Problem 2

Solve the diffusion-reaction equation with boundary conditions $u(0) = u_0 > 0, u(L) = 0$

$$-k\frac{d^2u}{dx^2} + bu = 0, \quad \text{where} \quad b = \beta^2 kL^{-2}$$

using a two parametric trial function for temperature u and

1. the Galerkin's method,
2. the least square method.

Draw the results with the values of $\beta = 1, 10, 100$.

Solution

The weak form of the diffusion-reaction problem is

$$\int_0^L \hat{u}\left(-k\frac{d^2u}{dx^2} + bu\right) dx = 0, \quad \text{where} \quad b = \beta^2 kL^{-2}.$$

A two-parametric trial function for temperature could be

$$u(\xi) = \phi_0(\xi)u_0 + \phi_1(\xi)\alpha_1 + \phi_2(\xi)\alpha_2 = (1-\xi)u_0 + \xi(1-\xi)\alpha_1 + \xi(1-\xi)(1-2\xi)\alpha_2,$$

where $\xi = x/L$ and a proper test function

$$\hat{u}(\xi) = \phi_1(\xi)\hat{\alpha}_1 + \phi_2(\xi)\hat{\alpha}_2.$$

Changing to the dimensionless co-ordinate ξ, the weak form can be written as ($dx = Ld\xi, d/dx = L^{-1}d/d\xi$):

$$\frac{k}{L}\int_0^1 \left(\hat{u}'u' + \beta^2\hat{u}\right) d\xi = 0,$$

since $\hat{u}(0) = \hat{u}(L) = 0$ and the prime now denotes differentiation with respect to the dimensionless co-ordinate ξ.

Case a: the Galerkin method. Testing with the function ϕ_i gives the equations

$$\int_0^1 \left[\phi_i'(\phi_0'u_0 + \phi_1'\alpha_1 + \phi_2'\alpha_2) + \beta^2\phi_i(\phi_0 u_0 + \phi_1\alpha_1 + \phi_2\alpha_2)\right] d\xi,$$

which after rearrangements has the form

$$\int_0^1 (\phi_i'\phi_1' + \beta^2\phi_i\phi_1)\, d\xi\, \alpha_1 + \int_0^1 (\phi_i'\phi_2' + \beta^2\phi_i\phi_2)\, d\xi\, \alpha_2 = -\int_0^1 (\phi_i'\phi_0' + \beta^2\phi_i\phi_0)\, d\xi\, u_0.$$

In short

$$\sum_{j=1}^{2} K_{ij}\alpha_j = f_i,$$

where

$$K_{ij} = \int_0^1 (\phi_i'\phi_j' + \beta^2\phi_i\phi_j)\,d\xi,$$
$$f_i = -\int_0^1 (\phi_i'\phi_0' + \beta^2\phi_i\phi_0)\,d\xi\,u_0.$$

Derivatives of the basis functions are:

$$\phi_0 = 1-\xi, \qquad \phi_0' = -1,$$
$$\phi_1 = \xi-\xi^2, \qquad \phi_1' = 1-2\xi,$$
$$\phi_2 = \xi-3\xi^2+2\xi^3 \qquad \phi_2' = 1-6\xi+6\xi^2.$$

Integration gives

$$K_{11} = \int_0^1 \left[(1-2\xi)^2 + \beta^2(\xi-\xi^2)^2\right]\,d\xi = \frac{11}{6} + \frac{1}{30}\beta^2,$$
$$K_{12} = K_{21} = 0,$$
$$K_{22} = \int_0^1 \left[(1-6\xi+6\xi^2)^2 + \beta^2(\xi-3\xi^2+2\xi^3)^2\right]\,d\xi = \frac{1}{5} + \frac{1}{210}\beta^2,$$
$$f_1 = -\int_0^1 \left[2\xi - 1\beta^2(\xi-2\xi^2+\xi^3)\right]\,d\xi\,u_0 = -\frac{7}{12}\beta^2 u_0,$$
$$f_2 = -\int_0^1 \left[-1+6\xi-6\xi^2+\beta^2(\xi-4\xi^2+5\xi^3-2\xi^4)\right]\,d\xi\,u_0 = -\frac{1}{60}\beta^2 u_0.$$

