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An anisotropic continuum damage model for concrete

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Summary. In this paper, a thermodynamic formulation for modelling anisotropic damage of elastic-brittle materials based on Ottosen's 4-parameter failure surface is proposed. The model is developed by using proper expressions for Gibb's free energy and the complementary form of the dissipation potential. The formulation predicts the basic characteristic behaviour of concrete well and results in a realistic shape for the damage surface.

Key words: damage, elastic-brittle material, the spesific Gibb's free energy, dissipation potential, Ottosen's 4-parameter criterion

Introduction

Concrete is a composite material composed mainly of water, aggregate and cement. It has relatively high compressive strength, but it has a low tensile strength, which is usually between 5-10% of the compressive strength.

A myriad of models have been proposed to model the mechanical behaviour of concrete. Earlier investigations have mainly focused in formulating the form of the ultimate failure surface in a way similar to the yield function of plasticity (e.g. [12], [21],[3], [11], [6], [17]). These models did not always take a proper account of the gradual degradation process prior to failure.

Recently, continuum damage mechanics is videly used for modelling the brittle behaviour of materials. The scalar damage variable which was first introduced by Kachanov [8] has been applied by several authors due to the simplicity of application together with plasticity, e.g. [10], [7],[13], [19]. Studies such as [5], [2] and [20] have proposed a mixed plasticity anisotropic damage model for concrete using higher order damage tensors.

Ottosen's four parameter criterion

In 1977, Ottosen proposed a failure surface for concrete which contains the three stress invariants and is capable of capturing the essential features of concrete's behaviour [14, 15, 16]. The failure criterion has four adjustable parameters and has the form

$$A\frac{J_2}{\sigma_{\rm c}} + \Lambda\sqrt{J_2} + BI_1 - \sigma_{\rm c} = 0, \tag{1}$$

where $\sigma_{\rm c}$ is the uniaxial compressive strength, $I_1 = \operatorname{tr}\boldsymbol{\sigma}$ the first invariant of the stress tensor and $J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s}$ the second invariant of the deviatoric stress tensor $\boldsymbol{s} = \boldsymbol{\sigma} - \sigma_{\rm m}\boldsymbol{I}$, where $\sigma_{\rm m}$ is

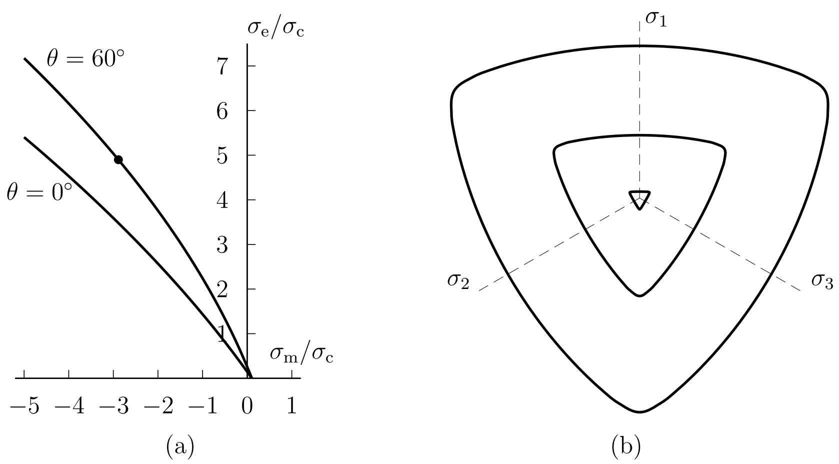

Figure 1. Ottosen's 4-parameter failure surface: (a) compressive and tensile meridian lines and (b) shape on the deviatoric plane when $\sigma_\mathrm{m} = 0, \sigma_\mathrm{m} = -\sigma_\mathrm{c}$ and $\sigma_\mathrm{m} = -2.887\sigma_\mathrm{c}$, strength ratios are $\sigma_\mathrm{t}/\sigma_\mathrm{c} = 0.1$, and $\sigma_\mathrm{bc}/\sigma_\mathrm{c} = 1.16$. $\sigma_\mathrm{e} = \sqrt{3J_2}$ is the effective stress.

the mean normal stress $\sigma_\mathrm{m} = \frac{1}{3}I_1$. Furthermore, the shape of the failure surface in the deviatoric plane is determined by the function $\Lambda = \Lambda(\theta)$ as

$$\Lambda = \begin{cases} k_1 \cos[\frac{1}{3}\arccos(k_2 \cos 3\theta)] & \text{if} \quad \cos 3\theta \geq 0 \\ k_1 \cos[\frac{1}{3}\pi - \frac{1}{3}\arccos(-k_2 \cos 3\theta)] & \text{if} \quad \cos 3\theta \leq 0 \end{cases}. \tag{2}$$

The Lode angle θ in the deviatoric plane can be expressed in terms of the deviatoric invariants as

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}, \tag{3}$$

where $J_3 = \det \boldsymbol{s}$ is the third invariant of the deviatoric stress tensor. Determination of the dimensionless parameters A and B, the size factor k_1 and the shape factor k_2 requires four tests, see [14, 16]. Shape of the failure surface is illustrated in Figure 1 in the meridian and deviatoric planes.

