## Problem 1

Let's investigate the numerical solution of the Maxwell-type creep problem

$$\sigma = E(\epsilon - \epsilon_c). \tag{1}$$

The creep strain rate $\dot{\epsilon}_c$ is obtained from

$$\dot{\epsilon}_c = \tau_{pr}^{-1}\left(\frac{\sigma}{\sigma_r}\right), \tag{2}$$

where $\tau_{pr}$ is the "pseudo"relaxation time (constant) and $\sigma_r$ is a reference stress (constant). The ralaxation time is $\tau = \tau_{pr}\epsilon_r$, where $\epsilon_r = \sigma_r/E$.

Formulate the problem as a first order ordinary differential equation for the stress and solve it numerically using the implicit Euler method. The loading is a constant strain rate: $\epsilon(t) = \tau^{-1}\epsilon_r t$. Integrate to the final time $t = 4\tau$ by using a time step $\Delta t = 2\tau$. **Hint:** Formulate the equation (1) is a dimensionless form using a dimensionless stress $y = \sigma/\sigma_r$. When you differentiate the equation (1) w.r.t. time, you can assume the Young's modulus $E$ to be a constant.

## Solution

Taking the time derivative of the constitutive equation (1) gives

$$\dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}_c). \tag{3}$$

Inserting the creep model (2) into it, gives

$$\dot{\sigma} = E\dot{\epsilon} - \frac{1}{\tau_{pr}}\frac{E}{\sigma_r}\sigma, \tag{4}$$

which after rearrangements has the form

$$\dot{\sigma} + \frac{1}{\tau}\sigma = \frac{E\epsilon_r}{\tau} \quad \rightarrow \quad \frac{\dot{\sigma}}{\sigma_r} + \frac{1}{\tau}\frac{\sigma}{\sigma_r} = \frac{1}{\tau}.$$

denoting $y = \sigma/\sigma_r$ we have the ordinary constant coefficient differential equation

$$\dot{y} + \frac{1}{\tau}y = \frac{1}{\tau}. \tag{5}$$

Assuming that the solution is known at time instance $t = t_n$, the implicit Euler method is obtained when the equation (5) is expressed at time $t_{n+1}$ and the time derivative at time instance $t_{n+1}$ is replaced by the backward difference expression:

$$\dot{y}_{n+1} + \frac{1}{\tau}y_{n+1} = \frac{1}{\tau}, \quad \dot{y}_{n+1} \approx \frac{y_{n+1} - y_n}{\Delta t},$$

which gives

$$\left(1 + \frac{\Delta t}{\tau}\right)y_{n+1} = y_n + \frac{Deltat}{\tau}.$$

Using the time step $\Delta t = 2\tau$ gives

$$\begin{aligned} y_0 &= 0, \\ y_1 &= \tfrac{1}{3}(0 + 2) = \tfrac{2}{3}, \\ y_2 &= \tfrac{1}{3}(\tfrac{2}{3} + 2) = \tfrac{8}{9}. \end{aligned}$$

The stress at time $t = 4\tau$ is thus $\sigma_2 = \frac{8}{9}\sigma_r$.

## Problem 2

Investigate the stability of the Crank-Nicolson scheme for the problem

$$\dot{y} + a(t)y = 0, \quad y(0) = y_0, \quad \text{where} \quad a(t) > 0.$$

## Solution

The Crank-Nicolson method, or the trapezoidal rule is obtained when the time derivative is evaluated as an average of the values at $t = t_n$ and at $t = t_{n+1}$, i.e.

$$\dot{y}_{n+\frac{1}{2}} \approx \frac{y_{n+1} - y_n}{\Delta t} = \tfrac{1}{2}(\dot{y}_{n+1} + \dot{y}_n) = -\tfrac{1}{2}(a_{n+1}y_{n+1} + a_n y_n),$$

which gives

$$y_{n+1} = \frac{1 - \frac{1}{2}a_n \Delta t}{1 + \frac{1}{2}a_{n+1}\Delta t} y_n.$$

The stability condition is

$$\left| \frac{1 - \frac{1}{2}a_n \Delta t}{1 + \frac{1}{2}a_{n+1}\Delta t} \right| < 1,$$

which is equivalent to

$$-1 < \frac{1 - \frac{1}{2}a_n \Delta t}{1 + \frac{1}{2}a_{n+1}\Delta t} < 1.$$

Two cases to be checked:

$$\frac{1 - \frac{1}{2}a_n \Delta t}{1 + \frac{1}{2}a_{n+1}\Delta t} > -1 \quad \Rightarrow \quad (a_n - a_{n+1})\Delta t < 4, \tag{6}$$

and

$$\frac{1 - \frac{1}{2}a_n \Delta t}{1 + \frac{1}{2}a_{n+1}\Delta t} < 1 \quad \Rightarrow \quad \tfrac{1}{2}(a_n + a_{n+1})\Delta t > 0. \tag{7}$$

Since $\Delta t > 0$ and $a(t) > 0$ the second condition (7) is always fulfilled. However, the condition (6) limits the time step when the coefficient $a$ is a decreasing function during the time period $(t_n, t_{n+1})$, i.e. when $a_{n+1} < a_n$. Therefore the scheme is only conditionally stable. If the coefficient $a(t)$ is an increasing function during the time step, then the Crank-Nicolson scheme is stable for arbitrary time steps.