A Novel Membrane Finite Element with Drilling Rotations

Reijo Kouhia [1]

Abstract. A new low order membrane finite element is presented. The element formulation is based on the variational principle of Hughes and Brezzi employing an independent rotation field. In the present work nonconforming interpolation is used for the drill rotation field. Both triangular and quadrilateral elements are considered. Combined with the Reissner-Mindlin plate bending element of Oñate, Zarate and Flores, where the rotations are interpolated by nonconforming functions, results in a simple shell element with three translational displacements at corner nodes and three rotation components at the midpoints of the element edges as degrees of freedom.

1 INTRODUCTION

In-plane rotational degrees of freedom, "drilling degrees of freedom", are particularly convenient in the analysis of shells. Typical shell elements have three translational and two rotational degrees of freedom at a node. This results in many difficulties of model construction and of programming and numerical ill - conditioning for certain types of element assemblages. Thus the presence of all three rotations at a node is advantageous from practical point of view.

Early attempts to construct membrane elements with drill rotations have been unsuccessful. Allman [1] and Bergan and Felippa [2] formulated successful membrane elements with drill rotations by using special interpolation patterns. Allman derived the interpolation scheme by choosing the normal and tangential displacement components to be quadratic and linear functions, respectively. The only deficiences of this remarkably simple element are the existing zero energy mode and that the vertex rotations are not true rotations. These shortcomings were removed in a later development by Allman [3], at the expense of having a cubic displacement field. Bergan and Felippa used the so-called free formulation of the finite element method to derive their element.

Recently, Hughes and Brezzi [4] presented a simple and beautiful variational formulation, which employs independent rotation field and is also stable in the discrete case. They also proved that elements based on their formulation are convergent for all standard interpolations including equal order interpolation for displacements and rotation. Numerical experiments reported by Hughes *et al.* [5] confirm the *a priori* theoretical convergence estimates. They compared linear and quadratic triangular and quadrilateral elements and also a bilinear element with incompatible modes. Incompatible modes significantly improve the coarse mesh accuracy of the bilinear element. On the other hand, the static condensation needed to eliminate the nodeless generalized displacements is awkward, especially in geometrically nonlinear problems.

Ibrahimbegović *et al.* [6] amended the displacement interpolation of the four node quadrilateral element by Allman type quadratic modes in order to improve the coarse mesh accuracy of the element. This considerably improves the bending behaviour of the element. They also added a hierarchical bubble interpolation mode to the displacement field.

A convenient interpolation scheme, advocated by Hughes and Brezzi, is to use the same functions for both displacements and rotations. This results in an element with equal number of degrees of freedom at each node.

In the present study, nonconforming interpolation is adopted for the drill rotation. Together with the nonconforming Reissner-Mindlin plate element of Oñate, Zarate and Flores [7], it gives an element with three degrees of freedom at each node. Thus, translational and rotational components appear at different nodes. It is argued that the increased freedom of the nonconforming interpolation is beneficial to the coarse mesh accuracy of the element, and still facilitating relatively simple implementation.

[1] Laboratory of Structural Mechanics, Helsinki University of Technology, Rakentajanaukio 4A, FIN-02150 Espoo, e - mail: Reijo.Kouhia@hut.fi

ECCOMAS 96.
Published in 1996 by John Wiley & Sons, Ltd.

2 ELEMENT FORMULATION

2.1 Membrane element

The variational equation of the Hughes-Brezzi formulation in the case of Dirichlet boundary conditions has the form

$$\begin{aligned}&\int_\Omega \delta\varepsilon : \mathbf{C} : \varepsilon d\Omega \\ &+\gamma \int_\Omega (\text{skew}\nabla\delta\mathbf{u} - \delta\psi)(\text{skew}\nabla\mathbf{u} - \psi) d\Omega \\ =& \int_\Omega \mathbf{f}\cdot\delta\mathbf{u} d\Omega \qquad (1)\end{aligned}$$

where $\varepsilon = \text{symm}\ \nabla\mathbf{u}$ is the strain tensor, i.e. the symmetric part of the displacement gradient and ψ is a skew symmetric tensor representing the in-plane rotation. The fourth order tensor $\mathbf{C}$ contains the material parameters and the body force vector is denoted by $\mathbf{f}$.

