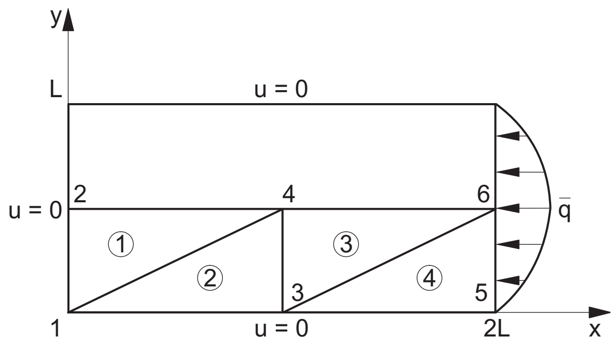
Problem 1

Solve the stationary two-dimensional heat transfer problem shown in the figure below by using linear elements. Use symmetry to reduce the problem size. The material is assumed to be homogeneous and isotropic with thermal conductivity k. The loading is given with prescribed heat flux on the boundary $x = L$ as $\vec{q}_s = -4\bar{q}_c(y/L)(1 - y/L)\vec{i}$.



Solution

The weak form is simply (now $f \equiv 0$)

$$\int_\Omega (\nabla w)^T k \nabla u \, dA = -\int_\Omega w q_n \, dA.$$

Using the elementwise interpolation $u^{(e)} = \boldsymbol{N}\boldsymbol{u}^{(e)}$ and $w^{(e)} = \boldsymbol{N}\boldsymbol{w}^{(e)}$, the element stiffness matrix has the form

$$\int_{\Omega^{(e)}} k\boldsymbol{B}^T\boldsymbol{B} \, dA,$$

where $\boldsymbol{B}$ is the discrete gradient operator matrix. In the case of linear elements, it has the form

$$\boldsymbol{B} = \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The elements 1 and 3 are equal as well as elements 2 and 4. The local nodes are numerated as shown in the table below thus for the elements 1 and 3 the coefficients which are needed are:

$$\begin{aligned}
b_1 &= y_2 - y_3 = 0, & c_1 &= x_3 - x_2 = -L, \\
b_2 &= y_3 - y_1 = \tfrac{1}{2}L, & c_2 &= x_1 - x_3 = 0, \\
b_3 &= y_1 - y_2 = -\tfrac{1}{2}L, & c_3 &= x_2 - x_1 = L.
\end{aligned}$$

	node		
elem.	1	2	3
1	1	4	2
2	1	3	4
3	3	6	4
4	3	5	6

The area of all elements is $A = \frac{1}{4}L^2$, thus

$$\mathbf{K}^{(1)} = \mathbf{K}^{(3)} = k \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} + k \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \frac{k}{4} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 1 & -1 \\ -4 & -1 & 5 \end{bmatrix}.$$

For elements 2 and 4:

$$\begin{array}{lllll} b_1 & = & y_2 - y_3 & = & -\frac{1}{2}L, \\ b_2 & = & y_3 - y_1 & = & \frac{1}{2}L, \\ b_3 & = & y_1 - y_2 & = & 0, \end{array} \qquad \begin{array}{lllll} c_1 & = & x_3 - x_2 & = & 0, \\ c_2 & = & x_1 - x_3 & = & -L, \\ c_3 & = & x_2 - x_1 & = & L. \end{array}$$

$$\mathbf{K}^{(2)} = \mathbf{K}^{(4)} = k \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} + k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{k}{4} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{bmatrix}.$$

Since the teperature is prescribed for nodes 1,2,3 and 5 there are only two active unknowns, i.e. u_4 and u_6. The local-global degrees of freedom are related as shown in the table below (same table as before, but now the essential boundary nodes are not shown)

	node		
elem.	1	2	3
1	-	4	-
2	-	-	4
3	-	6	4
4	-	-	6

The global stiffness matrix can be assembled from the element contributions as

$$\begin{aligned} K_{44} &= K_{22}^{(1)} + K_{33}^{(2)} + K_{33}^{(3)}, \\ K_{46} &= K_{32}^{(3)}, \\ K_{66} &= K_{22}^{(3)} + K_{33}^{(4)}, \end{aligned}$$

thus

$$\mathbf{K} = \begin{bmatrix} K_{44} & K_{46} \\ K_{64} & K_{66} \end{bmatrix} = \frac{k}{4} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}.$$

In the load vector there is only one non-zero component, which is

$$f_6 = -\int N_3 \vec{q} \cdot \vec{n} ds = \int N_3 4q_c (y/L)\left(1 - (y/L)\right) ds = \frac{L}{12}(q_c + 4\tfrac{1}{2}\tfrac{3}{4}q_c + 0) = \tfrac{5}{24}q_c L.$$