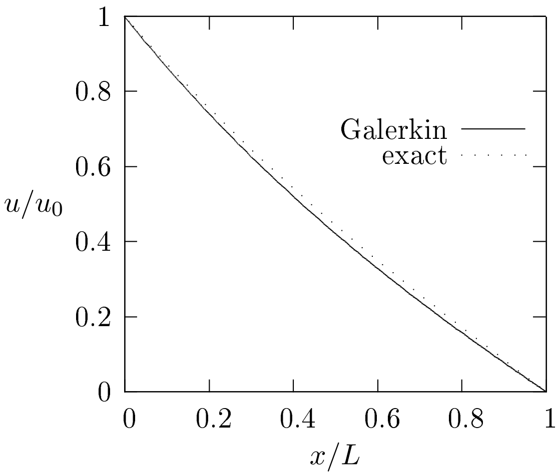
Solution is thus

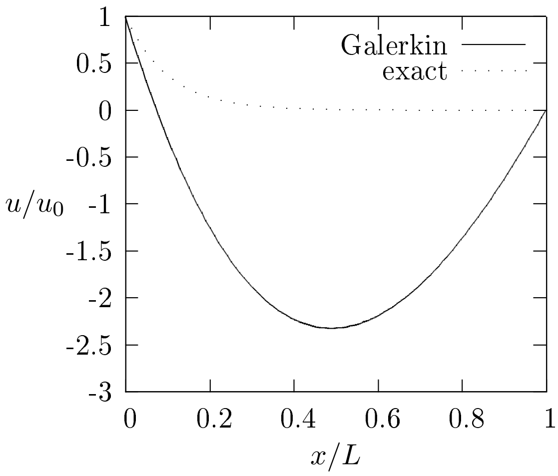
$$\alpha_1 = -\frac{105}{220+4\beta^2}\beta^2 u_0, \quad \alpha_2 = -\frac{7}{84+2\beta^2}\beta^2 u_0.$$

The limiting values occur when $\beta = 0$ or $\beta \to \infty$, giving

$$\beta = 0: \quad \alpha_1 = \alpha_2 = 0, \quad \text{or} \quad \beta\to\infty: \quad \alpha_1 \to -\frac{105}{4}u_0, \quad \alpha_2 \to -\frac{7}{2}u_0.$$

The solution is shown below for $\beta = 1, 10$ with the analytical solution, see exercise 1, problem 1b.





Case b: the least square method. Let's define the residual

$$\begin{aligned} R &= -k(\phi_{1,xx}\alpha_1 + \phi_{2,xx}\alpha_2) + b(\phi_0 u_0 + \phi_1\alpha_1 + \phi_2\alpha_2) \\ &= b\phi_0 u_0 + (-k\phi_{1,xx} + b\phi_1)\alpha_1 + (-k\phi_{2,xx} + b\phi_2)\alpha_2, \end{aligned}$$

and the least square interal

$$I = \tfrac{1}{2}\int_0^L R^2\,dx.$$

The residual can be transformed to a form

$$R = \frac{k}{L^2}\left[\beta^2\phi_0 u_0 + (-\phi_1'' + \beta^2\phi_1)\alpha_1 + (-\phi_2'' + \beta^2\phi_2)\alpha_2\right],$$

where the prime denotes differentiation with respect to ξ. Defining a non-dimensional residual $\tilde{R}$ and a non-dimensional least square integral as

$$\tilde{R} = \beta^2\phi_0 u_0 + (-\phi_1'' + \beta^2\phi_1)\alpha_1 + (-\phi_2'' + \beta^2\phi_2)\alpha_2, \quad \tilde{I} = \tfrac{1}{2}\int_0^1 \tilde{R}^2\,d\xi.$$

The existence of an extremum point requires

$$\frac{\partial\tilde{I}}{\partial\alpha_i} = \int_0^1 \tilde{R}\frac{\partial\tilde{R}}{\partial\alpha_i}\,d\xi = 0,$$

where

$$\frac{\partial\tilde{R}}{\partial\alpha_i} = -\phi_i'' + \beta^2\phi_i.$$

The resulting equation system is

$$\sum_{i=1}^{2} K_{ij} = f_i, \quad i = 1, 2,$$

and where

$$\begin{aligned} K_{ij} &= \int_0^1 (-\phi_i'' + \beta^2\phi_i)(-\phi_j'' + \beta^2\phi_j)\,d\xi, \\ f_i &= -\beta^2\int_0^1 \phi_0(-\phi_i'' + \beta^2\phi_i)\,d\xi\,u_0. \end{aligned}$$