Present model

Constitutive theory

The constitutive equations of the model are derived by using the second principle of thermodynamics and associated thermodynamic potentials. Considering isothermal elastic damaging material with small deformations the reversible behaviour of material is captured by the specific Gibb's free energy

$$\psi^\mathrm{c} = \psi^\mathrm{c}(\boldsymbol{\sigma}, \boldsymbol{D}, \kappa) \tag{4}$$

which is defined by the stress tensor $\boldsymbol{\sigma}$, the second order damage tensor $\boldsymbol{D}$ and the scalar variable κ that characterises the internal state of the material.

The second principle of thermodynamics is represented by the Clausius-Duhem inequality in the form

$$\gamma \geq 0, \quad \gamma = \rho_0 \dot{\psi}^\mathrm{c} - \dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} = \left(\rho_0 \frac{\partial \psi^\mathrm{c}}{\partial \boldsymbol{\sigma}} - \boldsymbol{\varepsilon}\right) : \dot{\boldsymbol{\sigma}} + \boldsymbol{Y} : \dot{\boldsymbol{D}} - K\dot{\kappa}, \tag{5}$$

where γ is the power of dissipation and

$$\boldsymbol{Y} = \rho_0 \frac{\partial \psi^\mathrm{c}}{\partial \boldsymbol{D}} \quad \text{and} \quad K = -\rho_0 \frac{\partial \psi^\mathrm{c}}{\partial \kappa} \tag{6}$$

are thermodynamic forces dual to the rates $\dot{\boldsymbol{D}}$ and $\dot{\kappa}$, respectively [18]. Density of the material is denoted by ρ_0.

The irreversible material behaviour is described through the dissipation potential

$$\varphi = \varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma}) \tag{7}$$

as a function of the dissipation variables $\boldsymbol{Y}$ and K. The dissipation potential is a non-smooth monotonic and subdifferentiable function from a linear space into $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. It determines the power of dissipation such that

$$\gamma \equiv \boldsymbol{B}_Y : \boldsymbol{Y} + B_K K, \quad (\boldsymbol{B}_Y, B_K) \in \partial\varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma}), \tag{8}$$

where $\boldsymbol{B}_Y$ and B_K are the components of the subgradient of $\varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma})$ and $\partial\varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma})$ is the subdifferential set of all subgradients [4].

Combining definition of γ in equations (5) and (8) results in equation

$$\left(\rho_0 \frac{\partial \psi^{\mathrm{c}}}{\partial \boldsymbol{\sigma}} - \boldsymbol{\varepsilon}\right) : \dot{\boldsymbol{\sigma}} + \left(\dot{\boldsymbol{D}} - \boldsymbol{B}_Y\right) : \boldsymbol{Y} + \left(-\dot{\kappa} - B_K\right) K = 0. \tag{9}$$

If the bracketed coefficients in equation (9) tend to zero, the equation holds for arbitrary $\dot{\boldsymbol{\sigma}}$, $\boldsymbol{Y}$ and K, and the following general constitutive equations are obtained

$$\boldsymbol{\varepsilon} = \rho_0 \frac{\partial \psi^{\mathrm{c}}}{\partial \boldsymbol{\sigma}}, \qquad \dot{\boldsymbol{D}} = \boldsymbol{B}_Y, \qquad \dot{\kappa} = -B_K. \tag{10}$$

Specific model

The specific Gibb's free energy function is formulated by using the representation theory of tensorial functions, which states that a scalar isotropic function depending on two symmetric second order tensors can be expressed by a combination of the invariants belonging to the integity basis

$$\left\{\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}(\boldsymbol{\sigma}^2), \mathrm{tr}(\boldsymbol{\sigma}^3), \mathrm{tr}\,\boldsymbol{D}, \mathrm{tr}(\boldsymbol{D}^2), \mathrm{tr}(\boldsymbol{D}^3), \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{D}), \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{D}^2), \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{D}), \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{D}^2)\right\}. \tag{11}$$