The global balance equations are

$$\frac{k}{4} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{5}{24} \end{Bmatrix} q_c L,$$

which has the solution

$$\begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} = \frac{1}{49} \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{5}{24} \end{Bmatrix} \frac{q_c L}{k} = \frac{5}{294} \begin{Bmatrix} 1 \\ 10 \end{Bmatrix} \frac{q_c L}{k}.$$

The heat flux is constant in each element

$$\boldsymbol{q}^{(e)} = -k\boldsymbol{B}\boldsymbol{u}^{(e)} = -\frac{k}{2A^{(e)}}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{Bmatrix}.$$

For element 4, $u_1^{(4)} = u_2^{(4)} = 0, u_3^{(4)} = u_6$, thus

$$\boldsymbol{q}^{(e)} = -\frac{k}{2A^{(e)}}\begin{Bmatrix} b_3 \\ c_3 \end{Bmatrix} u_6 = \begin{Bmatrix} 0 \\ -50/147\bar{q}_c \end{Bmatrix}.$$

Calculate the heat flux for other elements and draw the vectors!

Problem 2

Compute the St. Venant's torsion constant I_t for a beam having a square cross-section (side length L) and made of a homogeneous isotropic material, by using the finite element method and a triangular mesh as shown in the figure below. The problem can be formulated with St. Venant's stress function Φ as

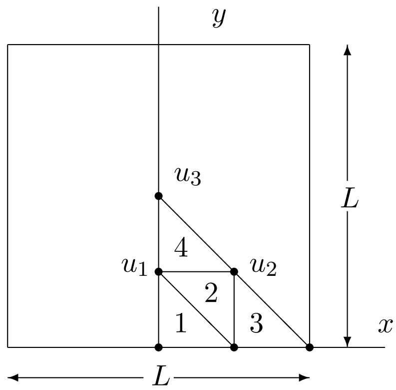
$$-\Phi_{,xx} - \Phi_{,yy} = 2G\theta,$$

with boundary conditions $\Phi = 0$. The torsional constant is obtained from equation

$$I_t = \frac{2}{G\theta}\int_\Omega \Phi(x,y)dA.$$

Determine also the shear stress distribution from a twist $\theta = 1/L$. The shear stresses can be computed from

$$\tau_{zx} = \Phi_{,y}, \qquad \tau_{zy} = -\Phi_{,x}.$$



Solution

The weak form of the PDE is

$$-\int_\Omega \hat{\Phi}\Delta\Phi\,dA = \int \hat{\Phi}2G\theta\,dA.$$

The first integral can be transformed as

$$-\int_\Omega \hat{\Phi}\,\Delta\Phi\,dA = -\int_\Omega \hat{\Phi}\;\nabla\cdot\nabla\Phi\,dA = -\oint_{\partial\Omega} \hat{\Phi}\;\nabla\Phi\cdot\mathbf{n}\,dS + \int_\Omega \nabla\hat{\Phi}\cdot\nabla\Phi\,dA = \int_\Omega \nabla\hat{\Phi}\cdot\nabla\Phi\,dA.$$

The stiffness matris is thus similar to the heat transfer problem when the thermal conductivity $k = 1$.

The element matrices from elements 1, 3 and 4 are identical. Thus it is necessary to form only the element matrices for elements 1 and 2. The constants b_i and c_i

$$\begin{aligned} b_1 &= y_2 - y_3, & c_1 &= x_3 - x_2 \\ b_2 &= y_3 - y_1, & c_2 &= x_1 - x_3 \\ b_3 &= y_1 - y_2, & c_3 &= x_2 - x_1 \end{aligned} \tag{1}$$

are calculated in the table below

	elements 1,3,4		element 2	
i	b_i	c_i	b_i	c_i
1	-$L/4$	-$L/4$	0	-$L/4$
2	$L/4$	0	$L/4$	$L/4$
3	0	$L/4$	-$L/4$	0

The area of all elements is $A^{(e)} = L^2/32$ and the element matrices are

$$\begin{aligned} \boldsymbol{K}^{(1)} &= \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{2}$$

$$\begin{aligned} \boldsymbol{K}^{(2)} &= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \end{aligned} \tag{3}$$

The local-global numbering is shown in the table below

	node		
elem.	1	2	3
1	-	-	1
2	-	2	1
3	-	-	2
4	1	2	3

The assembly of the global stiffness matrix is below

$$\begin{aligned} K_{11} &= K_{33}^{(1)} + K_{33}^{(2)} + K_{11}^{(4)}, \\ K_{12} &= K_{32}^{(2)} + K_{12}^{(4)}, \\ K_{13} &= K_{13}^{(4)}, \\ K_{22} &= K_{22}^{(2)} + K_{33}^{(3)} + K_{22}^{(4)}, \\ K_{23} &= K_{23}^{(4)}, \\ K_{33} &= K_{33}^{(4)}, \end{aligned} \tag{4}$$

