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A model for anisotropic magnetostriction

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Summary. In this paper, a coupled magnetoelastic model for isotropic ferromagnetic materials commonly used in electrical machines is presented. The constitutive equations are written on the basis of the total energy in which the right Cauchy-Green strain tensor and the Lagrangian form of the magnetic field strength are used as the basic state variables.

Key words: magnetostriction, anisotropy, integrity basis, total energy function

Introduction

In transformers and rotating electrical machines magnetostriction is known to generate vibrations and acoustic noise. Electrical steel used in these machines is known to behave anisotropically. In this paper a model for anisotropic magnetostriction is developed. The model is based on the formulation introduced by Dorfmann and Ogden [1, 2, 3] utilizing the concept of "total energy function".

Lagrangian fields

In magnetoelastostatics the three basic magnetic variables are the magnetic field $\boldsymbol{H}$, the magnetic induction $\boldsymbol{B}$ and the magnetization $\boldsymbol{M}$. The fields $\boldsymbol{H}$ and $\boldsymbol{B}$ are considered as the primary fields and $\boldsymbol{M}$ only as an auxiliary field [4], which can be defined in terms of $\boldsymbol{H}$ and $\boldsymbol{B}$.

In electromagnetics it is customary to work with the Eulerian frame, so the field $\boldsymbol{H}$ and $\boldsymbol{B}$ are related to the current configuration. To model anisotropic behaviour where the material orientation is important, the material description of motion is preferable and the Lagrangian forms of the primary magnetic fields are

$$\boldsymbol{H}_{\mathrm{L}} \equiv \boldsymbol{F}^T \boldsymbol{H}, \qquad \text{and} \qquad \boldsymbol{B}_{\mathrm{L}} \equiv J\boldsymbol{F}^{-1}\boldsymbol{B}, \tag{1}$$

where $\boldsymbol{F}$ is the deformation gradient and $J = \det \boldsymbol{F}$. For further details see [1, 2, 3].

Constitutive equations

General form

Denoting the complementary form of the total energy function as $\Omega^*(\boldsymbol{F}, \boldsymbol{H}_{\mathrm{L}})$, and using the standard Coleman-Noll procedure, the total stress $\boldsymbol{\tau}$ and the magnetic induction $\boldsymbol{B}$ can be obtained from equations

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\frac{\partial\Omega^*}{\partial\boldsymbol{F}}, \qquad \boldsymbol{B} = -J^{-1}\boldsymbol{F}\frac{\partial\Omega^*}{\partial\boldsymbol{H}_\mathrm{L}}. \tag{2}$$

The function $\Omega^*(\boldsymbol{F}, \boldsymbol{H}_\mathrm{L})$ is a partial Legendre transform of the total energy function $\Omega(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L})$, i.e.

$$\Omega^*(\boldsymbol{F}, \boldsymbol{H}_\mathrm{L}) = \Omega(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L}) - \boldsymbol{H}_\mathrm{L}\cdot\boldsymbol{B}_\mathrm{L}. \tag{3}$$

The total energy function Ω is related to the Helmholtz free energy per unit mass ψ as

$$\Omega \equiv \rho_0\Phi + \tfrac{1}{2}\mu_0^{-1}J\boldsymbol{B}\cdot\boldsymbol{B}, \qquad \text{where} \qquad \Phi(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L}) \equiv \psi(\boldsymbol{F}, J^{-1}\boldsymbol{F}\boldsymbol{B}_\mathrm{L}) \tag{4}$$

in which ρ_0, μ_0 are the density in the reference configuration and the magnetic permeability in vacuum, respectively.

Intregrity basis

For modelling ansiotropic magnetostriction, it has been chosen that the energy function depends on the right Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, the Lagrangian magnetic field $\boldsymbol{H}_\mathrm{L}$ and two direction vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, not necessarily orthogonal to each other. Integrity basis of a scalar function depending of a symmetric second order tensor and three vectors consist of the following 21 invariants [5]:

$$\begin{aligned}
&I_1 = \mathrm{tr}\,\boldsymbol{C}, && I_2 = \tfrac{1}{2}[(\mathrm{tr}\,\boldsymbol{C})^2 - \mathrm{tr}\,\boldsymbol{C}^2], && I_3 = \det\boldsymbol{C}, && I_4 = \boldsymbol{H}\cdot\boldsymbol{H}, && I_5 = \boldsymbol{H}\cdot\boldsymbol{a}_1,\\
&I_6 = \boldsymbol{H}\cdot\boldsymbol{a}_2, && I_7 = \boldsymbol{a}_1\cdot\boldsymbol{a}_2, && I_8 = \boldsymbol{H}\cdot\boldsymbol{C}\boldsymbol{H}, && I_9 = \boldsymbol{a}_1\cdot\boldsymbol{C}\boldsymbol{a}_1, && I_{10} = \boldsymbol{a}_2\cdot\boldsymbol{C}\boldsymbol{a}_2,\\
&I_{11} = \boldsymbol{H}\cdot\boldsymbol{C}\boldsymbol{a}_1, && I_{12} = \boldsymbol{H}\cdot\boldsymbol{C}\boldsymbol{a}_2, && I_{13} = \boldsymbol{a}_1\cdot\boldsymbol{C}\boldsymbol{a}_2, && I_{14} = \boldsymbol{H}\cdot\boldsymbol{C}^2\boldsymbol{H}, && I_{15} = \boldsymbol{a}_1\cdot\boldsymbol{C}^2\boldsymbol{a}_1,\\
&I_{16} = \boldsymbol{a}_2\cdot\boldsymbol{C}^2\boldsymbol{a}_2, && I_{17} = \boldsymbol{H}\cdot\boldsymbol{C}^2\boldsymbol{a}_1, && I_{18} = \boldsymbol{H}\cdot\boldsymbol{C}^2\boldsymbol{a}_2, && I_{19} = \boldsymbol{a}_1\cdot\boldsymbol{C}^2\boldsymbol{a}_2, && I_{20} = \boldsymbol{a}_1\cdot\boldsymbol{a}_1,\\
&I_{21} = \boldsymbol{a}_2\cdot\boldsymbol{a}_2.
\end{aligned} \tag{5}$$

Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unit vectors, i.e. $\boldsymbol{a}_1\cdot\boldsymbol{a}_1 = \boldsymbol{a}_2\cdot\boldsymbol{a}_2 = 1$, there are only 19 invariants in the anisotropic magnetoelastic model.

Total stress tensor and magnetic induction

From equations (2) expressions to the total stress tensor $\boldsymbol{\tau}$ and the magnetic induction vector $\boldsymbol{B}$ are

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\sum_{k=1}^{19}\frac{\partial\Omega^*}{\partial I_k}\frac{\partial I_k}{\partial\boldsymbol{F}}, \tag{6}$$

$$\boldsymbol{B} = -J^{-1}\boldsymbol{F}\sum_{k=1}^{19}\frac{\partial\Omega^*}{\partial I_k}\frac{\partial I_k}{\partial\boldsymbol{H}_\mathrm{L}}. \tag{7}$$

Evaluation of the derivatives $\partial\Omega^*/\partial\boldsymbol{F}$ and $\partial\Omega^*/\partial\boldsymbol{H}_\mathrm{L}$ gives

$$\begin{aligned}
\boldsymbol{\tau} =& J^{-1}[2\boldsymbol{b}\Omega_1^* + 2(I_1\boldsymbol{b} - \boldsymbol{b}^2)\Omega_2^* + 2I_3\Omega_3^* + 2\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{H}\Omega_8^* + 2\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{F}\boldsymbol{a}_1\Omega_9^* +\\
&+ 2\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{F}\boldsymbol{a}_2\Omega_{10}^* + (\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{b}\boldsymbol{H} + \boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{F}\boldsymbol{a}_1)\Omega_{12}^* + (\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{b}\boldsymbol{H} + \boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{F}\boldsymbol{a}_2)\Omega_{13}^* +\\
&+ 2(\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}^2\boldsymbol{H} + \boldsymbol{b}^2\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{H})\Omega_{14}^* + (\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{F}\boldsymbol{a}_1)\Omega_{15}^* +\\
&+ (\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{F}\boldsymbol{a}_2)\Omega_{16}^* + 2(\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{b}\boldsymbol{H})\Omega_{17}^* +\\
&+ 2(\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{b}\boldsymbol{H})\Omega_{18}^* +\\
&+ (\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1\otimes\boldsymbol{F}\boldsymbol{a}_2 + \boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1 + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2\otimes\boldsymbol{F}\boldsymbol{a}_1)\Omega_{19}^*],
\end{aligned} \tag{8}$$

$$\begin{aligned}
\boldsymbol{B} =& -J^{-1}(2\boldsymbol{b}\boldsymbol{H}\Omega_4^* + \boldsymbol{F}\boldsymbol{a}_1\Omega_5^* + \boldsymbol{F}\boldsymbol{a}_2\Omega_6^* + 2\boldsymbol{b}\boldsymbol{H}\Omega_8^* + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_1\Omega_{11}^* + \boldsymbol{b}\boldsymbol{F}\boldsymbol{a}_2\Omega_{12}^* +\\
&+ 2\boldsymbol{b}^3\boldsymbol{H}\Omega_{14}^* + \boldsymbol{b}^2\boldsymbol{F}\boldsymbol{a}_1\Omega_{17}^* + \boldsymbol{b}^2\boldsymbol{F}\boldsymbol{a}_2\Omega_{18}^*),
\end{aligned} \tag{9}$$

where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green deformation tensor, the notation Ω_i^* denotes the derivative $\Omega_i^* = \partial\Omega^*/\partial I_i$ and $\otimes$ is the standard tensor product. The specific form of the total energy function is now to be determined based on experimental evidence.

References

[1] A. Dorfmann and R.W. Ogden. Nonlinear magnetoelastic deformation of elastomers. *Acta Mechanica*, 167:13–28, 2004.

[2] A. Dorfmann and R.W. Ogden. Nonlinear magnetoelastic deformations. *Quarterly Journal of Mechanics and Applied Mathematics*, 57:599–622, 2004.

[3] A. Dorfmann and R.W. Ogden. Some problems in nonlinear magnetoelasticity. *Zeitschrift für Angewandte Mathematik und Physik*, 56:718–745, 2005.

[4] A. Kovetz. *Electromagnetic Theory*. Oxford University Press, 2000.

[5] A.J.M. Spencer. Theory of invariants. In A.C. Eringen, editor, *Continuum Physics*, volume 1, pages 239–353. Academic Press, 1971.