''_*\mathbf{q}_1\mathbf{q}_1 + 2\dot{\mathbf{f}}''_*\mathbf{q}_1 + \ddot{\mathbf{f}}'_*, \tag{19}$$

and the polynomial eigenvalue problem can be written as

$$\left(\mathbf{K}_{0|*} + \Delta\lambda\mathbf{K}_{1|*} + \tfrac{1}{2}(\Delta\lambda)^2\mathbf{K}_{2|*} + \cdots\right)\phi = \mathbf{0}, \tag{20}$$

In the classical linear stability analysis the reference state is the undeformed stress free configuration. Assuming also dead weight loading, i.e. $\dot{\mathbf{f}}' \equiv \mathbf{0}$, the matrices for the linear stability eigenvalue problem are simply the following:

$$\mathbf{K}_{0|0} = \mathbf{f}'(\mathbf{0}, 0) \tag{21}$$

$$\mathbf{K}_{1|0} = \mathbf{f}''(\mathbf{0}, 0)\mathbf{q}_1, \tag{22}$$

where $\mathbf{K}_{0|0}\mathbf{q}_1 = \mathbf{p}_r$. Therefore the strains are linear functions of the displacements $\mathbf{q}_1$ and the geometric stiffness matrix $\mathbf{K}_{1|0}$ is a linear function of the displacements $\mathbf{q}_1$.

It is seen from the definition of the $\mathbf{K}_{1|0}$ matrix that the "initial stress" state to the linear eigenvalue problem has to be linear with respect to the load parameter change. This is not true if the linear stability eigenvalue problem is solved from

$$\left(\mathbf{K}_{0|*} + s(\mathbf{K}_{0|*} - \mathbf{K}_{0|**})\right)\phi = \mathbf{0}, \tag{23}$$

where $\mathbf{K}_{0|*}$ and $\mathbf{K}_{0|**}$ are the tangent stiffness matrices from two consecutive equilibrium states. It will be a correct approximation to the linear eigenvalue problem only if the load increment $\Delta\lambda = \lambda_* - \lambda_{**}$ is small, i.e. $\mathbf{K}_{1|*} \approx (\Delta\lambda)^{-1}(\mathbf{K}_{0|*} - \mathbf{K}_{0|**})$.

2.3 Some computational issues

One bottleneck in applying the Newton's method to the extended system (3) or (6) is the need to compute the matrix $\mathbf{Z}$ or the vector $\mathbf{z}$ which requires evaluation of the second derivatives of the residual vector with respect to the state variables. If the iteration is started at the undeformed configuration, i.e. at $(\mathbf{q} = \mathbf{0}, \lambda = 0)$ the $\mathbf{Z}$ matrix is simply the initial stress or geometric stiffness matrix formed from the linearized stresses evaluated from a displacement field which is the initial guess of the eigenvector ϕ. This can be seen by comparing equations (5) and (22). A modified Newton scheme, where the $\mathbf{Z}$ matrix is not updated will completely avoid the evaluation of the second derivative terms. However, the rate of convergence can be slow. The vector $\mathbf{z}$ is simply the load stiffness matrix multiplied with the eigenmode approximation.

If direct solvers for the solution of the linearized system is used, a block elimination scheme is a feasible choise. Denoting the extended system (3) as

$$\begin{cases} \mathbf{f}(\mathbf{q},\lambda) = \mathbf{0} \\ \mathbf{h}(\mathbf{q},\boldsymbol{\phi},\lambda) \equiv \mathbf{f}'(\mathbf{q},\lambda)\boldsymbol{\phi} = \mathbf{0} \\ N(\boldsymbol{\phi},\lambda) = 0, \end{cases} \tag{24}$$

the Newton step can be written as

$$\begin{cases} \mathbf{K}\delta\mathbf{q} - \mathbf{p}_{\mathrm{r}}\delta\lambda = -\mathbf{f} \\ \mathbf{Z}\delta\mathbf{q} + \mathbf{K}\delta\boldsymbol{\phi} + \mathbf{z}\delta\lambda = -\mathbf{h} \\ \mathbf{n}^{\mathrm{T}}\delta\boldsymbol{\phi} + a\delta\lambda = -N. \end{cases} \tag{25}$$

Partitioning the iterative steps $\delta\mathbf{q}$ and $\delta\boldsymbol{\phi}$ as

$$\delta\mathbf{q} = \mathbf{q}_f + \mathbf{q}_p\delta\lambda, \quad \delta\boldsymbol{\phi} = \boldsymbol{\phi}_h + \boldsymbol{\phi}_z\delta\lambda, \tag{26}$$

where the vectors $\mathbf{q}_f, \mathbf{q}_p, \boldsymbol{\phi}_h$ and $\boldsymbol{\phi}_z$ can be solved from equations

$$\mathbf{K}\mathbf{q}_f = -\mathbf{f}, \quad \mathbf{K}\mathbf{q}_p = \mathbf{p}_{\mathrm{r}}, \quad \mathbf{K}\boldsymbol{\phi}_h = -\mathbf{h} - \mathbf{Z}\mathbf{q}_f, \quad \mathbf{K}\boldsymbol{\phi}_z = -\mathbf{z} - \mathbf{Z}\mathbf{q}_p. \tag{27}$$

The iterative change in the load parameter can be solved from the normalizing condition resulting in

$$\delta\lambda = -\frac{N + \mathbf{n}^{\mathrm{T}}\boldsymbol{\phi}_h}{a + \mathbf{n}^{\mathrm{T}}\boldsymbol{\phi}_z}. \tag{28}$$

3 GEOMETRICALLY EXACT BEAM MODEL

3.1 Description of rotation

A rotation motion can be represented by rotation operators $\mathbf{R}$ forming a special noncommutative Lie-group of the proper orthogonal linear transformations, which is defined as

$$SO(3) := \left\{ \mathbf{R} : \mathsf{E}^3 \to \mathsf{E}^3 \,\middle|\, \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I},\ \det\mathbf{R} = +1 \right\}, \tag{29}$$

where E^3 denotes the 3-dimensional Euclidean vector space. The rotation tensor can be represented minimally by three parameters, which parametrize rotation tensor only locally. It is well known that there exist no a single three-parametric global presentation of rotation tensor because the rotation group is a compact group. The rotation operator $\mathbf{R}$ can be written in terms of the rotation vector that is defined by $\boldsymbol{\Psi} := \psi\mathbf{n},\ \mathbf{n} \in \mathsf{E}^3, \psi \in \mathsf{R}_+$. This yields the expression of the rotation operator

$$\mathbf{R} := \mathbf{I} + \frac{\sin\psi}{\psi}\tilde{\boldsymbol{\Psi}} + \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\Psi}}^2 = \exp(\tilde{\boldsymbol{\Psi}}), \quad \psi = \|\boldsymbol{\Psi}\|, \tag{30}$$

