Problem

Solve the 1-D stationary heat transfer problem

$$-(ku')' = 0$$

of a wall $x \in (0, L)$ using FEM. Let's assume that the outside temperature at $x = 0$ is $u_0 > 0$. What is the power needed at $x = L$ to maintain the inside temperature $2u_0$? The conductivity of the wall is defined as

$$k(x) = \begin{cases} 34k_0, & x \in (0, L_1) = (0, \frac{3}{14}L), \quad \text{concrete}, \\ k_0, & x \in (L_1, L_2) = (\frac{3}{14}L, \frac{13}{14}L), \quad \text{glass wool}, \\ 4k_0, & x \in (L_2, L) = (\frac{13}{14}L, L), \quad \text{gypsum}. \end{cases}$$

What is the thermal transmittance (suom. lämmönläpäisykerroin), i.e. the U-value of the wall. The values are $k_0 = 0.05$ W/(mK) and $L = 0.28$ m ($L_1 = 6$ cm, $L_2 = 26$ cm).

Solution: Let's discretize the wall into three linear elements, one for the concrete, one for the glass wool and one for the gypsum plate. Since at the both boundaries essential boundary conditions are given, thus there are only two unknowns. The weak form is

$$\int_0^L \hat{u}' k u' \, dx = 0.$$

For linear element the stiffness matrix (conductivity matrix) is

$$\boldsymbol{K}^{(e)} = \frac{k^{(e)}}{h^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For the element of the concrete part:

$$\boldsymbol{K}^{(1)} = \frac{34k_0}{\frac{3}{14}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{476k_0}{3L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

glass wool

$$\boldsymbol{K}^{(2)} = \frac{k_0}{\frac{10}{14}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{7k_0}{5L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and for the gypsum plate

$$\boldsymbol{K}^{(3)} = \frac{4k_0}{\frac{1}{14}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{56k_0}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The global system is thus

$$\frac{k_0}{L} \begin{bmatrix} \frac{476}{3} & -\frac{476}{3} & 0 & 0 \\ -\frac{476}{3} & \frac{476}{3} + \frac{7}{5} & -\frac{7}{5} & 0 \\ 0 & -\frac{7}{5} & \frac{7}{5} + 56 & -56 \\ 0 & 0 & -56 & 56 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Since the boundary temperatures are prescribed, i.e. $u_1 = u_0$ and $u_4 = 2u_0$ there are only two active unknowns. The first and fourth equations above are unnecessary. Moving the terms related to u_1 and u_4 to the r.h.s. gives

$$\begin{bmatrix} 160.0\bar{6} & -1.4 \\ -1.4 & 57.4 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 158.\bar{6} \\ 112 \end{Bmatrix} u_0.$$

The solution is

$$u_2 = 1.008535u_0, \quad u_3 = 1.975818u_0.$$

The heat flux is constant over the wall, let's compute it from the element 1

$$q^{(1)} = -k^{(1)}u' = -34k_0\frac{(1.008535 - 1.0)u_0}{\frac{3}{14}L} = -1.5129\frac{k_0 u_0}{L}.$$

The flux is thus directing to the left. The required power inside is thus the absolute value of the flux.

If $u_0 = 1°\mathrm{C}$, then

$$q^{(1)} = -0.2418\frac{\mathrm{W}}{\mathrm{m}^2}$$

The U-value is thus 0.2418 $\mathrm{W/m^2K}$.

Problem:

Solve by FEM the following stationary 1-dimensional diffusion-reaction equation

$$-ku'' + cu = 0, \qquad u(0) = 0, u(L) = \bar{u}_L,$$

where k, c are positive constants $c = \beta^2 k L^{-2}$. Use three equal elements in the domain. Perform computations with the values $\beta = 1$ and 100.

Compute the problem also in the case where the part

$$\int c\hat{u}u dx$$

in the conductivity matrix is lumped. A lumped matrix is obtained as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} + a_{12} & 0 \\ 0 & a_{21} + a_{22} \end{bmatrix}.$$

What can be concluded?