Carrying out the integrations result in

$$\begin{aligned} K_{11} &= 4 - \tfrac{2}{3}\beta^2 + \tfrac{1}{30}\beta^4, \\ K_{12} &= K_{21} = 0, \\ K_{22} &= 12 + \tfrac{2}{5}\beta^2 + \tfrac{1}{210}\beta^4, \\ f_1 &= -\beta^2(1 + \tfrac{1}{12}\beta^2)u_0, \\ f_2 &= -\beta^2(1 + \tfrac{1}{60}\beta^2)u_0. \end{aligned}$$

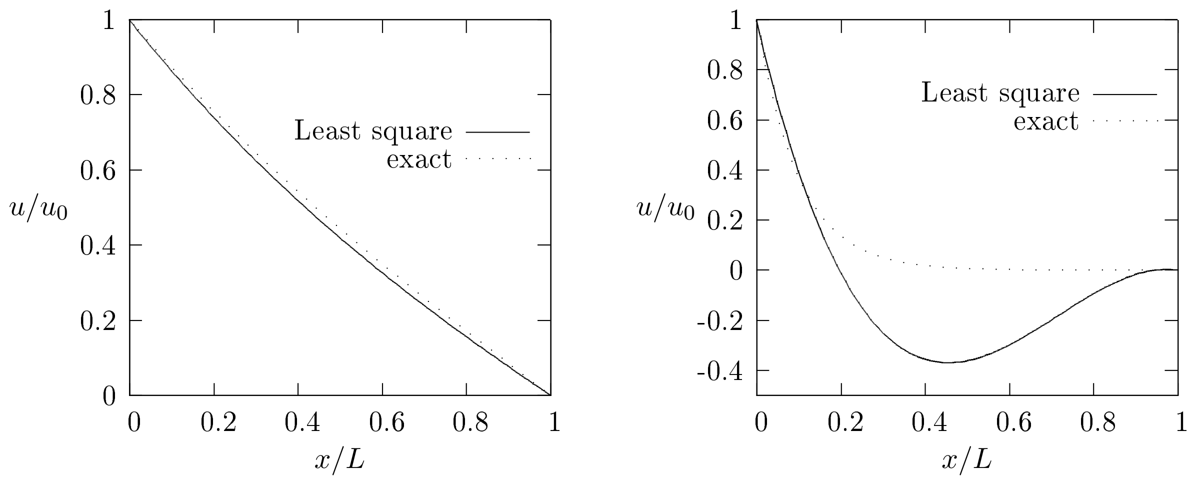
The solution is

$$\alpha_1 = -\frac{\beta^2(1 + \frac{1}{12}\beta^2)}{4 - \frac{2}{3}\beta^2 + \frac{1}{30}\beta^4}u_0, \quad \alpha_2 = -\frac{\beta^2(1 + \frac{1}{60}\beta^2)}{12 + \frac{2}{5}\beta^2 + \frac{1}{210}\beta^4}u_0.$$

The limiting values occur when $\beta = 0$ or $\beta \to \infty$, giving

$$\beta = 0: \quad \alpha_1 = \alpha_2 = 0, \quad \text{or} \quad \beta \to \infty: \quad \alpha_1 \to -\frac{5}{2}u_0, \quad \alpha_2 \to -\frac{7}{2}u_0.$$

The least square solution is shown below for $\beta = 1, 10$ with the analytical solution.



Problem 1

Solve the following beam-column problem:

$$
\begin{aligned}
&EI\frac{d^4v}{dx^4} + P\frac{d^2}{dx^2} = f = \text{constant},\\
&v(0) = v'(0) = 0, \quad M(L) = -EIv''(L) = 0,\\
&Q(L) - Pv'(L) = -EIv'''(L) - Pv'(L) = 0,
\end{aligned}
$$

using the Galerkin method using a two-parametric polynomial trial function. Draw the tip deflection as a function of the compressive load P.

If the transverse load $f = 0$, the problem is an eigenvalue problem. Solve the eigenvalues P and the corresponding eigenmodes (critical loads, and buckling modes).