where the skew-symmetric rotation tensor $\tilde{\boldsymbol{\Psi}}$, is defined formally $\tilde{\boldsymbol{\Psi}} := \tilde{\boldsymbol{\Psi}}\times$. Compound rotation can be defined by two different, nevertheless, equivalent ways: the material description, and the spatial description. The material description of a compound rotation is defined as

$$\mathbf{R}\mathbf{R}_{\mathrm{inc}}^{\mathrm{mat}} = \mathbf{R}\exp(\tilde{\boldsymbol{\Theta}}) \quad \mathbf{R}_{\mathrm{inc}}^{\mathrm{mat}}, \mathbf{R} \in SO(3), \tag{31}$$

where $\mathbf{R}_{\mathrm{inc}}^{\mathrm{mat}}$ is an incremental material rotation operator, and $\boldsymbol{\Theta}$ is an incremental material rotation vector with respect to the base point $\mathbf{R} \in SO(3)$. This description is called material, since the incremental rotation operator acts on a material vector space. Differentiating the material expression of the compound rotation $\mathbf{R}\exp(\eta\,\tilde{\boldsymbol{\Theta}})$ with respect to the parameter η and setting $\eta = 0$, yields the material tangent space at the base point $\mathbf{R} \in SO(3)$, defined as

$$ {}_{\mathrm{mat}}T_{\mathbf{R}}SO(3) := \{\tilde{\boldsymbol{\Theta}}_{\mathbf{R}} := (\mathbf{R}, \tilde{\boldsymbol{\Theta}}) \mid \text{for any } \ \tilde{\boldsymbol{\Theta}} \in so(3)\}, \tag{32} $$

where the skew-symmetric tensor $\tilde{\boldsymbol{\Theta}}_{\mathbf{R}} \in {}_{\mathrm{mat}}T_{\mathbf{R}}SO(3)$ is an element of the material tangent space. The notation $(\mathbf{R},\,\tilde{\boldsymbol{\Theta}})$, the pair of the rotation operator $\mathbf{R}$ and the skew-symmetric tensor $\tilde{\boldsymbol{\Theta}}$, represents the material skew-symmetric tensor at the base point $\mathbf{R} \in SO(3)$. Considering the material form of a compound rotation (31), with the aid of η-parametrized exponential mappings

$$ \exp(\tilde{\boldsymbol{\Psi}} + \eta\,\delta\tilde{\boldsymbol{\Psi}}) = \exp(\tilde{\boldsymbol{\Theta}})\exp(\eta\,\delta\tilde{\boldsymbol{\Theta}}_{\mathbf{R}}) \tag{33} $$

where we are finding an incremental rotation tensor, the virtual rotation tensor$\delta\tilde{\boldsymbol{\Psi}}$, such that it belongs to the same tangent space as the rotation tensor $\tilde{\boldsymbol{\Psi}}$, i.e. such that $\delta\tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{\Psi}} \in {}_{\mathrm{mat}}T_{\mathbf{I}}SO(3)$ with the identity as a base point omitted for simplicity. Taking the derivative of (33) with respect to the parameter η at $\eta = 0$ gives

$$ \delta\,\tilde{\boldsymbol{\Theta}}_{\mathbf{R}} = \mathbf{T}\cdot\delta\,\tilde{\boldsymbol{\Theta}} \tag{34} $$

$$ \mathbf{T} := \frac{\sin\psi}{\psi}\mathbf{I} - \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\Psi}} + \frac{\psi-\sin\psi}{\psi^3}\tilde{\boldsymbol{\Psi}}\otimes\ \tilde{\boldsymbol{\Psi}} \tag{35} $$

$$ \psi := \|\tilde{\boldsymbol{\Psi}}\|, \quad \mathbf{R} = \exp(\tilde{\boldsymbol{\Psi}}), \quad \lim_{\tilde{\boldsymbol{\Psi}}\to\mathbf{0}} \mathbf{T}(\tilde{\boldsymbol{\Psi}}) = \mathbf{I} \tag{36} $$

where the material tangential transformation $\mathbf{T} = \mathbf{T}(\boldsymbol{\Psi})$ is a linear mapping between the virtual material tangent spaces: ${}_{\mathrm{mat}}T_{\mathbf{I}}SO(3) \ \rightarrow\ {}_{\mathrm{mat}}T_{\mathbf{R}}SO(3)$. The virtual incremental rotation vector $\delta\boldsymbol{\Theta}_{\mathbf{R}}$ and the virtual total rotation vector $\delta\boldsymbol{\Psi}$ belong to different vector spaces on the manifold, since the tangential transformation $\mathbf{T}$ is equal to the identity only at $\boldsymbol{\Psi} = \mathbf{0}$. Note that the transformation $\mathbf{T}$ has an effect on the base points, changing the base point $\mathbf{I}$ into $\mathbf{R}$. For convenience, a material vector space on the rotation manifold at any point $\mathbf{R}$ is defined as

$$ {}_{\mathrm{mat}}T_{\mathbf{R}} := \left\{\boldsymbol{\Theta}_{\mathbf{R}} \in \mathsf{E}^3 \,\middle|\, \tilde{\boldsymbol{\Theta}}_{\mathbf{R}} \in {}_{\mathrm{mat}}T_{\mathbf{R}}SO(3)\right\} \tag{37} $$

where an element of the material vector space is $\boldsymbol{\Theta}_{\mathbf{R}} \in {}_{\mathrm{mat}}T_{\mathbf{R}}$, which is an affine space with the rotation vector $\boldsymbol{\Psi}$ as a base point and the incremental rotation vector $\boldsymbol{\Theta}$ as a tangent vector, then $\mathbf{T} : {}_{\mathrm{mat}}T_{\mathbf{I}} \rightarrow {}_{\mathrm{mat}}T_{\mathbf{R}}$. Definition (37) gives a practical notation for sorting rotation vectors in different tangent spaces.

