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R. Kouhia, J. Mäkinen, S. Pajunen and T. Saksala (Eds.)


# On the direct solution of critical equilibrium states

Anders Eriksson[1], Reijo Kouhia[2], Jari Mäkinen[3]

[1]KTH Mechanics, Osquars backe 18, SE-10044 Stockholm, Sweden
[2]Dep. of Mechanical Engineering and Industrial Design, Tampere University of Technology
P.O. Box 589, FI-33101 Tampere, Finland
[3] Department of Civil Engineering, Tampere University of Technology
P.O. Box 589, FI-33101 Tampere, Finland
anderi@kth.se, reijo.kouhia@tut.fi, jari.m.makinen@tut.fi

**Summary.** Determination of a critical point is the primary problem in structural stability analysis. Mathematically it means solution of a non-linear eigenvalue problem together with the equilibrium equations. Several techniques exist to compute the critical equilibrium states and the corresponding modes. In this paper direct algorithms to solve the critical equilibrium state are discussed and a hybrid algorithm is proposed, which hopefully has enlarged domain of convergence.

*Key words:* computational stability analysis, finite element method, critical points, eigenvalue problem, non-linear systems

## Introduction

Proper stability analysis of structures, or other systems in various physical disciplines, requires the computation of the critical point and its sensitivity analysis, which in structural stability analysis is called as imperfection sensitivity analysis. In practice, the stability analysis is most often performed with the following two-step procedure. First, the linearized stability eigenvalue problem is computed, where the linearization is performed with respect to the unloaded, undeformed state. Second, a full non-linear analysis is performed, with an imperfect structure through some continuation (path-following) algorithm. The imperfection is usually taken as a combination of the lowest critical modes. Success of such an approach is very much depending on the proper choice of the perturbation. In most commercial finite element (FE) codes, this is the only possible approach for stability analysis.

The non-linear stability eigenvalue problem consists of solving the equilibrium equations simultaneously with the criticality condition. The first appearance of this idea seems to be from 1973 by Keener and Keller [1]. In their approach the criticality condition is augmented as an eigenvalue equation. Similar approaches have been used also in Refs. [2, 3, 4, 5]. Another approach uses a scalar equation indicating the criticality [6, 7] or expansion to a higher order polynomial eigenvalue problem [8, 9]. In the context of parametric investigations of instability behavior, several methods for defining and handling criticality have been discussed in [10, 11].

## Stability eigenvalue problem

The problem of finding a critical point along an equilibrium path can be stated as: find the critical values of $\boldsymbol{q}, \lambda$ and the corresponding eigenvector $\boldsymbol{\phi}$ such that

$$\begin{cases} \boldsymbol{f}(\boldsymbol{q},\lambda) &= \boldsymbol{0} \\ \boldsymbol{f}'(\boldsymbol{q},\lambda)\boldsymbol{\phi} &= \boldsymbol{0} \end{cases} \tag{1}$$

where $\boldsymbol{f}$ is a vector defining the equilibrium equations and $\boldsymbol{f}'$ denotes the Gateaux derivative (Jacobian matrix) with respect to the state variables $\boldsymbol{q}$, i.e. the stiffness matrix. At the critical point the equilibrium equations $(1)_1$ has to be satisfied together with the criticality condition $(1)_2$, which states the zero stiffness in the direction of the critical eigenmode $\boldsymbol{\phi}$. Such a system is considered in Refs. [3, 12, 5].

The equilibrium equation $(1)_1$ constitutes the balance of internal forces $\boldsymbol{r}$ and external loads $\boldsymbol{p}$, which is usually parameterized by a single variable $\lambda$, the load parameter, defining the intensity of the load vector:

$$\boldsymbol{f}(\boldsymbol{q},\lambda) \equiv \boldsymbol{r}(\boldsymbol{q}) - \lambda \boldsymbol{p}_{\mathrm{r}}(\boldsymbol{q}). \tag{2}$$

If the loads do not depend on the deformations, like in dead-weight loading, the reference load vector $\boldsymbol{p}_{\mathrm{r}}$ is independent of the displacement field $\boldsymbol{q}$. A more general case, with $\boldsymbol{p} = \boldsymbol{p}(\lambda, \boldsymbol{q})$ is an obvious expansion of the above expressions. A discussion on different loading control variables, and their effects on stability conclusions is given in [13].