Assuming no crack interaction, only the linear terms in $\boldsymbol{D}$ are retained [1]. Furthermore, restricting to linear elasticity, only the linear and quadratic invariants of the stress tensor are included. Hence, the specific Gibb's free energy function describing the isothermal elastic behaviour of material with a reduction effect due to damage can be formulated as

$$\rho_0 \psi^{\mathrm{c}}(\boldsymbol{\sigma}, \boldsymbol{D}, \kappa) = \frac{1+\nu}{2E}\left[\mathrm{tr}\,\boldsymbol{\sigma}^2 + \mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{D}\right] - \frac{\nu}{2E}(1 + \tfrac{1}{3}\,\mathrm{tr}\,\boldsymbol{D})(\mathrm{tr}\,\boldsymbol{\sigma})^2 + \psi^{\mathrm{c},\kappa}(\kappa), \tag{12}$$

where ν and E stand for Poisson's ratio and elastic modulus, respectively. The function $\psi^{\mathrm{c},\kappa}(\kappa)$ denotes the damage hardening part of the specific Gibb's free energy. It should be noticed that in the case of isotropic damage, $\boldsymbol{D} = D\boldsymbol{I}$, where D is a scalar damage variable, and the model reduces into the form

$$\rho_0 \psi^{\mathrm{c}} = (1 + D)\left[\frac{1+\nu}{2E}\,\mathrm{tr}\,\boldsymbol{\sigma}^2 - \frac{\nu}{2E}(\mathrm{tr}\,\boldsymbol{\sigma})^2\right] + \psi^{\mathrm{c},\kappa}(\kappa). \tag{13}$$

The dissipation potential is defined by the non-smooth indicator function $\mathrm{I}_\Sigma = \mathrm{I}_\Sigma(\boldsymbol{Y}, K; \boldsymbol{\sigma})$ [4] such that

$$\varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma}) = \mathrm{I}_\Sigma(\boldsymbol{Y}, K; \boldsymbol{\sigma}), \quad \mathrm{I}_\Sigma(\boldsymbol{Y}, K; \boldsymbol{\sigma}) = \begin{cases} 0 & \text{if } (\boldsymbol{Y}, K) \in \Sigma \\ +\infty & \text{if } (\boldsymbol{Y}, K) \notin \Sigma \end{cases}, \tag{14}$$

where

$$\Sigma = \{(\boldsymbol{Y}, K) | f(\boldsymbol{Y}, K; \boldsymbol{\sigma}) \leq 0\} \tag{15}$$

is a convex set of admissible thermodynamic force $\boldsymbol{Y}$ and hardening variable K defined by the damage surface

$$f(\boldsymbol{Y}, K; \boldsymbol{\sigma}) = \frac{A\tilde{J}_2}{\sigma_{c0}} + \Lambda\sqrt{\tilde{J}_2} + BI_1 - (\sigma_{c0} + K) = 0, \tag{16}$$

where σ_{c0} denotes the initial elastic limit in uniaxial compression. The damage surface (16) is obtained by reformulating the deviatoric invariants of the Ottosen's 4-parameter failure surface in terms of the thermodynamic force $\boldsymbol{Y}$, hardening variable K, and stress $\boldsymbol{\sigma}$ as

$$\tilde{J}_2 = \frac{1}{1+\nu}\left[E \operatorname{tr} \boldsymbol{Y} - \tfrac{1}{6}(1-2\nu)(\operatorname{tr}\boldsymbol{\sigma})^2\right], \tag{17}$$

$$\tilde{J}_3 = \frac{2}{3(1+\nu)}\left\{E\left[\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{Y}) - \operatorname{tr}\boldsymbol{\sigma}\operatorname{tr}\boldsymbol{Y}\right] + \tfrac{1}{9}(1-2\nu)(\operatorname{tr}\boldsymbol{\sigma})^3\right\}. \tag{18}$$

The subdifferential of φ is defined by the set

$$\partial\varphi(\boldsymbol{Y}, K; \boldsymbol{\sigma}) = \begin{cases} \{(\boldsymbol{B}_Y, B_K)\}, & \text{if } (\boldsymbol{Y}, K) \in \Sigma, \\ \emptyset, & \text{if } (\boldsymbol{Y}, K) \notin \Sigma, \end{cases} \tag{19}$$

where $\boldsymbol{B}_Y$ and B_K are the components of the subgradient which is zero in the interior of the damage surface and equals to the the normal of the damage surface at point $(\boldsymbol{Y}, K)$ such that

$$(\boldsymbol{B}_Y, B_K) = \begin{cases} (\boldsymbol{0}, 0), & \text{if } f(\boldsymbol{Y}, K_\alpha; \boldsymbol{\sigma}) < 0, \\ \left(\dot{\lambda}\dfrac{\partial f}{\partial \boldsymbol{Y}}, \dot{\lambda}\dfrac{\partial f}{\partial K}\right), \ \dot{\lambda} \geq 0, & \text{if } f(\boldsymbol{Y}, K_\alpha; \boldsymbol{\sigma}) = 0. \end{cases} \tag{20}$$

The multiplier $\dot{\lambda}$ above can be calculated by use of the consistency condition $\dot{f} = 0$.