3.2 Virtual work expression for the Reissner's beam model

In the material representation, the internal virtual work expression for the Reissner's beam model is

$$ \delta W_{\mathrm{int}} = \int_L (\delta\boldsymbol{\Gamma}\cdot\mathbf{N} + \delta\mathbf{K}_{\mathbf{R}}\cdot\mathbf{M}_{\mathbf{R}})\,\mathrm{d}s, \tag{38} $$

where the material curvature tensor is defined by $\tilde{\mathbf{K}}_{\mathbf{R}} := \mathbf{R}^{\mathrm{T}}\mathbf{R}'$, and the prime denotes derivative with respect to the coordinate s along the beam's axis. The material internal force vector $\mathbf{N}$ and the material internal moment vector $\mathbf{M}_{\mathbf{R}}$ are related to the material strain and curvature vectors by a linear constitutive law

$$ \mathbf{N} = \mathbf{C}_{\mathrm{N}}\boldsymbol{\Gamma}, \quad \mathbf{M}_{\mathbf{R}} = \mathbf{C}_{\mathrm{M}}\mathbf{K}_{\mathbf{R}}. \tag{39} $$

The work conjugate of the material vector $\mathbf{N}$ is the variation of the material strain vector $\mathbf{\Gamma}$, defined by the formula and its variation:

$$\mathbf{\Gamma} := \mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}} - \mathbf{E}_1, \quad \delta\mathbf{\Gamma} = \mathbf{R}^{\mathrm{T}}\delta\mathbf{x}'_{\mathrm{c}} - \delta\tilde{\mathbf{\Theta}}\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}. \tag{40}$$

In the total Lagrangian formulation, the virtual work expression has to be written in terms of the total material rotation vector $\mathbf{\Psi}$ and its virtual counterpart $\delta\mathbf{\Psi}$. The virtual work of the internal forces (38) has the form

$$\delta W_{\mathrm{int}} = \int_L (\delta\mathbf{x}'_{\mathrm{c}} \cdot \mathbf{R}\mathbf{N})\,\mathrm{d}s + \int_L (\delta\mathbf{\Psi} \cdot (-\mathbf{T}^{\mathrm{T}}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{N} + \mathbf{C}_1^{\mathrm{T}}(\mathbf{\Psi}', \mathbf{\Psi})\,\mathbf{M}_{\mathbf{R}}) + \delta\mathbf{\Psi}' \cdot \mathbf{T}^{\mathrm{T}}\mathbf{M}_{\mathbf{R}})\,\mathrm{d}s, \tag{41}$$

where the tensor $\mathbf{C}_1$ is given in the Appendix. The internal virtual work can be written compactly as

$$\delta W_{\mathrm{int}} = \int_L \delta\hat{\mathbf{q}} \cdot \mathbf{B}^{\mathrm{T}}\mathbf{F}_{\mathrm{int}}\,\mathrm{d}s, \tag{42}$$

where $\delta\hat{\mathbf{q}} := (\delta\mathbf{x}'_{\mathrm{c}}, \delta\mathbf{\Psi}', \delta\mathbf{\Psi})$. The generalized internal force field $\mathbf{F}_{\mathrm{int}}$ and the kinematic tensor $\mathbf{B}$ are

$$\mathbf{F}_{\mathrm{int}} := \begin{pmatrix} \mathbf{N} \\ \mathbf{M}_{\mathbf{R}} \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} \mathbf{R}^{\mathrm{T}} & \mathbf{O} & \widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T} \\ \mathbf{O} & \mathbf{T} & \mathbf{C}_1(\mathbf{\Psi}', \mathbf{\Psi}) \end{pmatrix}. \tag{43}$$

Moreover, the field $\mathbf{F}_{\mathrm{int}}$ can be given in the term of the material strain and curvature vectors with the aid of the constitutive relations (39). Linearizing the internal work form δW_{int} at the point $\mathbf{q}_0^{\mathrm{T}} = (\mathbf{d}_0^{\mathrm{T}}, \mathbf{\Psi}_0^{\mathrm{T}})$ in the vector direction $\Delta\hat{\mathbf{q}}^{\mathrm{T}} = (\Delta\mathbf{x}'^{\mathrm{T}}_{\mathrm{c}}, \Delta\mathbf{\Psi}'^{\mathrm{T}}, \Delta\mathbf{\Psi}^{\mathrm{T}})$ will result in the equation

$$\mathrm{Lin}\,(\delta W_{\mathrm{int}}(\mathbf{q}; \delta\hat{\mathbf{q}})) = \delta W_{\mathrm{int}}(\mathbf{q}_0; \delta\hat{\mathbf{q}}) + \mathrm{D}_{\mathbf{q}}\delta W_{\mathrm{ext}}(\mathbf{q}_0, \delta\hat{\mathbf{q}}) \cdot \Delta\hat{\mathbf{q}}, \tag{44}$$

where the linear form $\mathrm{D}_{\hat{\mathbf{q}}}\delta W_{\mathrm{int}} \cdot \Delta\hat{\mathbf{q}}$ can be written using the material stiffness tensor $\mathbf{K}_{\mathrm{mat}}$ and the geometric stiffness tensor $\mathbf{K}_\sigma$ as

$$\mathrm{D}_{\hat{\mathbf{q}}}\delta W_{\mathrm{int}} \cdot \Delta\hat{\mathbf{q}} = \int_L (\mathbf{K}_{\mathrm{mat}} + \mathbf{K}_\sigma) : (\delta\hat{\mathbf{q}} \otimes \Delta\hat{\mathbf{q}})\,\mathrm{d}s, \quad \mathbf{K}_{\mathrm{mat}} := \mathbf{B}_0^{\mathrm{T}}\mathbf{C}_{\mathrm{NM}}\mathbf{B}_0, \tag{45}$$

$$\mathbf{K}_\sigma := \begin{pmatrix} \mathbf{O} & \mathbf{O} & -\mathbf{R}\tilde{\mathbf{N}}\mathbf{T} \\ \mathbf{O} & \mathbf{O} & \mathbf{C}_2(\mathbf{M}_{\mathbf{R}}, \mathbf{\Psi}_0) \\ \mathbf{T}^{\mathrm{T}}\tilde{\mathbf{N}}\mathbf{R}^{\mathrm{T}} & \mathbf{C}_2^{\mathrm{T}}(\mathbf{M}_{\mathbf{R}}, \mathbf{\Psi}_0) & \mathbf{K}_{\sigma 33} \end{pmatrix}, \quad \text{where}$$

$$\mathbf{K}_{\sigma 33} = \mathbf{C}_3(\mathbf{M}_{\mathbf{R}}, \mathbf{\Psi}'_0, \mathbf{\Psi}_0) + \mathbf{C}_2(\tilde{\mathbf{N}}\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}, \mathbf{\Psi}_0) + \mathbf{T}^{\mathrm{T}}\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T}. \tag{46}$$

The material stiffness tensor $\mathbf{K}_{\mathrm{mat}}$ arises from the linearization of the vector $\mathbf{F}_{\mathrm{int}}$ with the aid of kinematic relation, and the geometric stiffness tensor $\mathbf{K}_\sigma$ arises from the linearization of the kinematic operator $\mathbf{B}$. The tensors $\mathbf{C}_2$ and $\mathbf{C}_3$ are given in the Appendix. Note that the material stiffness tensor $\mathbf{K}_{\mathrm{mat}}$ is symmetric due to the symmetry of the elasticity tensor $\mathbf{C}_{\mathrm{NM}}$. In addition, the geometric stiffness $\mathbf{K}_\sigma$ is also a symmetric tensor. This symmetry of the stiffness tensor is due to the local parametrization of the rotation operator.