The system (1) consists of $2n+1$ unknowns, the displacement vector $\boldsymbol{q}$, the eigenmode $\boldsymbol{\phi}$ and the load parameter value $\lambda$ at the critical state. Since the eigenvector $\boldsymbol{\phi}$ is defined uniquely up to a constant, a normalizing condition can be added to the system (1). In addition, some stabilizing conditions might also be needed. In general, the full augmented system can be written as

$$\boldsymbol{g}(\boldsymbol{q},\boldsymbol{\phi},\boldsymbol{\lambda}) = \begin{cases} \hat{\boldsymbol{f}}(\boldsymbol{q},\boldsymbol{\lambda}) \equiv \boldsymbol{f}(\boldsymbol{q},\lambda) + \boldsymbol{f}_0(\boldsymbol{q},\boldsymbol{\lambda}) = \boldsymbol{0} \\ \boldsymbol{h}(\boldsymbol{q},\boldsymbol{\phi},\boldsymbol{\lambda}) \equiv \boldsymbol{f}'(\boldsymbol{q},\boldsymbol{\lambda})\boldsymbol{\phi} + \boldsymbol{h}_0(\boldsymbol{\phi},\boldsymbol{\lambda}) = \boldsymbol{0} \\ \boldsymbol{c}(\boldsymbol{q},\boldsymbol{\phi},\boldsymbol{\lambda}) = \boldsymbol{0}, \end{cases} \tag{3}$$

where $\boldsymbol{\lambda}$ is a vector of control and auxiliary parameters and $\boldsymbol{c}$ is a vector of constraint or stabilizing equations: the dimension of these vectors is $p \geq 1$. The additional functions $\boldsymbol{f}_0$ and $\boldsymbol{h}_0$ are chosen such that $\boldsymbol{f}_0 = \boldsymbol{h}_0 = \boldsymbol{0}$ at the solution point. A Newton step for the approximate solution of (2) can thereby be written as

$$\begin{bmatrix} \boldsymbol{K}_f & \boldsymbol{0} & \boldsymbol{P} \\ \boldsymbol{Z} & \boldsymbol{K}_h & \boldsymbol{N} \\ \boldsymbol{C}_q & \boldsymbol{C}_\phi & \boldsymbol{C}_\lambda \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{\phi} \\ \delta\boldsymbol{\lambda} \end{Bmatrix} = - \begin{Bmatrix} \hat{\boldsymbol{f}} \\ \boldsymbol{h} \\ \boldsymbol{c} \end{Bmatrix}, \tag{4}$$

where

$$\boldsymbol{Z} = \left[\boldsymbol{f}'\boldsymbol{\phi}\right]', \quad \boldsymbol{C}_q = \boldsymbol{c}' = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{q}}, \quad \boldsymbol{C}_\phi = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\phi}}, \quad \boldsymbol{C}_\lambda = \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\lambda}}. \tag{5}$$

$$\boldsymbol{K}_f = \boldsymbol{K} + \boldsymbol{f}_0', \quad \boldsymbol{K}_h = \boldsymbol{K} + \frac{\partial \boldsymbol{h}_0}{\partial \boldsymbol{\phi}}, \quad \boldsymbol{P} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\lambda}} \quad \text{and} \quad \boldsymbol{N} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\lambda}} \tag{6}$$

Computation of the matrix $\boldsymbol{Z}$ requires second order derivatives of the residual. In the literature, these are usually obtained by numerical differentiation. For the geometrically exact Reissner's beam model, an analytical derivation of the $\boldsymbol{Z}$-matrix is given in [14].

For the eigenvector normalization different constraint equations can be used, [5, 14, 15].

The system (4) is usually solved by a block elimination scheme together with direct linear solvers. Utilization of iterative linear solvers has been discussed in [15].