Finally, the specific constitutive equations can be obtained from equations (6), (10) and (20). In this work, the hardening variable K has the expression

$$K = -\rho_0\frac{\partial\psi^c}{\partial\kappa} = \frac{a_1\left(\kappa/\kappa_{\max}\right) + a_2\left(\kappa/\kappa_{\max}\right)^2}{1 + b\left(\kappa/\kappa_{\max}\right)^2}, \tag{21}$$

where $\kappa_{\max}$ corresponds to the value of κ when K reaches its maximum value $K_{\max}$ and a_1, a_2, b are material parameters to be determined. The expression (21) differs from the choice in [18].

Numerical example

The present model is used in analysis of a concrete specimen with the ultimate compressive strength of $\sigma_c = 32.8$ MPa. Parameters in the inital failure surface have been determined assuming the initial tensile strength $\sigma_{t0} = 1$ MPa and compressive strength $\sigma_{c0} = 18$ MPa, equibiaxial compressive strengh $1.16\,\sigma_{c0}$ and the point on the compressive meridian $(I_1, \sqrt{J_2}) = (-5\sqrt{3}\sigma_{c0}, 4\sigma_{c0}/\sqrt{2})$. Resulting values are $A = 2.6943, B = 5.4975, k_1 = 19.0829, k_2 = 0.9982$. The hardening parameters have the values $a_1 = 85.30\,\text{MPa}, a_2 = -12.65\,\text{MPa}$, $b = 0.7032$, $K_{\max} = 42.65\,\text{MPa}$ and $\kappa_{\max} = 4.41 \times 10^{-6}$.

Fig.2a shows the results obtained by the numerical model when the material is subjected to uniaxial compression. The result is compared to the available experimental data [9]. As can be seen in the figure, the results are in good agreement with the experiments. Damage evolution is shown in Fig.2b and the typical behaviour of concrete like materials in compression can be seen, i.e. damage in the planes parallel to the loading directions is dominating. The result shows the potential of the model to simulate the splitting failure of brittle materials.

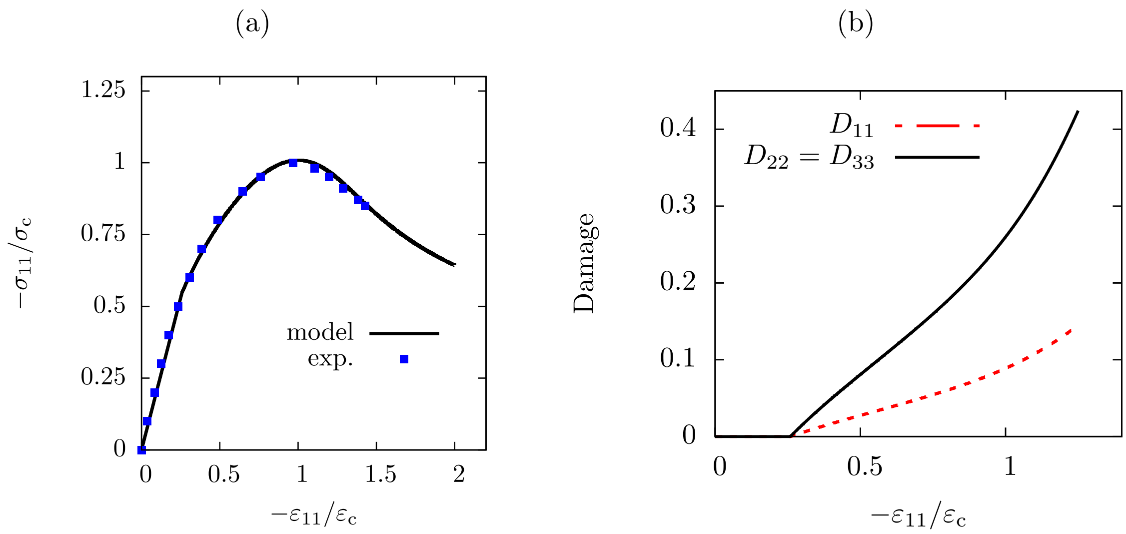

Figure 2. (a) Stress-strain diagram in uniaxial compression. Experimental data from Ref. [9]. (b) Damage evolution in uniaxial compression. Notice that damage is larger in the planes parallel to the loading direction indicating splitting failure mode.

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