The $\mathbf{Z}$-matrix, needed in the Jacobian (4) of the extended system (3) can be divided as

$$\mathbf{Z} = \mathbf{Z}_{\mathrm{mat}} + \mathbf{Z}_\sigma, \tag{47}$$

where the matrices can be defined as

$$\int_L \mathbf{Z}_{\mathrm{mat}}\,\mathrm{d}s := \int_L \mathrm{D}_{\hat{\mathbf{q}}}\mathbf{K}_{\mathrm{mat}}\hat{\boldsymbol{\phi}}\,\mathrm{ds} = \int_L \mathrm{D}_{\hat{\mathbf{q}}}\mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\hat{\boldsymbol{\phi}}\,\mathrm{d}s, \quad \int_L \mathbf{Z}_\sigma\,\mathrm{d}s := \int_L \mathrm{D}_{\hat{\mathbf{q}}}\mathbf{K}_\sigma\hat{\boldsymbol{\phi}}\,\mathrm{d}s. \tag{48}$$

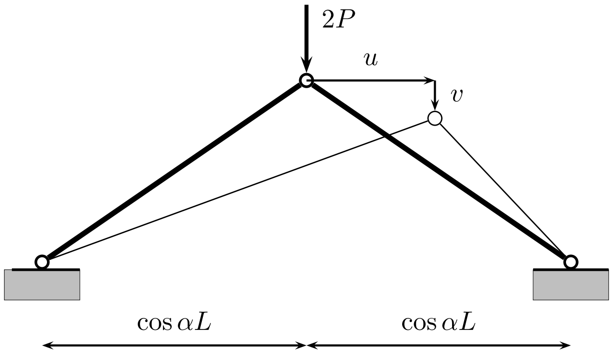

Figure 1: Von Mises truss.

Partitioning the eigenvector $\hat{\phi}$ similarly as the displacement vector $\hat{\mathbf{q}}$: $\hat{\boldsymbol{\phi}}^{\mathrm{T}} := [\boldsymbol{\phi}_1^{\mathrm{T}}, \boldsymbol{\phi}_2^{\mathrm{T}}, \boldsymbol{\phi}_3^{\mathrm{T}}] \sim \hat{\mathbf{q}}^{\mathrm{T}} := [\mathbf{x}'^{\mathrm{T}}_{\mathrm{c}}, \boldsymbol{\Psi}'^{\mathrm{T}}, \boldsymbol{\Psi}^{\mathrm{T}}]$, the matrices $\mathbf{Z}_{\mathrm{mat}}$ and $\mathbf{Z}_\sigma$ have expressions

$$
\begin{aligned}
\mathbf{Z}_{\mathrm{mat}} &= \mathbf{K}_\sigma(\mathbf{C}_{\mathrm{NM}}\mathbf{B}\hat{\boldsymbol{\phi}}) + \\
&+ \mathbf{B}^{\mathrm{T}}\mathbf{C}_{\mathrm{NM}} \begin{bmatrix} -\widetilde{\mathbf{T}\boldsymbol{\phi}_3}\mathbf{R}^{\mathrm{T}} & \mathbf{O} & \mathbf{R}^{\mathrm{T}}\widetilde{\boldsymbol{\phi}_1}\mathbf{T} + \widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{C}_1(\boldsymbol{\phi}_3, \boldsymbol{\Psi}) - \widetilde{\mathbf{T}\boldsymbol{\phi}_3}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T} \\ \mathbf{O} & \dot{\mathbf{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi}) & \mathbf{C}_1(\boldsymbol{\phi}_2, \boldsymbol{\Psi}) + \mathbf{C}_8(\boldsymbol{\phi}_3, \boldsymbol{\Psi}', \boldsymbol{\Psi}) \end{bmatrix},
\end{aligned} \tag{49}
$$

and

$$
\mathbf{Z}_\sigma = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{R}\tilde{\mathbf{N}}\widetilde{\mathbf{T}\boldsymbol{\phi}_3}\mathbf{T} - \mathbf{R}\tilde{\mathbf{N}}\mathbf{C}_1(\boldsymbol{\phi}_3, \boldsymbol{\Psi}) \\ \mathbf{O} & \mathbf{O} & \mathbf{C}_7(\boldsymbol{\phi}_3, \mathbf{M}, \boldsymbol{\Psi}) \\ \dot{\mathbf{T}}^{\mathrm{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi})\tilde{\mathbf{N}}\mathbf{R}^{\mathrm{T}} & \mathbf{C}_9(\boldsymbol{\phi}_3, \mathbf{M}, \boldsymbol{\Psi}) & \mathbf{Z}_{\sigma 33} \end{bmatrix} + \mathbf{Z}_{\mathrm{NM}}, \tag{50}
$$

where

$$
\begin{aligned}
\mathbf{Z}_{\sigma 33} &= \mathbf{T}^{\mathrm{T}}\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\boldsymbol{\phi}_1}\mathbf{T} + \mathbf{C}_1(\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\boldsymbol{\phi}_1}, \boldsymbol{\Psi}) + \mathbf{C}_3(\mathbf{M}, \boldsymbol{\phi}_2, \boldsymbol{\Psi}) + \mathbf{C}_{10}(\boldsymbol{\phi}_3, \mathbf{M}, \boldsymbol{\Psi}', \boldsymbol{\Psi}) + \dot{\mathbf{T}}^{\mathrm{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi})\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T} + \\
&+ \mathbf{C}_7(\boldsymbol{\phi}_3, \tilde{\mathbf{N}}\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}, \boldsymbol{\Psi}) + \mathbf{C}_2(\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T}\boldsymbol{\phi}_3, \boldsymbol{\Psi}) - \mathbf{T}^{\mathrm{T}}\tilde{\mathbf{N}}\widetilde{\mathbf{T}\boldsymbol{\phi}_3}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\tilde{\mathbf{N}}\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{C}_1(\boldsymbol{\phi}_3, \boldsymbol{\Psi})
\end{aligned} \tag{51}
$$

and

$$
\mathbf{Z}_{\mathrm{NM}} = \begin{bmatrix} \mathbf{R}\widetilde{\mathbf{T}\boldsymbol{\phi}_3} & \mathbf{O} \\ \mathbf{O} & \dot{\mathbf{T}}^{\mathrm{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi}) \\ \mathbf{Z}_{\mathrm{NM31}} & \dot{\mathbf{T}}^{\mathrm{T}}(\boldsymbol{\phi}_2, \boldsymbol{\Psi}) + \dot{\mathbf{C}}_1^{\mathrm{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi}', \boldsymbol{\Psi}) \end{bmatrix} \mathbf{C}_{\mathrm{NM}}\mathbf{B}, \tag{52}
$$

where

$$
\mathbf{Z}_{\mathrm{NM31}} = -\mathbf{T}^{\mathrm{T}}\widetilde{\mathbf{R}^{\mathrm{T}}\boldsymbol{\phi}_1} - \dot{\mathbf{T}}^{\mathrm{T}}(\boldsymbol{\phi}_3, \boldsymbol{\Psi})\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}} - \mathbf{T}^{\mathrm{T}}\mathrm{skew}(\widetilde{\mathbf{R}^{\mathrm{T}}\mathbf{x}'_{\mathrm{c}}}\mathbf{T}\boldsymbol{\phi}_3). \tag{53}
$$

4 NUMERICAL EXAMPLE

To demonstrate the behaviour of different scemes, a two bar truss, von Mises truss, is analysed, see fig. 1. Length and the initial angle of the bars at the initial state are L and α, respectively, the axial stiffness equals to EA, and the bars are assumed to be absolutely rigid in bending.