The key problem in solving the extended system (3) with the Newton's method starting from the unloaded undeformed equilibrium state is that the Newton's method is only locally convergent. Therefore the domain of attraction can be small and it is likely that the initial state does not belong to it.

## Hybrid algorithm

A simple way to circumvent the problem related to the small convergence domain is to use a continuation algorithm to get closer to the domain of attraction of the extended system. A single "continuation step algorithm" for critical point computation could be constructed in the following way.

1. Compute a crude approximation to the lowest critical load and the corresponding eigenvector.
2. Use that point as a starting point of the orthogonal trajectory method [16, 17, 18] to get a nearby point on the equilibrium path.
3. From the computed equilibrium state, use the extended system (3) for computing the critical point.

It is believed that such an algorithm is more robust, but not computationally more demanding than the pure direct procedure.

## References


[1] Keener, J. & Keller, H. Perturbed bifurcation theory. *Archive for Rational Mechanics and Analysis* **50**, 159–175 (1973).

[2] Keener, J. Perturbed bifurcation theory at multiple eigenvalues. *Archive for Rational Mechanics and Analysis* **56**, 348–366 (1974).

[3] Seydel, R. Numerical computation of branch points in nonlinear equations. *Numerische Mathematik* **33**, 339–352 (1979).

[4] Wriggers, P., Wagner, W. & Miehe, C. A quadratically convergent procedure for the calculation of stability points in finite element analysis. *Computer Methods in Applied Mechanics and Engineering* **70**, 329–347 (1988).

[5] Wriggers, P. & Simo, J. A general procedure for the direct computation of turning and bifurcation problems. *International Journal for Numerical Methods in Engineering* **30**, 155–176 (1990).

[6] Abbot, J. An efficient algorithm for the determination of certain bifurcation points. *Journal Computational and Applied Mathematics* **4**, 19–27 (1987).

[7] Battini, J.-M., Pacoste, C. & Eriksson, A. Improved minimal augmentation procedure for the direct computation of critical points. *Computer Methods in Applied Mechanics and Engineering* **192**, 2169–2185 (2003).

[8] Huitfeldt, J. & Ruhe, A. A new algorithm for numerical path following applied to an example from hydrodynamic flow. *SIAM Journal on Scientific and Statistical Computing* **11**, 1181–1192 (1990).

[9] Huitfeldt, J. Nonlinear eigenvalue problems - prediction of bifurcation points and branch switching. Tech. Rep. 17, Department of Computer Sciences, Chalmers University of technology (1991).

[10] Eriksson, A. Fold lines for sensitivity analyses in structural instability. *Computer Methods in Applied Mechanics and Engineering* **114**, 77–101 (1994).

[11] Eriksson, A. Structural instability analyses based on generalised path-following. *Computer Methods in Applied Mechanics and Engineering* **156**, 45–74 (1998).

[12] Werner, B. & Spence, A. The computation of symmetry-breaking bifurcation points. *SIAM Journal on Numerical Analysis* **21**, 388–399 (1984).

[13] Eriksson, A. & Nordmark, A. Instability of hyper-elastic balloon-shaped space membranes under pressure loads. *Computer Methods in Applied Mechanics and Engineering* **237–240**, 118–129 (2012).

[14] Mäkinen, J., Kouhia, R., Fedoroff, A. & Marjamäki, H. Direct computation of critical equilibrium states for spatial beams and frames. *International Journal for Numerical Methods in Engineering* **89**, 135–153 (2012).

[15] Kouhia, R., Tůma, M., Mäkinen, J., Fedoroff, A. & Marjamäki, H. Implementation of a direct procedure for critical point computations using preconditioned iterative solvers. *Computers and Structures* **108–109**, 110–117 (2012).

[16] Haselgrove, C. The solution of non-linear equations and of differential equations with two point boundary conditions. *Computer Journal* **4**, 225–259 (1961).

[17] Fried, I. Orthogonal trajectory accession to the equilibrium curve. *Computer Methods in Applied Mechanics and Engineering* **47**, 283–297 (1984).

[18] Allgower, E. & Georg, K. *Numerical Continuation Methods - An Introduction* (Springer-Verlag, 1990).