An appropriate value of the regularizing penalty parameter γ is chosen in accordance with the ellipticity condition. For the isotropic case the value $\gamma = \mu$ (shear modulus) seems to balance the terms in the stability estimate and thus seems reasonable [8]. However, the method is insensitive to the choice of the penalty parameter in the region $0 < \gamma \leq \mu$. Since the formulation above is extensively studied in Refs. [4],[8], the details are not repeated here.

Two interpolation schemes of the present element are considered. The first one is the standard linear or bilinear continuous interpolation for the translational displacements with nonconforming interpolation for the drilling rotation:

$$\begin{aligned} u &= \sum_{i=1}^{n} N_i u_i, \qquad v = \sum_{i=1}^{n} N_i v_i, \\ \psi &= \sum_{i=n+1}^{2n} N_i^{nc}\psi_i, \qquad (2)\end{aligned}$$

where n is the number of vertex nodes. In the second approach the displacement interpolation is amended by parabolic modes linked with the rotation.

For the triangular element the nonconforming interpolation at edge i (connecting nodes i and $i+$) is simply

$$N_i^{nc} = L_i + L_{i+} - L_{i-} \qquad (3)$$

where L_i is the area coordinate associated with node i. Nonconforming interpolation for the quadrilateral element is obtained by rotating the bilinear interpolation, resulting in

$$N_i^{nc} = \tfrac{1}{4}\left[1 + 2\xi_i\xi + 2\eta_i\eta + (\xi_i+\eta_i)^{i-4}(\xi^2 - \eta^2)\right], \quad (4)$$

where ξ_i and η_i $(i = 5, ..., 8)$ are the nodal point coordinates in the parent element. In order to improve the coarse mesh accuracy, the in-plane displacement interpolation of the quadrilateral element can be amended by parabolic modes linked to the drilling rotations in the following way:

$$\begin{aligned}\mathbf{u} &= \sum_{i=1}^{4} N_i(\xi,\eta)\mathbf{q}_i + N_5(\xi)\frac{L_{86}}{8}(\psi_8 - \psi_6)\mathbf{n}_{86} \\ &\quad + N_6(\eta)\frac{L_{57}}{8}(\psi_7 - \psi_5)\mathbf{n}_{57}, \qquad (5)\end{aligned}$$

where $\mathbf{q}_i$ contains the nodal translational (u, v)-degrees of freedom and $N_1, ..., N_4$ are the standard bilinear interpolation functions. The unit vectors $\mathbf{n}_{57}$ and $\mathbf{n}_{86}$ are normal to the lines connecting the nodes 5 and 7 and 8 and 6, respectively, see Fig. 1. The parabolic incompatible modes N_5 and N_6 are defined as

$$N_5(\xi) = 1 - \xi^2, \qquad N_6(\eta) = 1 - \eta^2. \qquad (6)$$

For the triangular element such an enhanced interpolation is not possible.

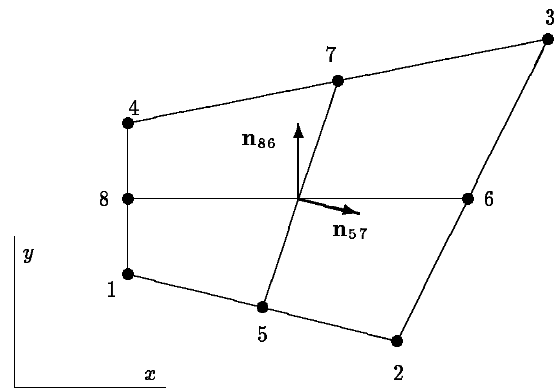


Figure 1. A quadrilateral element.