Using the Green-Lagrange definition for the strain and assuming linear relation between the axial force and the strain, $N_i = EA\epsilon_i$, the equilibrium equations take the form

$$
\mathbf{f}(\mathbf{q}) = \left\{ \begin{array}{ll} f_1 = 2c^2 q_1 + q_1^3 - 2sq_1q_2 + q_1q_2^2 & = 0 \\ f_2 = -sq_1^2 + q_1^2q_2 + 2s^2q_2 - 3sq_2^2 + q_2^3 - 2\lambda & = 0 \end{array} \right. , \tag{54}
$$

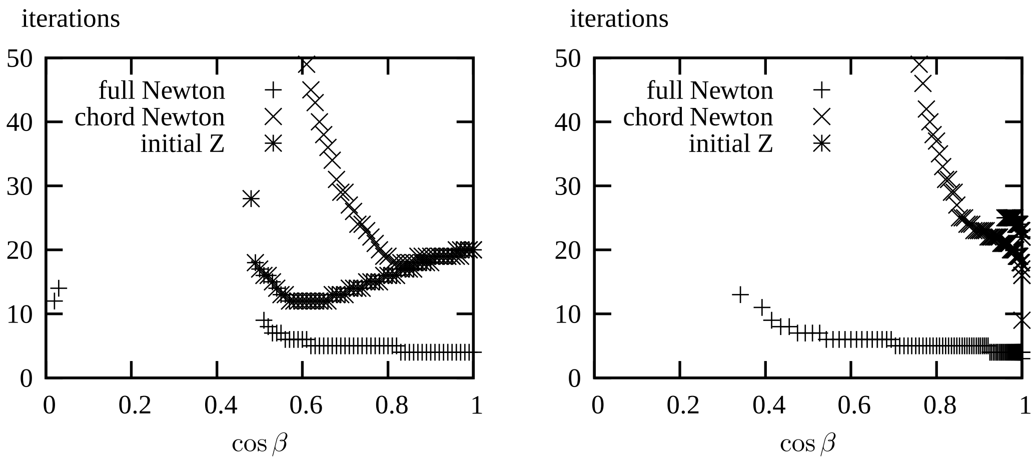

Figure 2: Number of iterations needed for convergence of the three iteration schemes with different intial values of the eigenvector: $\alpha = 30°$ limit point (l.h.s.) and $\alpha = 70°$ bifurcation.

where $\lambda = P/EA$ and the non-dimensional state variables are $q_1 = u/L, q_2 = v/L$ and $s = \sin\alpha$ and $c = \cos\alpha$. If the angle α is less than $\arctan\sqrt{2} \approx 54.75°$, the equilibrium path has a limit point at the load value $\lambda_{\mathrm{cr}} = \frac{1}{9}\sqrt{3}\sin^3\alpha$.

The effect of the initial value for the eigenvector on the convergence is shown in fig. 2. Simple vector norm condition, $N = \|\boldsymbol{\phi}\| - 1 = 0$ is used in the extended system (3). The angle between the initial eigenvector approximation and the converged eigenvector is denoted by β. It should be noted that in the chord Newton the stiffness matrix has to be evaluated at every iteration step due to the criticality condition. The strategy with constant **Z**-matrix do not work properly for the bifurcation point ($\alpha = 70°$, in fig. 2).

APPENDIX

The tensors $\mathbf{C}_i, i = 1, 2, 3$ are defined by directional derivatives

$$\mathbf{C}_1(\mathbf{V},\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}(\mathbf{T}\cdot\mathbf{V})\cdot\Delta\boldsymbol{\Psi}, \tag{55}$$

$$\mathbf{C}_2(\mathbf{V},\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}(\mathbf{T}^{\mathrm{T}}\cdot\mathbf{V})\cdot\Delta\boldsymbol{\Psi}, \tag{56}$$

$$\mathbf{C}_3(\mathbf{V},\boldsymbol{\Psi}',\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}(\mathbf{C}_1^{\mathrm{T}}(\boldsymbol{\Psi}',\boldsymbol{\Psi})\cdot\mathbf{V})\cdot\Delta\boldsymbol{\Psi}, \quad \forall\,\mathbf{V}\in E^3, \tag{57}$$

and have the following explicit expressions

$$\begin{aligned}
\mathbf{C}_1(\mathbf{V},\boldsymbol{\Psi}) &= c_1\mathbf{V}\otimes\boldsymbol{\Psi} - c_2(\tilde{\boldsymbol{\Psi}}\mathbf{V})\otimes\boldsymbol{\Psi} + c_3(\boldsymbol{\Psi}\cdot\mathbf{V})\boldsymbol{\Psi}\otimes\boldsymbol{\Psi} - c_4\tilde{\mathbf{V}} + c_5\left((\boldsymbol{\Psi}\cdot\mathbf{V})\mathbf{I} + \boldsymbol{\Psi}\otimes\mathbf{V}\right),\\
\mathbf{C}_2(\mathbf{V},\boldsymbol{\Psi}) &= c_1\mathbf{V}\otimes\boldsymbol{\Psi} + c_2(\tilde{\boldsymbol{\Psi}}\mathbf{V})\otimes\boldsymbol{\Psi} + c_3(\boldsymbol{\Psi}\cdot\mathbf{V})\boldsymbol{\Psi}\otimes\boldsymbol{\Psi} + c_4\tilde{\mathbf{V}} + c_5\left((\boldsymbol{\Psi}\cdot\mathbf{V})\mathbf{I} + \boldsymbol{\Psi}\otimes\mathbf{V}\right),\\
\mathbf{C}_3(\mathbf{V},\boldsymbol{\Psi}',\boldsymbol{\Psi}) &= \Big(c_1(\boldsymbol{\Psi}'\cdot\mathbf{V}) + c_2(\boldsymbol{\Psi}\cdot\tilde{\mathbf{V}}\boldsymbol{\Psi}') + c_3(\boldsymbol{\Psi}\cdot\boldsymbol{\Psi}')(\mathbf{V}\cdot\boldsymbol{\Psi})\Big)\,\mathbf{I} + c_2(\boldsymbol{\Psi}\underset{\mathrm{S}}{\otimes}\tilde{\mathbf{V}}\boldsymbol{\Psi}') +\\
&+ c_3(\boldsymbol{\Psi}\cdot\boldsymbol{\Psi}')(\boldsymbol{\Psi}\underset{\mathrm{S}}{\otimes}\mathbf{V}) + \frac{1}{\psi}\Big(c_1'(\boldsymbol{\Psi}'\cdot\mathbf{V}) + c_2'(\boldsymbol{\Psi}\cdot\tilde{\mathbf{V}}\boldsymbol{\Psi}') + c_3'(\boldsymbol{\Psi}\cdot\boldsymbol{\Psi}')(\mathbf{V}\cdot\boldsymbol{\Psi})\Big)(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}) +\\
&+ c_3(\mathbf{V}\cdot\boldsymbol{\Psi})(\boldsymbol{\Psi}\underset{\mathrm{S}}{\otimes}\boldsymbol{\Psi}') + c_5(\boldsymbol{\Psi}'\underset{\mathrm{S}}{\otimes}\mathbf{V}),
\end{aligned}$$