Solution: Let's multiply the equation by the weight function (or test function) $\hat{u}$ and integerate the resultin terms over the domain and integrate by parts the second order derivative term. Due to Dirichlet's boundary conditions we have $\hat{u}(0) = \hat{u}(L) = 0$ and we get

$$\begin{aligned} -ku'' + cu &= 0 \\ -\int_0^L ku''\hat{u}dx + \int_0^L cu\hat{u}dx &= 0 \\ -k\Big|_0^L u'\hat{u} + k\int_0^L u'\hat{u}'dx + c\int_0^L u\hat{u}dx &= 0 \\ k\int_0^L u'\hat{u}'dx + c\int_0^L u\hat{u}dx &= 0 \end{aligned}$$

Using the method of Galerkin and divide the domain into three equal elements $u = \sum_i N_i u_i, \quad \hat{u} = \sum_i N_i \hat{u}_i$.

$$\sum_e \left[\int_{x_0^e}^{x_1^e} k \sum_j \frac{d}{dx} N_j u_j \sum_i \frac{d}{dx} N_i \hat{u}_i dx + \int_{x_0^e}^{x_1^e} c \sum_j N_j u_j \sum_i N_i \hat{u}_i dx \right] = 0$$

substituting $\xi = \frac{2}{h}(x - x_c), \quad dx = \frac{h}{2} d\xi, \quad \frac{d}{dx} = \frac{2}{h}\frac{d}{d\xi}$

$$\sum_e \left[\sum_i \hat{u}_i \sum_j \left(\tfrac{2k}{h} \int_{-1}^1 N_i' N_j' d\xi + \tfrac{ch}{2} \int_{-1}^1 N_i N_j d\xi \right) u_i \right] = 0$$

$$\Rightarrow K_{ij}^{(e)} = \tfrac{2k}{h} \int_{-1}^1 N_i' N_j' d\xi + \tfrac{ch}{2} \int_{-1}^1 N_i N_j d\xi$$

Using linear elements, the interpolation functions are $N_1 = \frac{1}{2}(1-\xi)$ and $N_2 = \frac{1}{2}(1+\xi)$. The element stiffness matrix is

$$\begin{aligned}
\boldsymbol{K}^{(e)} &= \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{ch}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\beta^2 kh}{6L^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \frac{k}{h}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\beta^2}{6}\left(\frac{h}{L}\right)^2\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right).
\end{aligned}$$

In this case $h = L/3$ the contribution from one element is

$$\boldsymbol{K}^{(e)} = \frac{3k}{L}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\beta^2}{54}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right).$$

The global equilibrium equation is

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & K_{21}^{(3)} & K_{11}^{(3)} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ \bar{u}_L \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

which can be written as

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -K_{12}^{(3)}\bar{u}_L \end{Bmatrix}.$$

Let's insert the parameter values

$$\begin{bmatrix} 108 + 4\beta^2 & \beta^2 - 54 \\ \beta^2 - 54 & 108 + 4\beta^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 54 - \beta^2 \end{Bmatrix}\bar{u}_L.$$

with $\beta = 1$ and $\beta = 100$ we get the equations

$$\begin{bmatrix} 112 & -53 \\ -53 & 112 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 53 \end{Bmatrix}\bar{u}_L$$

and

$$\begin{bmatrix} 40108 & 9947 \\ 9947 & 40108 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -9947 \end{Bmatrix}\bar{u}_L$$

solutions for nodal temperatures u_2 and u_3 are:

$$(\beta = 1): \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.2885 \\ 0.6098 \end{Bmatrix}\bar{u}_L, \qquad (\beta = 100): \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.0655 \\ -0.2643 \end{Bmatrix}\bar{u}_L.$$

For large values of the reaction coefficient c the FE solution starts to oscillate. The solutions are drawn in figure 1.

If the submatrix corresponding to the reaction term is lumped, i.e. the consistent matrix

$$\frac{ch}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

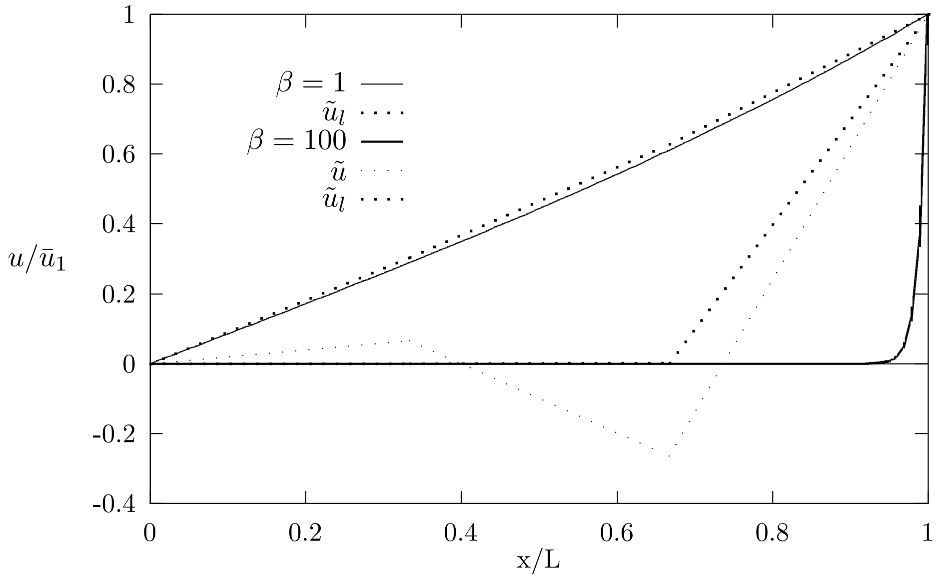

Figure 1: Numerical solutions by linear FEM as compared to the exact solution (solid line). Solution with the standard Galerkin is drawn by dashed line and the "lumped" solution by dotted line.