If the conventional conforming interpolation is used for the drill rotation, then the Allman-type amendment for the in-plane displacement components can be used for both the triangular $(n = 3)$ and the quadrilateral $(n = 4)$ element:

$$\begin{aligned}\mathbf{u} &= \sum_{i=1}^{n} N_i(\xi,\eta)\mathbf{q}_i \\ &\quad + \sum_{i=1}^{n} N_{n+i}(\xi,\eta)\frac{L_i}{8}(\psi_{i+} - \psi_i)\mathbf{n}_i \qquad (7)\end{aligned}$$

where $\mathbf{n}_i = \begin{bmatrix} \cos\alpha_i & \sin\alpha_i \end{bmatrix}^T$ is the unit outward normal vector of the element edge i.

2.2 Plate bending element

Combining the membrane element described above with the nonconforming Reissner-Mindlin plate bending element introduced by Oñate, Zarate and Flores [7], results in a facet shell element where the translational degrees of freedom are located at the vertex nodes and the rotational degrees of freedom at the midside nodes.

In order to avoid shear locking, the MITC type of reduction operator is used to evaluate the shear strains. The reduction gives constant tangential shear strains on element egdes, which are set to be equal to the shear strains obtained from the original displacement variables at the midpoints of element edges.

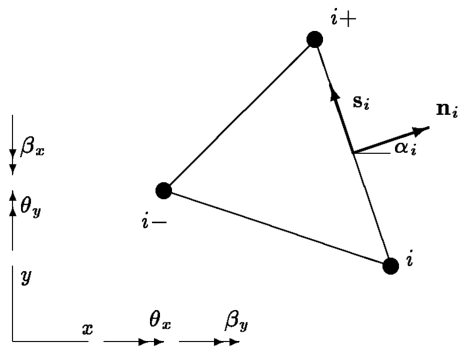


Figure 2. Notations.

The standard linear or bilinear interpolation is used for the deflection but nonconforming one for each rotation component β_x and β_y:

$$w = \sum_{i=1}^{n} N_i w_i,$$
$$\beta_x = \sum_{i=n+1}^{2n} N_i^{nc} \beta_{xi}, \qquad \beta_y = \sum_{i=n+1}^{2n} N_i^{nc} \beta_{yi}. \quad (8)$$

See Fig. 2 for the definition of the rotations. A convenient way to implement the shear reduction operator is to define a new rotation field from which the shear strains are evaluated, i.e.

$$\beta_x^S = \sum_{i=1}^{n} N_i \beta_{xi}^S, \qquad \beta_y^S = \sum_{i=1}^{n} N_i \beta_{yi}^S. \quad (9)$$

Here, N_i's are the standard linear/bilinear interpolation functions. Note, that this interpolation is defined separately in each element. The additional rotation field (9) gives two additional degrees of freedom for each edge. They can be eliminated using the following two conditions:

- The tangential shear strain is constant at each element edge
$$\gamma_s = w_{,s} - \mathbf{s}^T \beta^S, \quad (10)$$
- and is equal to the shear evaluated from the original displacement fields at the midpoints of the element edges
$$w_{,s} - \mathbf{s}^T \beta^S = w_{,s} - \mathbf{s}^T \beta(0). \quad (11)$$

The resulting form of the rotation β^S is

$$\beta_x^S = \sum_{i=1}^{n} \Big[\big(\frac{C_{i+} S_i}{E_{i+}} N_{i+} - \frac{C_{i-} S_i}{E_i} N_i \big) \beta_{xi} \big(\frac{C_i C_{i-}}{E_i} N_i - \frac{C_{i+} C_i}{E_{i+}} N_{i+} \big) \beta_{yi} \Big],$$
$$\beta_y^S = \sum_{i=1}^{n} \Big[\big(\frac{S_{i+} S_i}{E_{i+}} N_{i+} - \frac{S_{i-} S_i}{E_i} N_i \big) \beta_{xi} \big(\frac{C_i S_{i-}}{E_i} N_i - \frac{S_{i+} C_i}{E_{i+}} N_{i+} \big) \beta_{yi} \Big], \quad (12)$$

where
$$E_i = C_i S_{i-} - C_{i-} S_i, \quad (13)$$
and C_i, S_i are abbreviations for $\cos\alpha_i, \sin\alpha_i$.