The global stiffness matrix is thus

$$\boldsymbol{K} = \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ -1 & 2 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \tag{5}$$

The inverse of the stiffness matrix is

$$\boldsymbol{K}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & 3 \end{bmatrix}. \tag{6}$$

The load vector is

$$\begin{aligned} f_1 &= f_3^{(1)} + f_3^{(2)} + f_1^{(4)} = 3 \cdot \tfrac{2}{3} G\theta \tfrac{1}{32} L^2 = \tfrac{1}{16} G\theta L^2, \\ f_2 &= \tfrac{1}{16} G\theta L^2, \\ f_3 &= \tfrac{1}{48} G\theta L^2. \end{aligned} \tag{7}$$

Finally the nodal values of the stress function Φ_i can be computed as

$$\begin{aligned} \Phi_1 &= \tfrac{11}{96} GL^2\theta \approx 0.1146 GL^2\theta, \\ \Phi_2 &= \tfrac{17}{192} GL^2\theta \approx 0.0885 GL^2\theta, \\ \Phi_3 &= \tfrac{5}{32} GL^2\theta \approx 0.1563 GL^2\theta. \end{aligned} \tag{8}$$

Let's compute the integral

$$I_t = \frac{2}{G\theta} \int_\Omega \Phi dA = 8 \frac{2}{G\theta} \sum_{e=1}^{4} \int_{\Omega^{(e)}} \Phi dA.$$

Since the stress function Φ is linear in each element, the element integrals are easy to compute

$$\int_{\Omega^{(e)}} \Phi dA = \sum_{i=1}^{3} \int_{\Omega^{(e)}} N_i \Phi_i dA = \frac{1}{3} A^{(e)} \sum_{i=1}^{3} \Phi_i,$$

and ($A^{(e)} = L^2/32$)

$$\begin{aligned} \frac{1}{G\theta} \int_{\Omega^{(1)}} \Phi dA &= \frac{1}{3} \frac{L^4}{32} \frac{11}{96}, \\ \frac{1}{G\theta} \int_{\Omega^{(2)}} \Phi dA &= \frac{1}{3} \frac{L^4}{32} \left(\frac{11}{96} + \frac{17}{192} \right), \\ \frac{1}{G\theta} \int_{\Omega^{(3)}} \Phi dA &= \frac{1}{3} \frac{L^4}{32} \frac{17}{192}, \\ \frac{1}{G\theta} \int_{\Omega^{(2)}} \Phi dA &= \frac{1}{3} \frac{L^4}{32} \left(\frac{11}{96} + \frac{17}{192} + \frac{5}{32} \right). \end{aligned}$$

The value for the torsion constant is thus

$$I_t = 16 \tfrac{1}{3} \tfrac{1}{32} L^4 \left(3 \tfrac{11}{96} + 3 \tfrac{17}{192} + \tfrac{5}{32} \right) = \tfrac{147}{1152} L^4 \approx 0.1276 L^4.$$

The exact solution with three significant digits is $I_t = 0.141 L^4$.

let's finally compute the shear stresses, which are constants in each element since we have used linear elements:

$$\tau_{zx} = \Phi_{,y}, \qquad \tau_{zy} = -\Phi_{,x}.$$

Element 1:

$$\begin{aligned} \Phi &= N_3 \Phi_1, \\ \Phi_{,x} &\equiv 0, \\ \Phi_{,y} &= \frac{c_3}{2A} \Phi_1 = \frac{11}{24} GL\theta. \end{aligned}$$

Element 2:

$$
\begin{aligned}
\Phi &= N_2\Phi_2 + N_3\Phi_1, \\
\Phi_{,x} &= \frac{b_2}{2A}\Phi_2 + \frac{b_3}{2A}\Phi_1 = -\frac{5}{48}GL\theta, \\
\Phi_{,y} &= \frac{c_2}{2A}\Phi_2 + \frac{c_3}{2A}\Phi_1 = \frac{17}{48}GL\theta.
\end{aligned}
$$

Element 3:

$$
\begin{aligned}
\Phi &= N_3\Phi_2, \\
\Phi_{,x} &\equiv 0, \\
\Phi_{,y} &= \frac{c_3}{2A}\Phi_2 = \frac{17}{48}GL\theta.
\end{aligned}
$$

The element 4 correspondingly.