Solution

Let's multiply the differential equation with the test function $\hat{v}$ and integrate over the domain

$$\int_0^L \hat{v}(EIv^{(4)} + Pv'' - f)\,dx = 0.$$

Integrating by parts will result in the form

$$\Big|_0^L \hat{v}(EIv''' + Pv') - \Big|_0^L \hat{v}'EIv'' + \int_0^L (\hat{v}''EIv'' - \hat{v}'Pv' - \hat{v}f)\,dx = 0.$$

Since at the boundary $x = 0$ essential boundary conditions are set for both the deflection and rotation, the test function has to satisfy $\hat{v}(0) = \hat{v}'(0) = 0$, thus

$$\hat{v}(L)\left[EIv'''(L) + Pv'(L)\right] - \hat{v}'(L)EIv''(L) + \int_0^L (\hat{v}''EIv'' - \hat{v}'Pv' - \hat{v}f)\,dx = 0.$$

Finally we obtain

$$\int_0^L (\hat{v}''EIv'' - \hat{v}'Pv' - \hat{v}f)\,dx = 0.$$

Notice the minus sign in the second term of the integral.

Proper two-parametric trial function and the corresponding test function are

$$v(x) = (x/L)^2\alpha_1 + (x/L)^3\alpha_2, \quad \hat{v}(x) = (x/L)^2\hat{\alpha}_1 + (x/L)^3\hat{\alpha}_2.$$

Testing with $(x/L)^2$ results in the equation

$$\int_0^L \left[\frac{2}{L^2}EI\left(\frac{2}{L^2}\alpha_1 + \frac{6x}{L^3}\alpha_2\right) - \frac{2x}{L^2}P\left(\frac{2x}{L^2}\alpha_1 + \frac{3x^2}{L^3}\alpha_2\right) - \frac{x^2}{L^2}f\right]dx = 0.$$

Testing with $(x/L)^3$ results in the equation

$$\int_0^L \left[\frac{6x}{L^3}EI\left(\frac{2}{L^2}\alpha_1 + \frac{6x}{L^3}\alpha_2\right) - \frac{3x^2}{L^3}P\left(\frac{2x}{L^2}\alpha_1 + \frac{3x^2}{L^3}\alpha_2\right) - \frac{x^3}{L^3}f\right]dx = 0.$$

Let's define the compressive axial force in the form

$$P = \lambda\frac{EI}{L^2},$$

where λ is a dimensionless load parameter. Integrating the expressions above results in

$$\frac{EI}{L^3}\begin{bmatrix} 4-\frac{4}{3}\lambda & 6-\frac{3}{2}\lambda \\ 6-\frac{3}{2}\lambda & 12-\frac{9}{5}\lambda \end{bmatrix}\left\{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} = \left\{\begin{array}{c}\frac{1}{3}\\ \frac{1}{4}\end{array}\right\} fL. \tag{1}$$

Determinant of the dimensionless stiffness matrix is

$$\det(\tilde{\mathbf{K}}) = (4-\tfrac{4}{3}\lambda)(12-\tfrac{9}{5}\lambda) - (6-\tfrac{3}{2}\lambda)^2 = \tfrac{3}{20}\lambda^2 - \tfrac{26}{5}\lambda + 12.$$

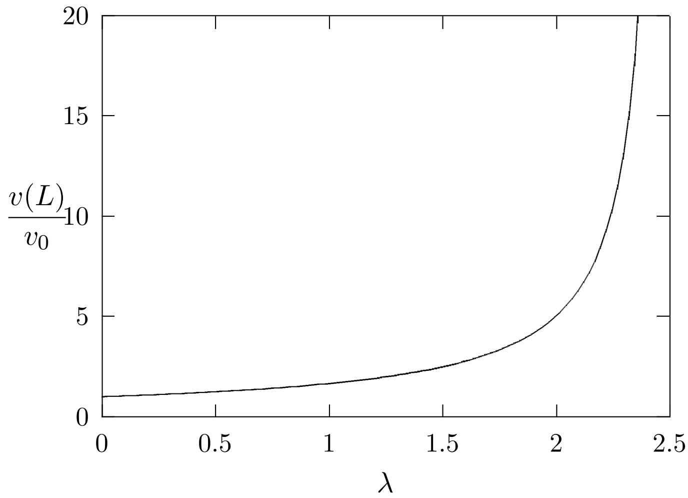
Solution of the discrete equilibrium equations (1) is

$$\left\{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} = \frac{1}{\frac{3}{20}\lambda^2 - \frac{26}{5}\lambda + 12}\begin{bmatrix} 12-\frac{9}{5}\lambda & \frac{3}{2}\lambda - 6 \\ \frac{3}{2}\lambda - 6 & 4-\frac{4}{3}\lambda \end{bmatrix}\left\{\begin{array}{c}\frac{1}{3}\\ \frac{1}{4}\end{array}\right\}\frac{fL^4}{EI}.$$