In the stabilized finite element methods for the Reissner-Mindlin model, shear forces are computed from the expressions

$$Q_x = \frac{k\mu t}{1 + \alpha(h/t)^2} (w_{,x} - \beta_x^S),$$
$$Q_y = \frac{k\mu t}{1 + \alpha(h/t)^2} (w_{,y} - \beta_y^S), \quad (14)$$

where α is a positive stabilization factor, t is the thickness of the plate and h is a characteristic length of an element, i.e. the length of the longest edge.

The behaviour of this simple nonconforming Reissner-Mindlin plate bending element is not particularly good. Mathematical analysis also reveals that the triangular element is not convergent for a fixed plate thickness, as shown recently by Arnold and Falk [9]. However, optimal order convergence, uniform in t, is obtained if the plate thickness t tends to zero as the mesh is refined. Thus the element exhibits an unusual type of locking. Perhaps, some technique can remove this deficiency. Some preliminary calculations by the triangular element with constant shear, evaluated from the original displacement variables, have shown better performance. However, mathematical analysis is needed to assess the applicability of this element.

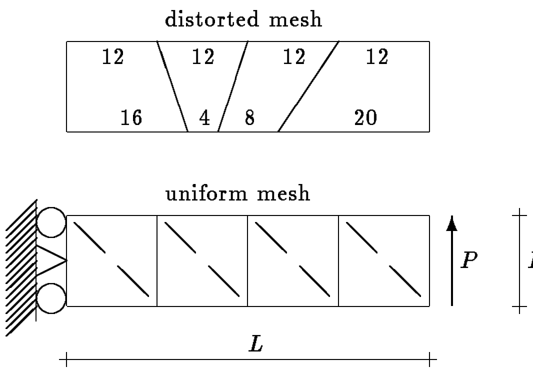
In a triangular case there is an interesting connection of this nonconforming Reissner-Mindlin plate element to the well known Morley's constant moment Kirchhoff plate element [7], [9]. Morley's element [10] can be obtained from interpolations (8) by enforcing the Kirchhoff constraint at the midpoints of the element edges. Thus one of the two rotation components can be eliminated.

3 EXAMPLES

3.1 A cantilever beam

A shear loaded cantilever beam is a common test case for membrane elements. The problem definition is shown in Fig. 3. Computed tip displacements are tabulated in Table 1, where also results of Allman [1], MacNeal and Harder [11] and Ibrahimbegović *et al.* [6] are shown. Only a minimum number of restraints is imposed, hence no drilling degrees of freedom are restrained. The penalty parameter γ is chosen to be equal to the Young's modulus. The reference solution for the tip displacement with the properties shown in Fig.3 is defined in Ref. [6] as

$$v = \frac{PL^3}{3EI} + \frac{(4+5\nu)PL}{2EH} = 0.3553. \qquad (15)$$



$E = 30000$, $\nu = 0.25$, $P = 40$, $L = 48$, $H = 12$

Figure 3. Short cantilever beam.

The behaviour of the nonconforming elements is comparable to the corresponding one of the conforming elements. The nonconforming drill rotation elements are also insensitive to the value of the penalty parameter γ, see Fig.4, where the normalized tip displacements (with respect to (15)) are shown as a function of γ. The results are similar to those reported by Hughes *et al.* [8]. The spectral condition number as a function of the penalty parameter is shown in Fig. 5 for the Q4ND4 element. Other elements based on the Hughes-Brezzi formulation behave almost identically. It seems that the range $10^{-4} < \gamma/\mu < 1$ can be used with no danger of deterioration of numerical conditioning of the equation system.

The nonconforming quadrilateral with incompatible modes behaves like the elements QM6D4-1 and QM6D4-2 in Ref. [8] and without the incompatible modes like Q4D4. Thus, it seems to be a promising alternative for the membrane interpolation of a facet type shell element. It will be tested in the next example.