where $\underset{S}{\otimes}$ denotes the symmetric tensor product $(\mathbf{a}\underset{S}{\otimes}\mathbf{b}) := (\mathbf{a}\otimes\mathbf{b}) + (\mathbf{b}\otimes\mathbf{a}), \quad \forall \mathbf{a},\mathbf{b}\in E^3$. The coefficients c_i and their derivatives are given by

$$
\begin{aligned}
&c_1 := (\psi\cos\psi - \sin\psi)\psi^{-3}, && c_2 := (\psi\sin\psi + 2\cos\psi - 2)\psi^{-4},\\
&c_3 := (3\sin\psi - 2\psi - \psi\cos\psi)\psi^{-5}, && c_4 := (\cos\psi - 1)\psi^{-2},\\
&c_5 := (\psi - \sin\psi)\psi^{-3}, &&
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
c_1' &= (3\sin\psi - \psi^2\sin\psi - 3\psi\cos\psi)\psi^{-4},\\
c_2' &= (\psi^2\cos\psi - 5\psi\sin\psi - 8\cos\psi + 8)\psi^{-5},\\
c_3' &= (7\psi\cos\psi + 8\psi + \psi^2\sin\psi - 15\sin\psi)\psi^{-6},
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
c_1'' &= (-\psi^3\cos\psi + 5\psi^2\sin\psi + 12\psi\,\cos\psi - 12\sin\psi)\psi^{-5},\\
c_2'' &= -(\psi^3\sin\psi + 8\psi^2\cos\psi - 28\psi\sin\psi - 40\cos\psi + 40)\psi^{-6},\\
c_3'' &= (\psi^3\cos\psi - 11\psi^2\sin\psi - 50\psi\cos\psi - 40\psi + 90\sin\psi)\psi^{-7}.
\end{aligned} \tag{60}
$$

The limit processes for the tensors $\mathbf{C}_i$ give

$$\lim_{\boldsymbol{\Psi}\to\mathbf{0}} \mathbf{C}_1(\mathbf{V},\boldsymbol{\Psi}) = \tfrac{1}{2}\tilde{\mathbf{V}}, \qquad \lim_{\boldsymbol{\Psi}\to\mathbf{0}} \mathbf{C}_2(\mathbf{V},\boldsymbol{\Psi}) = -\tfrac{1}{2}\tilde{\mathbf{V}}, \tag{61}$$

$$\lim_{\boldsymbol{\Psi}\to\mathbf{0}} \mathbf{C}_3(\mathbf{V},\boldsymbol{\Psi}',\boldsymbol{\Psi}) = -\tfrac{1}{3}(\boldsymbol{\Psi}'\cdot\mathbf{V})\mathbf{I} + \tfrac{1}{6}(\boldsymbol{\Psi}'\underset{S}{\otimes}\mathbf{V}) \tag{62}$$

The time derivative of the transformation $\mathbf{T}$ is

$$
\begin{aligned}
&\dot{\mathbf{T}}(\dot{\boldsymbol{\Psi}},\boldsymbol{\Psi}) =\\
&c_1\left(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}}\right)\mathbf{I} - c_2\left(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}}\right)\tilde{\boldsymbol{\Psi}} + c_3\left(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}}\right)\boldsymbol{\Psi}\otimes\boldsymbol{\Psi} + c_4\dot{\tilde{\boldsymbol{\Psi}}} + c_5\left(\dot{\boldsymbol{\Psi}}\otimes\boldsymbol{\Psi} + \boldsymbol{\Psi}\otimes\dot{\boldsymbol{\Psi}}\right),
\end{aligned} \tag{63}
$$

where the coefficients are given in (58). The limit value of the tensor $\dot{\mathbf{T}}$ is

$$\lim_{\boldsymbol{\Psi}\to\mathbf{0}} \dot{\mathbf{T}}(\dot{\boldsymbol{\Psi}},\boldsymbol{\Psi}) = -\tfrac{1}{2}\dot{\tilde{\boldsymbol{\Psi}}}. \tag{64}$$

The tensors $\mathbf{C}_i, i = 7,\ldots,10$ are defined by directional derivatives

$$\mathbf{C}_7(\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}\left(\mathbf{C}_2(\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{W}\right)\cdot\Delta\boldsymbol{\Psi}, \tag{65}$$

$$\mathbf{C}_8(\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}\left(\mathbf{C}_1(\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{W}\right)\cdot\Delta\boldsymbol{\Psi}, \tag{66}$$

$$\mathbf{C}_9(\mathbf{X},\mathbf{W},\boldsymbol{\Psi})\cdot\Delta\mathbf{V} := \mathrm{D}_{\mathbf{V}}\left(\mathbf{C}_3(\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{X}\right)\cdot\Delta\mathbf{V}, \tag{67}$$

$$\mathbf{C}_{10}(\mathbf{X},\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\Delta\boldsymbol{\Psi} := \mathrm{D}_{\boldsymbol{\Psi}}\left(\mathbf{C}_3(\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{X}\right)\cdot\Delta\boldsymbol{\Psi}, \tag{68}$$

$$\mathrm{D}_{\mathbf{W}}\left(\mathbf{C}_3(\mathbf{W},\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{X}\right) = \dot{\mathbf{C}}_1^{\mathrm{T}}(\mathbf{X},\mathbf{V},\boldsymbol{\Psi}), \tag{69}$$

$$\mathrm{D}_{\mathbf{V}}\left(\mathbf{C}_1(\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{W}\right) = \dot{\mathbf{T}}(\mathbf{W},\boldsymbol{\Psi}), \tag{70}$$