The tip deflection is

$$v(L) = \alpha_1 + \alpha_2 = \frac{\frac{3}{2} - \frac{7}{120}\lambda}{\frac{3}{20}\lambda^2 - \frac{26}{5}\lambda + 12}\frac{fL^4}{EI},$$

which is shown as a function of the load parameter λ in the following figure. The displacement is normalized to the tip deflection without compressive load, i.e. when $\lambda = 0, \Rightarrow v(L) = v_0 = \frac{1}{8}fL^4/EI$.



Notice that the displacements start to increase rapidly when $\lambda > 2$, which is about 20 % of the critical buckling load $\lambda_{\text{cr}} = \frac{1}{4}\pi^2 \approx 2.467$. The superposition principle is not valid for the axial load P. Why?

When $f \equiv 0$ the proble is an linear eigenvalue problem. The critical load P_{cr} can be solved from the generalized linear eigenvalue problem

$$\left(\frac{EI}{L^3}\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} - \frac{P}{L}\begin{bmatrix} \frac{4}{3} & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{5} \end{bmatrix}\right)\left\{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} = \left\{\begin{array}{c}0\\ 0\end{array}\right\}, \tag{2}$$

or expressed in an dimensionless form

$$\frac{EI}{L^3}\left(\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} - \lambda\begin{bmatrix} \frac{4}{3} & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{5} \end{bmatrix}\right)\left\{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} = \left\{\begin{array}{c}0\\ 0\end{array}\right\}. \tag{3}$$

A homogeneous linear equation has a non-trivial solution only if the coefficeint matrix is singular, i.e. if $\det(\mathbf{K}) = 0$, which gives

$$\lambda = \tfrac{52}{3} \pm \sqrt{\left(\tfrac{52}{3}\right)^2 - 80} \approx 2.4860 \quad (\text{or} \quad 32.18).$$

The error to the exact value is only 0.8 %.

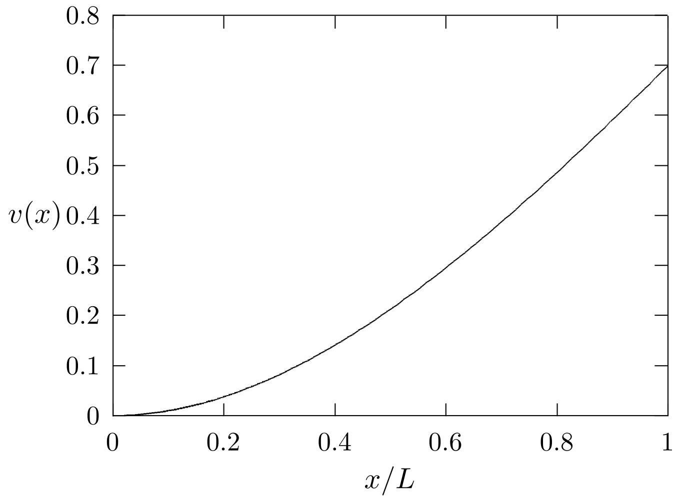
The buckling mode can be solved when substituting the critical value to the equation (2) or (3), giving

$$0.6853\alpha_1 + 2.271\alpha_2 = 0.$$

The solution for the buckling mode is naturally non-unique. Only the form of the deflection can be determined, the absolute values are undetermined. Therefore the buckling mode is of the form

$$v_1(x) = \alpha \left(\frac{x}{L}\right)^2 \left(1 - 0.302\frac{x}{L}\right),$$

which is shown below



The higher, practically irrelevant bucling load is $\lambda_{\text{cr},2} \approx 32.18$, which gives

$$-38.9\alpha_1 - 42.27\alpha_2 = 0,$$

and the mode has the form

$$v_2(x) = \alpha \left(\frac{x}{L}\right)^2 \left(1 - 1.087\frac{x}{L}\right).$$