Table 1. Short cantilever beam, tip deflections.

	uniform meshes			distorted
element	4 x 1	8 x 2	16 x 4	4 x 1
T3	0.0934	0.1977	0.2982	0.0900
T3D3	0.0851	0.1872	0.2882	0.0819
T3D3-A	0.2104	0.3170	0.3465	0.2020
Q4	0.2434	0.3180	0.3481	0.2138
Q4D4	0.2179	0.3058	0.3433	0.1988
Q4D4-A	0.3311	0.4301	0.3709	0.3230
Q4ND4 3x3	0.2250	0.3060	0.3423	0.1992
Q4ND4 4pq	0.2216	0.3050	0.3420	0.1979
Q4ND4-IC*	0.3295	0.3479	0.3550	0.2735
T3ND3	0.0855	0.1842	0.2850	0.0822
[1]	0.3026	0.3394	0.3512	-
[6]	0.3445	0.3504	0.3543	0.3064
[11]	0.3409	-	-	0.2978

T, Q = triangular, quadrilateral element
D = Hughes-Brezzi formulation for drill rotation
A = Allman type interpolation (7)
N = nonconforming, IC = incompatible modes (5)
3x3 = 3 by 3 Gaussian quadrature
4pq = special four point quadrature
$(\xi_i, \eta_i) = (\pm\sqrt{2/3}, 0), (0, \pm\sqrt{2/3})$, weights $\equiv 1$,
integrates exactly monomials $\xi^i \eta^j$ where $i + j \leq 3$.
$*$ = 3x3 rule and the special four point rule give equal results

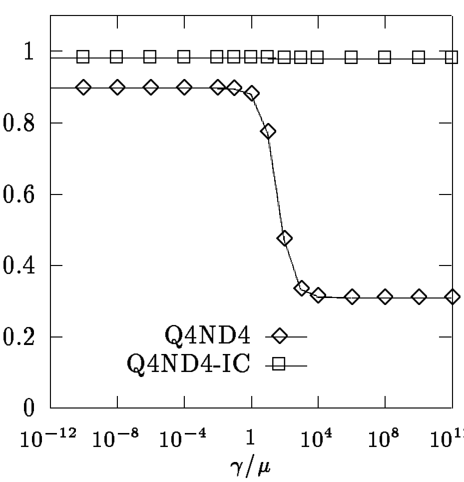


Figure 4. Normalized tip deflection as a function of γ. Uniform 8×2 mesh.

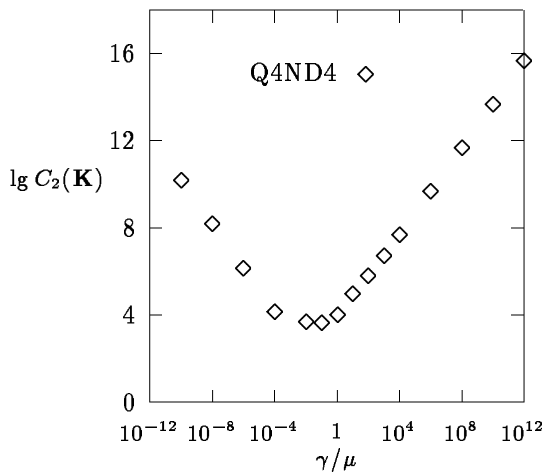


Figure 5. Logarithm of the spectral condition number of the stiffness matrix as a function of γ. Uniform 8×2 mesh.

3.2 A folded plate problem

A cantilever beam structure with a thin-walled I-cross section is analyzed. Same material constants, length and the height of the beam between the flange centroids are used as in the previous example. The wall thickness of all parts of the beam is chosen to be $t = 0.2 = L/240$, and the width of the flanges are equal to the height H.

All the shell elements compared are of flat facet type, where the membrane interpolations are as in the previous example and the plate-bending part is modelled by the stabilized MITC type of elements, see Refs. [12], [13].