$$\mathrm{D}_{\mathbf{V}}\left(\mathbf{C}_2(\mathbf{V},\boldsymbol{\Psi})\cdot\mathbf{W}\right) = \dot{\mathbf{T}}^{\mathrm{T}}(\mathbf{W},\boldsymbol{\Psi}), \quad \forall \mathbf{V},\mathbf{W},\mathbf{X}\in E^3, \tag{71}$$

and have the following explicit expressions

$$
\begin{aligned}
&\mathbf{C}_7(\mathbf{W},\mathbf{V},\boldsymbol{\Psi}) = c_1\mathbf{V}\otimes\mathbf{W} - c_2(\boldsymbol{\Psi}\cdot\mathbf{W})\tilde{\mathbf{V}} + c_2\tilde{\boldsymbol{\Psi}}\mathbf{V}\otimes\mathbf{W} + c_3(\boldsymbol{\Psi}\cdot\mathbf{W})\boldsymbol{\Psi}\otimes\mathbf{V} + c_3(\boldsymbol{\Psi}\cdot\mathbf{V})\boldsymbol{\Psi}\otimes\mathbf{W} +\\
&\quad + \left(c_3(\boldsymbol{\Psi}\cdot\mathbf{W})(\boldsymbol{\Psi}\cdot\mathbf{V}) + c_5(\mathbf{W}\cdot\mathbf{V})\right)\mathbf{I} + c_5\mathbf{W}\otimes\mathbf{V} + \frac{c_1'}{\psi}(\boldsymbol{\Psi}\cdot\mathbf{W})\mathbf{V}\otimes\boldsymbol{\Psi} + \frac{c_2'}{\psi}(\boldsymbol{\Psi}\cdot\mathbf{W})\tilde{\boldsymbol{\Psi}}\mathbf{V}\otimes\boldsymbol{\Psi} +\\
&\quad + \left(\frac{c_3'}{\psi}(\boldsymbol{\Psi}\cdot\mathbf{V})(\boldsymbol{\Psi}\cdot\mathbf{W}) + c_3(\mathbf{W}\cdot\mathbf{V})\right)\boldsymbol{\Psi}\otimes\boldsymbol{\Psi} - c_2\tilde{\mathbf{V}}\mathbf{W}\otimes\boldsymbol{\Psi} + c_3(\boldsymbol{\Psi}\cdot\mathbf{V})\mathbf{W}\otimes\boldsymbol{\Psi}
\end{aligned} \tag{72}
$$

$$\mathbf{C}_8(\mathbf{W},\mathbf{V},\mathbf{\Psi}) = c_1\mathbf{V}\otimes\mathbf{W} + c_2(\mathbf{\Psi}\cdot\mathbf{W})\tilde{\mathbf{V}} - c_2\tilde{\mathbf{\Psi}}\mathbf{V}\otimes\mathbf{W} + c_3(\mathbf{\Psi}\cdot\mathbf{W})\mathbf{\Psi}\otimes\mathbf{V} + c_3(\mathbf{\Psi}\cdot\mathbf{V})\mathbf{\Psi}\otimes\mathbf{W} +$$
$$+\left(c_3(\mathbf{\Psi}\cdot\mathbf{W})(\mathbf{\Psi}\cdot\mathbf{V}) + c_5(\mathbf{W}\cdot\mathbf{V})\right)\mathbf{I} + c_5\mathbf{W}\otimes\mathbf{V} + \frac{c_1'}{\psi}(\mathbf{\Psi}\cdot\mathbf{W})\mathbf{V}\otimes\mathbf{\Psi} - \frac{c_2'}{\psi}(\mathbf{\Psi}\cdot\mathbf{W})\tilde{\mathbf{\Psi}}\mathbf{V}\otimes\mathbf{\Psi} +$$
$$+\left(\frac{c_3'}{\psi}(\mathbf{\Psi}\cdot\mathbf{V})(\mathbf{\Psi}\cdot\mathbf{W}) + c_3(\mathbf{W}\cdot\mathbf{V})\right)\mathbf{\Psi}\otimes\mathbf{\Psi} + c_2\tilde{\mathbf{V}}\mathbf{W}\otimes\mathbf{\Psi} + c_3(\mathbf{\Psi}\cdot\mathbf{V})\mathbf{W}\otimes\mathbf{\Psi} \tag{73}$$

$$\mathbf{C}_9(\mathbf{X},\mathbf{W},\mathbf{\Psi}) = c_1\mathbf{X}\otimes\mathbf{W} + c_2\mathbf{\Psi}\otimes\tilde{\mathbf{\Psi}}\mathbf{W} + c_2\mathbf{X}\otimes\tilde{\mathbf{\Psi}}\mathbf{W} + \left(c_3(\mathbf{X}\cdot\mathbf{W}) + \frac{c_3'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{\Psi}\cdot\mathbf{W})\right)\mathbf{\Psi}\otimes\mathbf{\Psi} +$$
$$+ c_3(\mathbf{\Psi}\cdot\mathbf{W})\mathbf{X}\otimes\mathbf{\Psi} + c_3(\mathbf{\Psi}\cdot\mathbf{W})\mathbf{\Psi}\otimes\mathbf{X} + c_5\mathbf{W}\otimes\mathbf{X} + \left(c_5(\mathbf{W}\cdot\mathbf{X}) + c_3(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{\Psi}\cdot\mathbf{W})\right)\mathbf{I} +$$
$$+ \frac{c_1'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{\Psi}\otimes\mathbf{W} + \frac{c_2'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{\Psi}\otimes\tilde{\mathbf{\Psi}}\mathbf{W} + c_2(\mathbf{\Psi}\cdot\mathbf{X})\tilde{\mathbf{W}} + c_3(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{W}\otimes\mathbf{\Psi} \tag{74}$$