is replaced by the lumped one

$$\frac{ch}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we get the element stiffness matrix (conductivity matrix)

$$\begin{aligned} \boldsymbol{K}^{(e)} &= \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{ch}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{k}{h}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\beta^2}{2}\left(\frac{h}{L}\right)^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right). \end{aligned}$$

The global equilibrium equation is thus

$$\begin{bmatrix} 54+2\beta^2 & -27 \\ -27 & 54+2\beta^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 27 \end{Bmatrix}\bar{u}_L.$$

It can be seen that the influence of the reaction term has vanished from the off-diagonal terms and also from the rhs vector (=same thing)

$$(\beta = 1): \quad \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.3029 \\ 0.6282 \end{Bmatrix}\bar{u}_L, \qquad (\beta = 100): \quad \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.0000 \\ 0.0013 \end{Bmatrix}\bar{u}_L.$$

Clearly the solution is stable. What about the accuracy? Determine the errors in nodal temperatures!

Problem:

Solve the 1-D stationary heat transfer problem

$$-ku'' = f_0$$

with boundary conditions

$$q(0) = -ku'(0) = -q_0 \quad \text{and} \quad q(L) = -ku'(L) = \alpha q_0.$$

The conductivity k and heat source f_0 are constants ($q_0 = \frac{1}{2}f_0 L$) and α is a positive dimensionless constant. Solve the problem by using a single quadratic element. Does the problem have a solution for arbitrary values of α ($\alpha \geq 0$)? Explain the sitution physically.

Solution:

$$\begin{aligned}
-ku'' &= f_0 \\
-\int ku''\hat{u}dx &= \int f_0\hat{u}dx \\
-\Big| ku'\hat{u} + \int ku'\hat{u}'dx &= \int f_0\hat{u}dx \\
\int ku'\hat{u}'dx &= \int f_0\hat{u}dx + ku'(L)\hat{u}(L) - ku'(0)\hat{u}(0)
\end{aligned}$$

Using hierarchic interpolation $u = \sum_{j=1}^{3} N_j u_j$ and $\hat{u} = \sum_{i=1}^{3} N_i\hat{u}_i$, where $N_1 = \frac{1}{2}(1-\xi)$, $N_2 = \frac{1}{2}(1+\xi)$ ja $N_3 = \frac{\sqrt{6}}{4}(\xi^2 - 1)$, and substituting $q_0 = \frac{1}{2}f_0 L$ and $h = L$ we get

$$\sum_i \hat{u}_i \left[\frac{2k}{h} \int_{-1}^{1} N_i' \sum N_j' u_j d\xi \right] = \sum_i \hat{u}_i \left[\frac{f_0 h}{2} \int_{-1}^{1} N_i d\xi - \alpha q_0 N_i(L) - q_0 N_i(0) \right]$$

$$\Rightarrow \frac{2k}{L} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \Delta u_3 \end{Bmatrix} = \frac{f_0 L}{2} \begin{Bmatrix} 1 \\ 1 \\ -\frac{\sqrt{6}}{3} \end{Bmatrix} - \alpha q_0 \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} - q_0 \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \frac{f_0 L}{2} \begin{Bmatrix} 0 \\ 1-\alpha \\ -\frac{\sqrt{6}}{3} \end{Bmatrix}$$

From the first equation $\frac{k}{L}(u_1 - u_2) = 0$ we get $u_1 = u_2$. The second equation $\frac{k}{L}(-u_1 + u_2) = \frac{f_0 L}{2}(1-\alpha)$ requires hence $\alpha = 1$. In other words, the heat produced by the source term will flow out equally from both boundaries. The balance law $q_0 + q_L + f_0 L = 0$ has to be fulfilled since this is a stationary case.

From the lowest equation we get the amplitude for the bubble mode $\Delta u_3 N_3(\xi = 0.5) = -\frac{\sqrt{6}}{12} f_0 L^2/k \left(-\frac{\sqrt{6}}{4}\right) = \frac{1}{8} f_0 L^2/k$. For the given boundary conditions we cannot solve the temperature uniquely.