The reference solution for the tip displacement is computed from the Timoshenko beam theory as

$$v = \frac{PL^3}{3EI} + \frac{(1+\nu)PL}{2EHt} = 0.3133. \qquad (16)$$

In the computations all dof's at the support are supressed. Only quadrilateral elements are compared and the values for the stabilization parameters used are $\alpha = 0.1$ and $\gamma = \mu$.

3.3 A pinched cylinder

The pinched cylinder with rigid end diaphragms is frequently analyzed test case to shell elements, see Fig. 6. One octant of the shell is modelled and the normalized displacement under the point loads is recorded in Table 3, where also some results from Refs. [14] [15], [16] are shown. The reference displacement is [14]:

$$w_c = 164.24\frac{P}{Et} = 1.82489 \times 10^{-5}. \qquad (17)$$

Table 2. I-beam, tip deflections.

	uniform meshes		
element	4×5	8×10	16×20
degrees of freedom	144	528	2016
Q4D4 + MITC4	0.2374	0.2879	0.3068
Q4D4-A + MITC4	0.3003	0.3088	0.3136
degrees of freedom	204	768	2979
Q4ND4 + MITCN4	0.2408	0.2899	0.3095
Q4ND4-IC + MITCN4	0.3027	0.3106	0.3165

For the triangular elements, computations with constant shear stress, evaluated from the original displacement variables, are also shown. In this case the element stiffness matrix can be exactly integrated by the one point rule.

The stabilization parameter used is also shown in Table 3. It is selected according to the results of Ref.[13] ($\alpha = 0.1$ for the quadrilateral and $\alpha = 0.4$ for the triangular element). It can be seen that the displacement value is influenced on the choice of the parameter. However, one should not concentrate too much on the accuracy of displacements, which is usually good enough. As shown numerically by Lyly *et al.* [13] the moments are practically unchanged when the stabilization parameter is varied, while the shear forces become better when a larger value is chosen. Also some nonlinear computations done by the author suggest to use a larger value ($\alpha = 0.4$ for the quadrilateral) for the stabilization parameter, than the one giving the minimum error for the displacement.

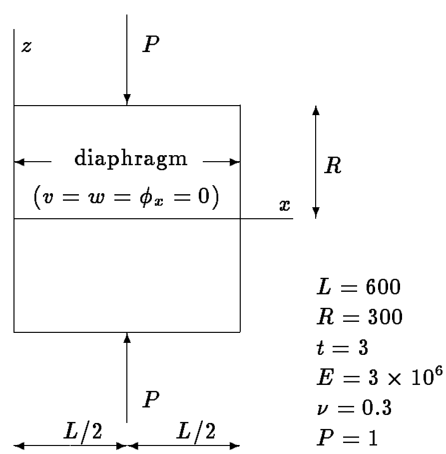


Figure 6. Pinched cylinder.

Table 3. Pinched cylinder, normalized radial displacement under loads.

element	α	4×4	8×8	16×16	32×32
		uniform meshes			
Q	0.4	0.6810	1.0416	1.1203	1.0584
Q	0.1	0.4863	0.8445	0.9807	1.0064
T 3pq	0.4	0.6880	1.0384	1.1042	1.0528
T 1pq	0.4	0.7392	1.0630	1.1119	1.0552
TN 3pq	0.4	0.6591	1.4347	1.4489	1.2221
TN 3pq	0	0.6069	1.2310	1.2881	1.1438
QN	0	1.0998	1.4999	1.3573	1.2982
TN 1pq *	0	0.4722	0.6352	0.5314	0.6177
TN 1pq *	0.4	0.6276	1.3425	1.2561	1.1576
[15]	-	0.535	0.857	0.980	-
[16]	-	0.599	0.909	0.973	-
		uniform meshes			
		5×5	10×10	20×20	
[14]	-	0.51	0.83	0.96	-

T = T3D3-A + stabilized MITC3
Q = Q4D4-A + stabilized MITC4
TN = T3ND3 + stabilized nonconforming MITC3
QN = Q4ND4-IC + stabilized nonconforming MITC4
* = constant shear from the original displacement variables.
1pq = one point quadrature
3pq = three point quadrature

4 CONCLUSIONS

A simple triangular and quadrilateral membrane element with drilling rotations is proposed. The Hughes-Brezzi formulation is used with nonconforming interpolation for the drill rotation. It is shown numerically that the element is comparable to the conforming elements with drilling rotations.