$$\mathbf{C}_{10}(\mathbf{X},\mathbf{V},\mathbf{\Psi}',\mathbf{\Psi}) = \left(c_2(\tilde{\mathbf{V}}\mathbf{\Psi}'\cdot\mathbf{X}) + c_3(\mathbf{\Psi}\cdot\mathbf{\Psi}')(\mathbf{V}\cdot\mathbf{X}) + c_3(\mathbf{\Psi}\cdot\mathbf{V})(\mathbf{\Psi}'\cdot\mathbf{V})\right)\mathbf{I} +$$
$$+ \frac{1}{\psi^2}\left(c_1''(\mathbf{\Psi}'\cdot\mathbf{V}) + c_2''(\mathbf{\Psi}\cdot\tilde{\mathbf{V}}\mathbf{\Psi}') + c_3''(\mathbf{\Psi}\cdot\mathbf{\Psi}')(\mathbf{V}\cdot\mathbf{\Psi})\right)(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{\Psi}\otimes\mathbf{\Psi} +$$
$$+ \frac{1}{\psi}\left(c_3'(\mathbf{V}\cdot\mathbf{\Psi})(\mathbf{\Psi}'\cdot\mathbf{X}) + c_3'(\mathbf{\Psi}'\cdot\mathbf{\Psi})(\mathbf{V}\cdot\mathbf{X}) + c_2'(\tilde{\mathbf{V}}\mathbf{\Psi}'\cdot\mathbf{X})\right)\mathbf{\Psi}\otimes\mathbf{\Psi} +$$
$$+ \frac{1}{\psi}\left(c_1'(\mathbf{\Psi}'\cdot\mathbf{V}) + c_2'(\mathbf{\Psi}\cdot\tilde{\mathbf{V}}\mathbf{\Psi}') + c_3'(\mathbf{\Psi}\cdot\mathbf{\Psi}')(\mathbf{V}\cdot\mathbf{\Psi})\right)(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{I} - \frac{1}{\psi^2}\mathbf{\Psi}\otimes\mathbf{\Psi}) + c_2\mathbf{X}\underset{S}{\otimes}\tilde{\mathbf{V}}\mathbf{\Psi}' +$$
$$+ \left(c_3(\mathbf{V}\cdot\mathbf{X}) + \frac{c_3'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{V}\cdot\mathbf{\Psi})\right)\mathbf{\Psi}\underset{S}{\otimes}\mathbf{\Psi}' + c_3(\mathbf{V}\cdot\mathbf{\Psi})\mathbf{X}\underset{S}{\otimes}\mathbf{\Psi}' + c_3(\mathbf{\Psi}\cdot\mathbf{\Psi}')\mathbf{X}\underset{S}{\otimes}\mathbf{V} + c_3(\mathbf{X}\cdot\mathbf{\Psi})\mathbf{V}\underset{S}{\otimes}\mathbf{\Psi} +$$
$$+ \frac{1}{\psi}\left(c_1'(\mathbf{\Psi}'\cdot\mathbf{V}) + c_2'(\mathbf{\Psi}\cdot\tilde{\mathbf{V}}\mathbf{\Psi}') + c_3'(\mathbf{\Psi}\cdot\mathbf{\Psi}')(\mathbf{V}\cdot\mathbf{\Psi})\right)\mathbf{\Psi}\underset{S}{\otimes}\mathbf{X} +$$
$$+ \left(c_3(\mathbf{\Psi}'\cdot\mathbf{X}) + \frac{c_3'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{\Psi}\cdot\mathbf{\Psi}')\right)\mathbf{\Psi}\underset{S}{\otimes}\mathbf{V}\frac{1}{\psi}c_2'(\mathbf{X}\cdot\mathbf{\Psi})\mathbf{\Psi}\underset{S}{\otimes}\tilde{\mathbf{V}}\mathbf{\Psi}' \tag{75}$$

$$\dot{\mathbf{C}}_1(\mathbf{X},\mathbf{V},\mathbf{\Psi}) = c_1\mathbf{V}\otimes\mathbf{X} - c_2(\tilde{\mathbf{X}}\mathbf{V})\otimes\mathbf{\Psi} - c_2(\tilde{\mathbf{\Psi}}\mathbf{V})\otimes\mathbf{X} + \left(c_3(\mathbf{X}\cdot\mathbf{V}) + \frac{c_3'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{\Psi}\cdot\mathbf{V})\right)\mathbf{\Psi}\otimes\mathbf{\Psi} +$$
$$+ c_3(\mathbf{\Psi}\cdot\mathbf{V})\mathbf{X}\otimes\mathbf{\Psi} + c_3(\mathbf{\Psi}\cdot\mathbf{V})\mathbf{\Psi}\otimes\mathbf{X} + \left(c_5(\mathbf{X}\cdot\mathbf{V}) + c_3(\mathbf{\Psi}\cdot\mathbf{X})(\mathbf{\Psi}\cdot\mathbf{V})\right)\mathbf{I} + c_5\mathbf{X}\otimes\mathbf{V} +$$
$$+ \frac{c_1'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{V}\otimes\mathbf{\Psi} - \frac{c_2'}{\psi}(\mathbf{\Psi}\cdot\mathbf{X})(\tilde{\mathbf{\Psi}}\mathbf{V})\otimes\mathbf{\Psi} + c_2(\mathbf{\Psi}\cdot\mathbf{X})\tilde{\mathbf{V}} + c_3(\mathbf{\Psi}\cdot\mathbf{X})\mathbf{\Psi}\otimes\mathbf{V} \tag{76}$$

The limit processes for the tensors $\mathbf{C}_i, i = 7, \ldots, 10$ give

$$\lim_{\mathbf{\Psi}\to\mathbf{0}}\mathbf{C}_7(\mathbf{W},\mathbf{V},\mathbf{\Psi}) = -\tfrac{1}{3}\mathbf{V}\otimes\mathbf{W} + \tfrac{1}{6}\left(\mathbf{W}\otimes\mathbf{V} + (\mathbf{W}\cdot\mathbf{V})\mathbf{I}\right) \tag{77}$$
$$\lim_{\mathbf{\Psi}\to\mathbf{0}}\mathbf{C}_8(\mathbf{W},\mathbf{V},\mathbf{\Psi}) = -\tfrac{1}{3}\mathbf{V}\otimes\mathbf{W} + \tfrac{1}{6}\left(\mathbf{W}\otimes\mathbf{V} + (\mathbf{W}\cdot\mathbf{V})\mathbf{I}\right), \tag{78}$$
$$\lim_{\mathbf{\Psi}\to\mathbf{0}}\mathbf{C}_9(\mathbf{X},\mathbf{W},\mathbf{\Psi}) = -\tfrac{1}{3}\mathbf{X}\otimes\mathbf{V} + \tfrac{1}{6}\left((\mathbf{X}\cdot\mathbf{V})\mathbf{I} + \mathbf{V}\otimes\mathbf{X}\right) \tag{79}$$
$$\lim_{\mathbf{\Psi}\to\mathbf{0}}\mathbf{C}_{10}(\mathbf{X},\mathbf{V},\mathbf{\Psi}',\mathbf{\Psi}) = -\tfrac{1}{12}(\tilde{\mathbf{V}}\mathbf{\Psi}'\cdot\mathbf{X})\mathbf{I} - \tfrac{1}{12}\mathbf{X}\underset{S}{\otimes}\tilde{\mathbf{V}}\mathbf{\Psi}' \tag{80}$$
$$\lim_{\mathbf{\Psi}\to\mathbf{0}}\dot{\mathbf{C}}_1(\mathbf{X},\mathbf{V},\mathbf{\Psi}) = -\tfrac{1}{3}\mathbf{V}\otimes\mathbf{X} + \tfrac{1}{6}\left((\mathbf{X}\cdot\mathbf{V})\mathbf{I} + \mathbf{X}\otimes\mathbf{V}\right) \tag{81}$$

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