The proposed membrane interpolation is also implemented in a flat facet shell element in combination with a specific Reissner-Mindlin plate element with nonconforming rotations. Since this plate bending element exhibits a peculiar type of locking problem (when $t > h$) additional work is needed to overcome this deficiency. In comparison to the conforming elements, the present nonconforming element has more global dof's when using the same element division.

ACKNOWLEDGEMENTS

The author would like to thank Pekka Valjento for his help in programming the nonconforming shell element and doing some of the computations presented here.

REFERENCES

[1] D.J. Allman. A compatible triangular element including vertex rotations for plane elasticity analysis. *Computers and Structures*, 19:1–8, 1984.

[2] P.G. Bergan and C.A. Felippa. A triangular membrane element with rotational degrees of freedom. *Computer Methods in Applied Mechanics and Engineering*, 50:25–69, 1985.

[3] D.J. Allman. Evaluation of the constant strain triangle with drilling rotations. *International Journal for Numerical Methods in Engineering*, 26:2645–2655, 1988.

[4] T.J.R. Hughes and F. Brezzi. On drilling degrees of freedom. *Computer Methods in Applied Mechanics and Engineering*, 72:105–121, 1989.

[5] T.J.R. Hughes, F. Brezzi, A. Masud, and I. Harari. Finite elements with drilling degrees of freedom: theory and numerical evaluations. In R. Gruber, J. Periaux, and R.P. Shaw, editors, *Proceedings of the fifth International Symposium on Numerical Methods in Engineering*, pages 3–17, Springer-Verlag, 1989.

[6] A. Ibrahimbegović, R.L. Taylor, and E.L. Wilson. A robust quadrilateral membrane finite element with drilling degrees of freedom. *International Journal for Numerical Methods in Engineering*, 30:445–457, 1990.

[7] E. Oñate, F. Zarate, and F. Flores. A simple triangular element for thick and thin plate and shell analysis. *International Journal for Numerical Methods in Engineering*, 37:2569–2582, 1994.

[8] T.J.R. Hughes, A. Masud, and I. Harari. Numerical assessment of some membrane elements with drilling degrees of freedom. *Computers and Structures*, 55:297–314, 1995.

[9] D.N. Arnold and R.S. Falk. Analysis of a linear-linear finite element for the Reissner-Mindlin plate. preprint.

[10] L.S.D. Morley. The constant moment plate bending element. *Journal of Strain Analysis*, 6:20-24, 1971.

[11] R.H. MacNeal and R.L. Harder. A refined four-noded membrane element with rotational degrees of freedom. *Computers and Structures*, 18:75–84, 1988.

[12] F. Brezzi, M. Fortin, and R. Stenberg. Error analysis of mixed-interpolated elements for Reissner-Mindlin plates. *Mathematical Models and Methods in Applied Sciences*, 1:125–151, 1991.

[13] M. Lyly, R. Stenberg, and T. Vihinen. A stable bilinear element for Reissner-mindlin plate model. *Computer Methods in Applied Mechanics and Engineering*, 110:343–357, 1993.

[14] E.N. Dvorkin and K.-J. Bathe. A continuum mechanics based four-node shell element for general non-linear analysis. *Engineering Computations*,1:77–88, 1984.

[15] P.G. Bergan and M.K. Nygård. Nonlinear shell analysis using free formulation finite elements. In K.J. Bathe, P.G. Bergan, and W. Wunderlich, editors, *Europe-US Symposium on Finite Elements in Nonlinear Problems*, Vol II, Springer-Verlag, 1986.

[16] D.J. Allman. A basic flat facet finite element for the analysis of general shells. *International Journal for Numerical Methods in Engineering*, 37:19–35, 1994.